PRIVATE
DEPOSITORY

SHL
WITHDRAWN

Digitized by the Internet Archive
in 2026

https://archive.org/details/london-mathematical-society-proceedings_1930-1931_32

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 32

LONDON:
PRINTED AND PUBLISHED FOR THE SOCIETY BY
C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C. 2.

1931

BIBL.
LONDIN.
UNIV.

CONTENTS.

Papers published in the *Proceedings*, pp. 1–556.

Index of Papers published in this Volume arranged according to the names of Authors, p. 557.

Index according to Subjects, p. 558.

Corrigenda, p. 558.

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON SUFFICIENT CONDITIONS FOR A FUNCTION TO BE ANALYTIC, AND ON BEHAVIOUR OF ANALYTIC FUNCTIONS IN THE NEIGHBOURHOOD OF NON-ISOLATED SINGULAR POINTS

By A. S. Besicovitch.

[Received 12 October, 1929.—Read 14 November, 1929.]

In this note are given generalizations of the following two theorems :—

Riemann's theorem. *If a function $f(z)$ of a complex variable defined in an open simply connected domain D is known to be bounded in the domain and is regular at all points of the domain, except possibly at a point $z = a$, then $\lim_{z \to a} f(z)$ exists, and the function $f(z)$ defined at the point $z = a$ by the value of this limit is also regular at the point $z = a$.*

Osgood's theorem*. *If a function $f(z)$ of a complex variable defined in an open simply connected domain D is known to be continuous in the domain and regular at all points except at those of a finite number of regular curves, then it is regular at all points of the domain.*

The two theorems can be considered from the point of view of studying the nature of singularities of analytic functions, and also from the point of view of a possibly smaller number of conditions for a function to be regular in a domain. The last problem has been considered in a series

* W. F. Osgood, *Bull. American Math. Soc.* (2), 2 (1896), 296.

of papers by various authors*. Each of them also gives a contribution to the first problem. I also consider these two problems under somewhat different conditions.

Let $f(z)$ be a function of a complex variable defined in a domain D, and let $f(z)$ have a finite derivative at the points of a subset G of the domain D. Under what conditions can we conclude the existence of the derivative at all other points of D, *i.e.* at the points of the set $E = D - G$?

Riemann's theorem states that, in the case of a bounded function, the set E may consist of one and consequently of any finite number of points.

Osgood's theorem states that, in the case of a continuous function, the set E may consist of any finite number of regular curves.

We can state immediately the bounds beyond which the generalization of these theorems cannot go. In the case of Riemann's conditions, the set E cannot be an arbitrary set of positive linear measure. For, defining $f(z)$ inside a circle belonging to D by values of one analytic function and outside the circle by values of another, we shall have the set E consisting of the points of the circumference and the function $f(z)$ non-analytic in the domain D. It is not difficult also to show that, in the case of Osgood's conditions, the set E cannot be an arbitrary set of $\aleph_c$ regular curves, even of plane measure zero.

The theorems that will be proved in this note show that the above limits are just the exact bounds of possible generalizations.

The theorems to be proved are the following :—

THEOREM 1. *If a function $f(z)$ of a complex variable defined in an open simply connected domain D is known to be bounded in the domain and to be differentiable (i.e. to have a finite derivative) at all points of the domain except possibly at the points of a set E of linear measure zero, then for every point a of E the limit of $f(z)$, as z tends to a through values of $D-E$, exists, and the function $f(z)$, defined at the points of E by values of these limits, is also differentiable at the points of E, and thus is holomorphic in the domain D.*

---

* P. Montel, (*a*) "Sur les suites infinies de fonctions", *Ann. sci. de l'Ecole Norm. sup.* (3), 24 (1907), 233–334 (298–334); (*b*) "Sur les différentielles totales et les fonctions monogènes", *Comptes Rendus*, 156 (1913), 1820–1822. L. Lichtenstein, (*a*) "Sur la definition générale des fonctions analytiques, *Comptes Rendus*, 150 (1910), 1109–1110; (*b*) "Über einige Integrabilitätsbedingungen zweigliedriger Differentialausdrücke mit einer Anwendung auf den Cauchyschen Integralsatz", *Sitzungsberichte der Berliner Math. Ges.*, 9 (1910), 84–100. Ch. de la Vallée-Poussin, "Réduction des intégrales doubles de Lebesgue. Application à la définition des fonctions analytiques", *Acad. R. de Belgique, Bull. de la classe des. sci.* (1910), 11, 793–798. H. Rademacher, "Bemerkungen zu den Cauchy-Riemannschen Differentialgleichungen und zum Moreraschen Satz", *Math. Zeitschrift*, 4 (1919), 177–185. S. Pollard, "On the conditions for Cauchy's theorem", *Proc. London Math. Soc.* (2), 21 (1923), 456–482.

THEOREM 2. *If a function $f(z)$ of a complex variable defined in an open simply connected domain D is known to be continuous at all points of D and to be differentiable at all points except possibly at the points of a set E of finite or of enumerably infinite linear measure*, then $f(z)$ is also differentiable at the points of E, and thus is holomorphic in the domain D.*

Theorem 1 enables us to generalize the Weierstrass theorem on behaviour of a single-valued analytic function in the neighbourhood of an isolated essential singularity to the case of a non-isolated one. In fact, we shall prove the following theorem :—

THEOREM 3. *If the set E of essential singularities of an analytic function $f(z)$ is of linear measure zero, then the set of values of $f(z)$ in the neighbourhood of each of the points of E is everywhere dense on the complex plane.*

Proof of Theorem 3. Suppose that the theorem is false. Then there exists a point a of E and a number c such that, in the neighbourhood of a, the function $f(z)$ does not take values arbitrarily near c. Then the function $[f(z)-c]^{-1}$ is bounded in the neighbourhood of a, say in a domain D, including the point a. The singular points of $[f(z)-c]^{-1}$ are the points of E. Thus in the domain D the function $[f(z)-c]^{-1}$ is bounded and regular at all points except those of the set E. The set E being of measure zero, on account of Theorem 1 we conclude that $[f(z)-c]^{-1}$ is regular also at all points of E, which is impossible.

Proof of Theorem 1. We shall prove the theorem by proving that the integral $\int f(z)\,dz$ taken along the boundary of any square completely included in the domain D is zero. Suppose that the theorem is false, and that there exists a square A_1 of side a such that

$$\left|\int_{A_1} f(z)\,dz\right| = m_1 > 0 \tag{1}$$

(on account of the Morera theorem).

We introduce the following notation. If to each element of a finite or an enumerably infinite set G is related a value of a symbol C, then we denote by $\sum_G C$ the sum of values of C corresponding to all elements of G.

Now let M denote the upper bound of $|f(z)|$ in the domain D. Since the measure of the set E is zero, we can define a finite or an enumerably infinite set U of convex areas such that each point of E is an interior

* A set E is said to be of enumerably infinite linear measure if it can be split into an enumerably infinite set of sets of finite linear measure.

point of at least one of these areas, and such that, if d denotes the diameter of an area, the following inequality is satisfied :

$$\sum_U d < \frac{m_1}{64\,M}. \tag{2}$$

Let $n_1 < n_2 < n_3 \ldots$ be a sequence of positive integers. Denote by U_1 the set of those areas of U whose diameter is greater than or equal to $2^{1-n_1}a$, by U_2 the set of those areas of U whose diameter is less than $2^{1-n_1}a$ and greater than or equal to $2^{1-n_2}a$, and so on. We have

$$U = U_1 + U_2 + U_3 + \ldots. \tag{3}$$

From (2) we conclude that the numbers n_1, n_2, ... can be so chosen (and we shall assume that they are so chosen) that the inequality

$$16 \sum_{U_1} d + 64 \sum_{U_2} d + 256 \sum_{U_3} d + \ldots < \frac{m_1}{2M} \tag{4}$$

is satisfied. We now divide A_1 into $s_1 = 2^{2n_1}$ equal squares $A_{11}, A_{12}, \ldots, A_{1s_1}$, by parallels to the sides of A_1. To each area of U_1 there corresponds the least rectangle containing it inside (in the strict sense) and formed by these parallels. Obviously the perimeter of the rectangle corresponding to an area of diameter d is less than or equal to $8d$. In the general case the rectangles $u_{11}, u_{12}, \ldots, u_{1k}$ corresponding to the areas of U_1 are overlapping. In this case we form a system of non-overlapping figures $v_{11}, v_{12}, \ldots, v_{1k}$ in the following way. We take u_{11} for v_{11}; further, v_{12} is the part of u_{12} which is outside u_{11}; v_{13} is the part of u_{13} which is outside u_{11} and u_{12}, and so on. It is clear that the sum of the perimeters of the figures $v_{11}, v_{12}, \ldots$ is less than the double sum for the rectangles $u_{11}, u_{12}, \ldots, u_{1k}$, and thus is less than $16 \sum_{U_1} d$. Consequently

$$\left| \sum_{i=1}^{k} \int_{v_{1i}} f(z)\,dz \right| < 16M \sum_{U_1} d. \tag{5}$$

Now we write

$$\int_{A_1} f(z)\,dz = \sum_i{}' \int_{A_{1i}} f(z)\,dz + \sum_i{}'' \int_{A_{1i}} f(z)\,dz, \tag{6}$$

where the summation Σ' is extended over those squares A_{1i} which are outside all figures $v_{11}, v_{12}, \ldots, v_{1k}$, and Σ'' over the other squares. We have

$$\Sigma'' \int_{A_{1i}} f(z)\,dz = \sum_{i=1}^{k} \int_{v_{1i}} f(z)\,dz. \tag{7}$$

By (1), (5), (6), (7),

$$\left|\Sigma'\int_{A_{1i}} f(z)\,dz\right| > m_1 - 16M\sum_{U_1} d. \tag{8}$$

Denoting by (U_jA_{1i}) $(j>1)$ those of the areas of U_j which have points common with A_{1i}, we write

$$\sum_i{}'\left\{\left|\int_{A_{1i}} f(z)\,dz\right| - 4^2M\sum_{(U_2A_{1i})} d - 4^3M\sum_{(U_3A_{1i})} d - \ldots\right\} \tag{9}$$
$$= \sum_i{}'\left|\int_{A_{1i}} f(z)\,dz\right| - 4^2M\sum_i{}'\sum_{(U_2A_{1i})} d - 4^3M\sum_i{}'\sum_{(U_3A_{1i})} d - \ldots.$$

We observe now that no area of U_2, U_3, ... can have points common with more than four squares A_{1i}. Hence

$$\sum_i{}'\sum_{(U_jA_{1i})} d \leqslant 4\sum_{U_j} d \quad (j=2,\ 3,\ \ldots). \tag{10}$$

By (8), (9), (10),

$$\sum_i{}'\left\{\left|\int_{A_{1i}} f(z)\,dz\right| - 4^2M\sum_{(U_2A_{1i})} d - 4^3M\sum_{(U_3A_{1i})} d - \ldots\right\} \tag{11}$$
$$> m_1 - 4^2M\sum_{U_1} d - 4^3M\sum_{U_2} d - \ldots.$$

Hence outside the figures v_{11}, v_{12}, ... there exists at least one square A_{1i}, say A_2, such that the inequality

$$\left|\int_{A_2} f(z)\,dz\right| - 4^2M\sum_{(U_2A_2)} d - 4^3M\sum_{(U_3A_2)} d - \ldots \tag{12}$$
$$> 2^{-2n_1}\left\{\left|\int_{A_1} f(z)\,dz\right| - 4^2M\sum_{U_1} d - 4^3M\sum_{U_2} d - \ldots\right\}$$

is satisfied.

Thus, starting from the square A_1, we have arrived at a square A_2 outside the set U_1 (and, in fact, outside the rectangles u_{11}, u_{12}, ..., u_{1k}), for which the inequality (12) *is satisfied.*

We now divide the square A_2 into $2^{2(n_2-n_1)}$ equal squares by parallels to the sides of A_2, and, as before, we define the least rectangles u_{21}, u_{22}, ..., u_{2l}, formed by these parallels and by the sides of A_2, which contain the parts of the areas U_2 belonging to A_2. In the same way, as before, we define a square A_3 outside the set U_2, and such that the following inequality is satisfied:

$$\left|\int_{A_3} f(z)\,dz\right| - 4^2M\sum_{(U_3A_3)} d - 4^3M\sum_{(U_4A_3)} d - \ldots \tag{13}$$
$$> 2^{-2(n_2-n_1)}\left\{\left|\int_{A_2} f(z)\,dz\right| - 4^2M\sum_{(U_2A_2)} d - 4^3M\sum_{(U_3A_2)} d - \ldots\right\}.$$

Proceeding in this way we arrive at a sequence of squares

(14) $$A_1,\ A_2,\ A_3,\ \ldots,$$

such that, for any i,

(i) A_i is of side $2^{-n_{i-1}}a$;

(ii) A_i is contained in A_{i-1};

(iii) A_i is outside all sets $U_1, U_2, \ldots, U_{i-1}$;

(iv) the following inequality is satisfied:

(15) $$\left|\int_{A_i} f(z)\,dz\right| - 4^2 M \sum_{(U_i A_i)} d - 4^3 M \sum_{(U_{i+1} A_i)} d - \ldots$$
$$> 2^{-2(n_{i-1}-n_{i-2})}\left\{\left|\int_{A_{i-1}} f(z)\,dz\right| - 4^2 M \sum_{(U_{i-1} A_{i-1})} d - 4^3 M \sum_{(U_i A_{i-1})} d - \ldots\right\}.$$

From (12), (13), (15), (4), we conclude that

(16) $$\left|\int_{A_i} f(z)\,dz\right| - 4^2 M \sum_{(U_i A_i)} d - 4^3 M \sum_{(U_{i+1} A_i)} d - \ldots > 2^{-2n_{i-1}} m_1,$$

and, *a fortiori*,

(17) $$\left|\int_{A_i} f(z)\,dz\right| > 2^{-2n_{i-1}} m_1.$$

Now we can apply the well known argument employed by Goursat for the proof of Cauchy's theorem. The sequence (14) defines a limit point c which cannot be an interior point of any of the sets $U_1, U_2, \ldots$, and consequently cannot belong to the set E. Thus at the point c the function $f(z)$ has a finite derivative. We write

(18) $$f(z) = f(c) + f'(c)(z-c) + \phi(z)(z-c),$$

where

(19) $$\lim \phi(z) = 0, \quad \text{as } z \to c.$$

Denote by η_i the upper bound of $|\phi(z)|$ on the boundary of A_i and take i so large that

(20) $$\eta_i < \frac{m_1}{8a^2}.$$

We now write

(21) $$\int_{A_i} f(z)\,dz = \int_{A_i} \phi(z)(z-c)\,dz.$$

Since the point c belongs to A_i, $|z-c|<2a\,2^{-n_{i-1}}$, and consequently

$$\left|\int_{A_i} f(z)\,dz\right| < \int_{A_i} |\phi(z)|\,.\,|z-c|\,.\,|dz| < \eta_i\, 2a2^{-n_{i-1}}\,.\,4a2^{-n_{i-1}}. \tag{22}$$

By (20), (22),

$$\left|\int_{A_i} f(z)\,dz\right| < m_1\, 2^{-2n_{i-1}}; \tag{23}$$

(17) and (23) being contradictory, the theorem is proved.

Proof of Theorem 2. The proof of Theorem 2 is similar to that of Theorem 1, and is only slightly more complicated in details. We shall prove the theorem for the case of the set E of infinite measure. The case of finite measure will be an immediate corollary.

We represent the set E as the sum

$$E = E_1+E_2+E_3+\ldots, \tag{24}$$

the measure of each E_i being equal to 1.

We assume, as before, that the theorem is not true and that there exists a square A_1 of side a such that

$$\left|\int_{A_1} f(z)\,dz\right| = m_1 > 0. \tag{25}$$

We now define a decreasing sequence

$$\epsilon_1 > \epsilon_2 > \ldots, \quad \epsilon_n \to 0, \tag{26}$$

such that

$$\epsilon_i < 2^{-i-7}\, m_1 \quad (i = 1, 2, \ldots). \tag{27}$$

Define now the numbers δ_1, δ_2, ... by the condition that, for each i,

$$|f(z_1)-f(z_2)| < \epsilon_i \text{ if only } |z_1-z_2| < 3\delta_i. \tag{28}$$

Corresponding to each E_i we construct a finite or an enumerable set $U(E_i, \delta_i)$ of convex areas, each of diameter less than δ_i and such that every point of E_i is an interior point of at least one of these areas, and that the inequality

$$\sum_{U(E_i,\,\delta_i)} d < 2 \tag{29}$$

is satisfied. We write

$$U = U(E_1,\, \delta_1)+U(E_2,\, \delta_2)+\ldots. \tag{30}$$

U is an enumerable set of convex areas. We attach the values of two numbers to each area of U : the value d of its diameter and the value i of

$U(E_i, \delta_i)$, to which the area belongs. We consider $\sum_U \epsilon_i d$. On account of (27) and (29) we have

$$\sum_U \epsilon_i d = \sum_{i=1}^{\infty} \epsilon_i \sum_{U(E_i, \delta_i)} d < 2 \sum_{i=1}^{\infty} \epsilon_i < 2^{-6} m_1. \tag{31}$$

In the same way as before we introduce numbers $n_1 < n_2 < \ldots$, and we divide U into partial sets

$$U = U_1 + U_2 + \ldots. \tag{32}$$

Each of the sets U_1, U_2, ... contains only a finite number of convex areas.

On account of (31) we conclude that we can choose the numbers n_1, n_2, ..., so that the inequality

$$\sum_{U_1} \epsilon_i d + 4 \sum_{U_2} \epsilon_i d + 4^2 \sum_{U_3} \epsilon_i d + \ldots < 2^{-5} m_1 \tag{33}$$

is satisfied.

Suppose that the whole of U_1 is formed of some of the areas of the sets $U(E_1, \delta_1)$, $U(E_2, \delta_2)$, ..., $U(E_j, \delta_j)$, and denote by $c_1^{(l)}, c_2^{(l)}, \ldots, c_{k_l}^{(l)}$ the areas common to U_1 and $U(E_l, \delta_l)$, so that

$$U_1 \times U(E_l, \delta_l) = c_1^{(l)} + c_2^{(l)} + \ldots + c_{k_l}^{(l)} \quad (l = 1, 2, \ldots, j). \tag{34}$$

By the definition of U_1 the diameter of each of its areas is greater than or equal to $2a/2^{n_1}$. We now divide the square A_1 into 2^{2n_1} equal squares by parallels to the sides of A_1 and form the least rectangles

$$u_1^{(1)}, \; u_2^{(1)}, \; \ldots, \; u_{k_1}^{(1)}, \; u_1^{(2)}, \; u_2^{(2)}, \; \ldots, \; u_{k_j}^{(j)}, \tag{35}$$

including the areas $c_1^{(1)}, c_2^{(1)}, \ldots$. Denoting by $bd(A)$ and $d(A)$ the length of the boundary and the diameter of the figure A, we have

$$bd(u_m^{(l)}) < 8d(c_m^{(l)}), \quad d(u_m^{(l)}) < 3d(c_m^{(l)}). \tag{36}$$

We construct the system of figures

$$v_1^{(1)}, \; v_2^{(1)}, \; \ldots, \; v_{k_1}^{(1)}, \; v_1^{(2)}, \; v_2^{(2)}, \; \ldots, \; v_{k_j}^{(j)} \tag{37}$$

in the same way as before, *i.e.* we take $u_1^{(1)}$ for $v_1^{(1)}$, the part of $u_2^{(1)}$ outside $u_1^{(1)}$ for $v_2^{(1)}$, and so on. We now compare the sums

$$\Sigma 2\epsilon_l \, bd(u_m^{(l)}), \quad \Sigma \epsilon_l \, bd(v_m^{(l)}), \tag{38}$$

where the summations are respectively extended over all terms of (35) and of the system (37). The boundaries of the figures (37) are formed of segments of the boundaries of the rectangles (35). No three figures of (37) have a common segment in their boundaries. If a segment λ belongs to the boundary of only one figure $v_m^{(l)}$, then it belongs to $u_m^{(l)}$,

and thus the contributions made by this segment to the sums (38) are respectively $2\epsilon_l\lambda$ and $\epsilon_l\lambda$. If a segment λ belongs to two figures $v_m^{(l)}$, $v_{m'}^{(l')}$, of which $v_m^{(l)}$ precedes $v_{m'}^{(l')}$ in (37), then the contributions made by λ to the sums will be $2\epsilon_l\lambda$ and $\epsilon_l\lambda+\epsilon_{l'}\lambda$, where $\epsilon_{l'} \leqslant \epsilon$, since $l' \geqslant l$. Thus in both cases the contribution made by λ to the first sum is not less than the contribution made to the second sum. Hence

$$\Sigma\, \epsilon_l\, bd(v_m^{(l)}) \leqslant \Sigma\, 2\epsilon_l\, bd(u_m^{(l)}). \tag{39}$$

Consider now an integral $\int_C f(z)\,dz$ taken along a closed contour C. Let z_1 be a point of C, and let $O\{f(z)\ C\}$ denote the upper bound of $|f(z'')-f(z')|$ when z' and z'' lie on the contour C. We have

$$\int_C f(z)\,dz = \int_C \{f(z)-f(z_1)\}\,dz, \tag{40}$$

and consequently

$$\left|\int_C f(z)\,dz\right| \leqslant O\{f(z), C\}\,.\,bd(C). \tag{41}$$

We apply this to the integrals $\int_{v_m^{(l)}} f(z)\,dz$. We obviously have

$$d(v_m^{(l)}) \leqslant d(u_m^{(l)}),$$

and so, by (36),

$$d(v_m^{(l)}) < 3d(c_m^{(l)}) < 3\delta_l. \tag{42}$$

We have, on account of (28) and (42),

$$O\{f(z), v_m^{(l)}\} < \epsilon_l, \tag{43}$$

and thus, on account of (41),

$$\left|\int_{v_m^{(l)}} f(z)\,dz\right| < \epsilon_l\, bd(v_m^{(l)}). \tag{44}$$

Hence

$$\left|\Sigma\int_{v_m^{(l)}} f(z)\,dz\right| < \Sigma\, \epsilon_l\, bd(v_m^{(l)}), \tag{45}$$

where the summation is extended over all terms of (37).

By (45), (39), (36),

$$\left|\Sigma\int_{v_m^{(l)}} f(z)\,dz\right| < \Sigma\, 16\epsilon_l\, d(c_m^{(l)}) = \sum_{U_1} 16\epsilon_i\, d. \tag{46}$$

The inequality (46) is analogous to the inequality (5) of the proof of Theorem 1, and now the proof of Theorem 2 may be completed in the same way as that of Theorem 1.

THE GROUPS OF THE REGULAR SOLIDS IN n DIMENSIONS

By D. E. LITTLEWOOD.

[Received 25 September, 1929.—Read 14 November, 1929.]

1. The groups of the three-dimensional solids play so important a rôle in group theory that no apology is felt to be necessary for a study of the same problem in n dimensions. The case of four dimensions will be considered first : the group of the 600-hedroid* and its reciprocal, the 120-hedroid, and of the 24-hedroid reciprocal to itself, will be examined. The other regular four-dimensional solids, namely the pentahedroid, the hypercube, and the 16-hedroid, will be examined incidentally as particular cases of the simplex, the orthogonal, and the reciprocal to the latter. Since these last three, existent in any number of dimensions†, constitute the only regular solids in more than four dimensions, every regular solid will thus have been considered.

2. We need first a short investigation into the nature of rotations in four dimensions. In three dimensions the resultant of any number of rotations about different axes is always equivalent to a single rotation about some axis, but in four dimensions this is not the case. Take orthogonal axes $OX_1X_2X_3X_4$. Then two rotations in the X_1X_2 and in the X_3X_4 planes respectively, taken together, will leave no line invariant, and so cannot be equivalent to any plane rotation. Nevertheless, these two plane rotations may take place together with perfect smoothness, and we shall speak of such a change in position as a rotation. If a rotation leaves one line invariant, it reduces to a rotation in three dimensions. In this sub-space another line is invariant, and the plane containing

* Schoute, *Mehrdimensionale Geometrie* (1902), Teil 2.

† Manning, *Geometry of four dimensions* (1918), ch. 8.

these two lines is an invariant plane. The rotation takes place in the plane perpendicular to this plane, and is referred to as a plane rotation to distinguish it from the more general case. It should be noted that perpendicular is here used in a stronger sense than is usual in three dimensions. We say that two planes are perpendicular if, and only if, every pair of lines, one being in each plane, is perpendicular. This defines in four dimensions a unique plane perpendicular to a given plane through any given point.

Rotations may be resolved, as is usual in three dimensions, into rotations about the axial planes. Of these there are six, and a rotation may be regarded as a six-vector. We use f_{ij} to denote a unit rotation in the X_iX_j plane, from OX_i towards OX_j. Thus $f_{ij} = -f_{ji}$. If the a_{ij}'s are scalar, $\Sigma a_{ij}f_{ij}$ represents any rotation, the various plane rotations indicated being supposed to take place not consecutively, but simultaneously. There are two independent invariants of a six-vector, and the invariants of the rotation will be Σa_{ij}^2 and $a_{12}a_{34}+a_{13}a_{42}+a_{14}a_{23}$. We shall call these F and G respectively, and we shall consider ϕ the invariant ratio $2G/F$.

Since $\Sigma(a_{12}-a_{34})^2$ and $\Sigma(a_{12}+a_{34})^2$ are essentially positive, we have $-F \leqslant 2G \leqslant F$, and hence ϕ varies between -1 and $+1$.

If $\phi = +1$, then

$$a_{12} = a_{34}, \quad a_{13} = a_{42}, \quad a_{14} = a_{23},$$

and if $\phi = -1$, then

$$a_{12} = -a_{34}, \quad a_{13} = -a_{42}, \quad a_{14} = -a_{23}.$$

Rotations for which $\phi = +1$ will be spoken of as right-handed, and those for which $\phi = -1$ as left-handed rotations. For plane rotations $\phi = 0$.

If we consider a rotation as separated into two parts, as $a_{12}f_{12}+a_{13}f_{13}+a_{14}f_{14}$ and $a_{34}f_{34}+a_{42}f_{42}+a_{23}f_{23}$, we see that a right-handed rotation consists of a rotation in one plane together with an equal rotation in the plane perpendicular to it, though the choice of the pair of planes is somewhat arbitrary. For left-handed rotations, the direction of the rotation in the second plane is reversed.

If, for two rotations, ϕ is respectively $+1$ and -1, then, for the combined rotation, ϕ takes a value between these, *i.e.* it is neither $+1$ nor -1.

Let the rotations be $\Sigma a_{ij}f_{ij}$ and $\Sigma a'_{ij}f_{ij}$, and the combined rotation $\Sigma(a_{ij}+a'_{ij})f_{ij}$, the invariants being F_1, G_1, ϕ_1; F_2, G_2, ϕ_2; and F_3, G_3, ϕ_3 respectively.

Then $F_1 = 2G_1$, $F_2 = -2G_2$,

$$2G_3 = 2\Sigma(a_{12}+a'_{12})(a_{34}+a'_{34}) = \Sigma(a_{12}+a'_{12})(a_{12}-a'_{12})$$
$$= \Sigma a_{12}^2 - \Sigma a_{12}'^2,$$

$$F_3 = \Sigma(a_{12}+a'_{12})^2 = \Sigma a_{12}^2 + \Sigma a_{12}'^2 + \Sigma a_{12}a'_{12}$$
$$= \Sigma a_{12}^2 + \Sigma a_{12}'^2,$$

since $$a_{12}a'_{12} = -a_{34}a'_{34}.$$

Thus clearly $$\phi_3 = \frac{\Sigma a_{12}^2 - \Sigma a_{12}'^2}{\Sigma a_{12}^2 + \Sigma a_{12}'^2}$$

lies between -1 and $+1$.

Associate with each point in four space, with coordinates x_1, x_2, x_3, x_4, the quaternion $Q = x_1+ix_2+jx_3+kx_4$. The equation $Q' = AQB$, where A and B are quaternions, represents a bi-uniform mapping of the four space upon itself. If A and B are of unit norm, this represents the most general orthogonal transformation of unit determinant*, and therefore the most general rotation of the axes.

Consider first $Q' = AQ$. We shall show that this represents a left-hand rotation, *i.e.* for this rotation $\phi = -1$.

Let $A = \cos\theta + V\sin\theta$, where V is a vector with unit norm. Since, from the nature of quaternions, all directions in the $X_2X_3X_4$ sub-space are equivalent, we may, without loss of generality, take $V = i$.

Thus we have $Q' = (\cos\theta + i\sin\theta)Q$.

$$Q = x_1+ix_2+jx_3+kx_4.$$

Hence
$$Q' = (x_1\cos\theta + x_2\sin\theta) + (x_2\cos\theta - x_2\sin\theta)\,i$$
$$+(x_3\cos\theta - x_4\sin\theta)\,j + (x_4\cos\theta + x_3\sin\theta)\,k.$$

The rotation is equivalent to a rotation through an angle θ in the X_1X_2 plane together with a rotation through an angle $-\theta$ in the X_3X_4 plane. Thus the invariant $\phi = -1$.

Similarly it can be shown that, for the rotation represented by $Q'' = Q'B$, $\phi = +1$.

Clearly the two rotations $Q' = AQ$ and $Q'' = Q'B$ are commutative, for, if the two are applied successively in either order, $Q'' = AQB$ results. And for this transformation ϕ is neither $+1$ nor -1, unless A or B is real. The condition that it represents a plane rotation is that the scalar parts of A and B are equal.

* A. Hurwitz, *Zahlentheorie der Quaternionen* (1919), 66.

Thus all right-handed rotations may be represented by the transformation $Q' = QB$, and all left-handed rotations by the transformation $Q' = AQ$; and conversely these represent right- and left-hand rotations respectively.

As a corollary we have the interesting theorem that every right-handed rotation is commutative with every left-handed rotation.

We are now in a position to consider our main problem.

3. We shall first examine the 24-hedroid. Suppose it to be inscribable in a unit hypersphere. Then if the centre be the origin, and the solid be properly oriented, the vertices will be those points represented by the quaternions

$$\pm 1,\ \pm i,\ \pm j,\ \pm k, \quad \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k). \tag{1}$$

i.e. the twenty-four quaternion units.

If A represents any of these units, the transformation $Q' = A^{-1}Q$ represents some rotation which rotates the vertex at A into the vertex at $Q = 1$, and the 24-hedroid into itself, for the product of any two units is also a unit, and every vertex becomes a vertex.

The transformations representing rotations of the 24-hedroid into itself, which leave the vertices $Q = \pm 1$ unaltered, will be of the form $Q' = B^{-1}QB$. These particular rotations, leaving the scalar axis unchanged, take place in a three-dimensional sub-space, and rotate the cube of vertices $\frac{1}{2}(1 \pm i \pm j \pm k)$ into itself, and therefore form an octahedral group, of order 24. If B be changed to $-B$, the transformation is unaltered. If, then, B is confined to the twenty-four units, the resulting group, necessarily a sub-group of the above octahedral group, will be of order 12. This must be the tetrahedral group. Another operation is necessary. We may take $B = (1+i)/\sqrt{2}$. Combining these values, we see that, for the octahedral group which we require, B must belong to the set

$$\pm 1,\ \pm i,\ \pm j,\ \pm k, \quad \tfrac{1}{2}(\pm 1 \pm i \pm j \pm k), \tag{2}$$

$$\frac{1}{\sqrt{2}}(\pm 1 \pm i), \quad \frac{1}{\sqrt{2}}(\pm 1 \pm j), \quad \frac{1}{\sqrt{2}}(\pm 1 \pm k),$$

$$\frac{1}{\sqrt{2}}(\pm i \pm j), \quad \frac{1}{\sqrt{2}}(\pm i \pm k), \quad \frac{1}{\sqrt{2}}(\pm j \pm k).$$

A change in the sign of B does not alter the transformation. All rotations of the 24-hedroid into itself may thus be represented by the transformation

$$Q' = B^{-1}AQB,$$

where A belongs to the set (1) and B to the set (2). Or, if we call the members of the set (1) positive, and the other members of the set (2) negative, then every rotation of the 24-hedroid into itself may be represented by the transformation

$$Q' = A^{-1}QB,$$

where A and B are either both positive or both negative members of the set (2). (This A is different from that used in the previous equation.)

All such transformations represent different rotations of the 24-hedroid into itself, except that the rotation is unaltered if the signs of A and B are simultaneously changed.

Let us denote the above rotation by the notation $\{AB\}$. Then two consecutive rotations $\{A_1B_1\}$ and $\{A_2B_2\}$ will be equivalent to a rotation $\{A_3B_3\}$ if $A_3 = A_1A_2$ and $B_3 = B_1B_2$.

The group of the 24-hedroid, then, is the quotient group of the group of couples $\{AB\}$, where A and B are either both positive or both negative members of the set (2), by the self-conjugate sub-group $\{1, 1\}$ and $\{-1, -1\}$. It is easy to see that the set of quaternions (2) forms a group.

Since the octahedral group may be represented by the transformations $Q' = B^{-1}QB$, the quotient group of the group (2) by the sub-group containing 1 and -1 is the octahedral group.

The group of pairs $\{A, B\}$ is the direct square of the group (2). The group of pairs $\{A, B\}$, where A and B are either both positive or both negative, is the positive sub-group of the direct square of the group (2).

Thus the group of the 24-hedroid is the quotient group of the positive group of the direct square of the group (2), by the sub-group $\{1, 1\}$ and $\{-1, -1\}$. It is of order 576.

A close connection between the group of the 24-hedroid and the direct square of the octahedral group is indicated. In fact, the quotient group of the group of the 24-hedroid, by that sub-group containing the rotation of every vertex to that vertically opposite, is simply isomorphic with the positive group of the direct square of the octahedral group.

The generators of the group of the 24-hedroid are as follows.

Let -1 denote the rotation $Q' = -Q$,

T the rotation $Q' = Qi$,

T' the rotation $Q' = -iQ$,

S the rotation $Q' = \frac{1}{2}Q(1+i+j+k)$,

S' the rotation $Q' = \frac{1}{2}(1-i-j-k)\,Q$,

R the rotation $Q' = \frac{1}{2}(1-i)\,Q(1+i)$.

Then the following equations are satisfied :—

$$T^2 = -1, \quad S^3 = (S^2T)^3 = (TS^2)^3 = -1, \quad (ST)^3 = (TS)^3 = 1.$$

Similar equations hold for S' and T'. S and T are commutative with S' and T',

$$R^2 = TT',$$

R is commutative with T and T',

$$R^{-1}SR = S^2T.$$

These equations define the group.

The geometrical significance is as follows.

S denotes a right-handed rotation of the hedroid through 60° in the plane containing $Q = 1$ and $Q = i+j+k$, and in the plane perpendicular; S' an equal left-handed rotation in the same planes.

T denotes a right-handed rotation through 90° in the plane containing $Q = 1$ and $Q = i$, and in the perpendicular plane; T' an equal left-handed rotation. R denotes a plane rotation through a right angle in the j, k plane.

4. *The* 600*-hedroid**.

This solid has 120 vertices, and is therefore reciprocal to the 120-hedroid.

If it is inscribed in a unit hypersphere, with vertices at $Q = 1$ and $Q = -1$, the structure is as follows.

The vertex $Q = 1$ is connected to a regular icosahedron lying in the hyperplane $x_1 = \cos 36°$. These vertices are connected to the vertices of a regular dodecahedron in the hyperplane $x_1 = \cos 60°$, the vertices of these two figures are connected to the vertices of an icosahedron in the hyperplane $x_1 = \cos 72°$, but the edges of this icosahedron do not appear as edges of the 24-hedroid. The vertices of this figure and of the dodecahedron are connected to a thirty-pointed figure in the hyperplane $x_1 = 0$. The vertices of this figure correspond to the mid-points of the sides of the icosahedrons or the dodecahedron. The reflection of these edges and vertices in the hyperplane $x_1 = 0$, together with edges connect-

* S. L. van Oss, "Das regelmässige Sechshundertzell und seine selbsdeckenden Bewegungen", *Amst. Verh.* 7, No. 1 (1901), 1–18.

ing the corresponding vertices of the icosahedrons in the hyperplanes $x_1 = \cos 72^\circ$ and $x_1 = \cos 108^\circ$, completes the figure.

Consider the set H of quaternions representing these vertices.

Then, if A belongs to H, the transformation $Q' = A^{-1}QA$ represents a rotation of the 600-hedroid into itself that leaves the vertices $Q = \pm 1$ invariant.

It is easy to see that these vertices are invariant. The transformation is unaltered if the sign of A is changed, so that we may assume that the scalar part of $A \geqslant 0$.

If $A = 1$, the result is obvious.

If the scalar part of A is $\cos 36^\circ$, put $A = \cos 36^\circ + V \sin 36^\circ$, where V is a vector of unit norm parallel to a diagonal of the icosahedron. The transformation $Q' = A^{-1}QA$ represents a rotation of 72° about this line, rotating the icosahedron, and therefore the 600-hedroid, into itself.

If $A = \cos 72^\circ + V \sin 72^\circ$, the transformation represents a rotation about the same axis through 144°.

If $A = \cos 60^\circ + V \sin 60^\circ$, the transformation represents a rotation through 120° about a diagonal of the dodecahedron.

Lastly, if A represents a vertex in the hyperplane $x_1 = 0$, the corresponding rotation is through 180° about the line through the centre to the mid-point of the side of the dodecahedron or icosahedron.

All these rotations clearly rotate the icosahedrons and dodecahedrons into themselves, and therefore the 600-hedroid.

In particular, these are three-dimensional rotations, and, since they rotate the icosahedron in the hyperplane $x_1 = \cos 36^\circ$ into itself, they must form the icosahedral group.

Two rotations $Q' = A_1^{-1}QA_1$ and $Q'' = A_2^{-1}Q'A_2$ are equivalent to a third rotation $Q'' = A_3^{-1}QA_3$, if $A_3 = A_1A_2$.

Hence the quaternions of the set H must form a group of order 120, and the quotient group of this by the invariant sub-group, 1, -1, is simply isomorphic with the icosahedral group.

The 600-hedroid, then, is itself a representation of the icosahedral group, the vertices, in opposite pairs, representing the operators of the group.

The vertices $Q = \pm 1$ represent identity. The icosahedron of vertices in the hyperplane $x_1 = \cos 36^\circ$ represents twelve operations of period 3. The dodecahedron of vertices in the hyperplane $x_1 = \cos 60^\circ$ represents twenty operations of period 3. For the vertices in the hyperplane $x_1 = \cos 72^\circ$ we have twelve more operations of period 5, and the thirty vertices in the hyperplane $x_1 = 0$, taken in pairs, give us fifteen operations of period 2. Each vertex is associated with that vertically opposite.

This clearly indicates the well known fact that the operations of period 5 are separated into two categories of twelve associates.

Now, since H forms a group, if A belongs to H, the transformation $Q' = AQ$ must change every vertex of the 600-hedroid into another vertex, as also must the transformation $Q' = QB$.

Every transformation of the form $Q' = AQB$, where A and B belong to the set H, must rotate the 600-hedroid into itself. Further, every rotation may be so represented. If A be the vertex which is rotated to the position $Q = 1$, then the rotation must be obtainable by any rotation which rotates this vertex into the required position, and a further rotation of the icosahedron in the hyperplane $x_1 = \cos 36^\circ$ into itself. These rotations may be

$$Q' = A^{-1}Q, \quad Q'' = B^{-1}Q'B,$$

giving $Q'' = B^{-1}A^{-1}QB$, a transformation of the type given above.

We represent the rotation $Q' = A^{-1}QB$ by the notation $\{A, B\}$. Then the result of two rotations $\{A_1, B_1\}$, $\{A_2, B_2\}$ is $\{A_3, B_3\}$ if $A_3 = A_1A_2$ and $B_3 = B_1B_2$. Two different pairs $\{A_1, B_1\}$ $\{A_2, B_2\}$ represent different rotations unless $A_1 = -A_2$ and $B_1 = -B_2$.

The group of pairs $\{AB\}$ is the direct square of the group H.

The quotient group of the group H by the invariant sub-group containing 1 and -1 is the icosahedral group.

The quotient group of the direct square of the group H by the invariant sub-group containing $\{1, 1\}$ and $\{-1, -1\}$ is the group of the 600-hedroid.

A close connection with the direct square of the icosahedral group is indicated. In fact, the quotient group of the group of the 600-hedroid by that invariant sub-group containing the rotation of every vertex to that vertically opposite is the direct square of the icosahedral group.

The order of the group of the 600-hedroid is 7200.

The generators of the icosahedral group are usually described in the following way*.

Suppose that α represents a vertex of the icosahedron, and that β represents an adjacent vertex. S represents a rotation through 72° about the diagonal through α. T represents a rotation through 180° about the line through the centre and the mid-point of $\alpha\beta$. U is a generator dependent on these, representing a rotation through 180° about the axis perpendicular to the central plane through α and β. Then $U = S^2TS^3TS^2T$.

$$S^5 = 1, \quad T^2 = 1, \quad U^2 = 1, \quad (TS)^3 = 1, \quad (TS^2)^3 = 1.$$

* Klein, *Lectures on the icosahedron*, translated by G. G. Morrice (1913), 27-29.

We obtain the generators of the group of the 600-hedroid as follows.

Let S_1 be a right-handed rotation through 36° from the scalar axis $x_2 = x_3 = x_4 = 0$ towards the axis of S, and perpendicular to this plane, S_2 a left-handed rotation through 36° from the axis of S towards the scalar axis, and perpendicular to this plane.

Then $S = S_1 S_2$.

Similarly, let $T = T_1 T_2$, where T_1 represents a right-handed rotation through 90° from the scalar axis to the axis of T, and perpendicularly, and T_2 the corresponding left-handed rotation.

Similarly $U = U_1 U_2$,

$$U_1 = S_1^2 T_1 S_1^3 T_1 S_1^2 T_1, \quad U_2 = S_2^2 T_2 S_2^3 T_2 S_2^2 T_2,$$

S_1, T_1, and U_1 are commutative with S_2, T_2, and U_2.

$$S_1^5 = T_1^2 = -1 = S_2^5 = T_2^2,$$

also $U_1^2 = U_2^2 = -1$. These relations define the group.

Thus for each operation of the icosahedral group, representing a rotation through an angle θ about a given axis, there are two operators of the group of the 600-hedroid representing right-handed and left-handed rotations respectively through an angle $\frac{1}{2}\theta$, from the scalar axis to the axis of the icosahedral rotation. Care is needed to find whether the corresponding operation, as generated by S_1 and T_1, or S_2 and T_2, denotes a rotation through an angle $\frac{1}{2}\theta$ or $\pi + \frac{1}{2}\theta$, but this can be found from a calculation of the corresponding quaternions.

5. *The Simplex.*

The other regular four-dimensional solids are the hypercube, the pentahedroid, and the 16-hedroid reciprocal to the hypercube.

The pentahedroid is a particular case of the simplex, a regular solid with $n+1$ vertices in n dimensions. Name the vertices $A_1, A_2, A_3, \ldots, A_{n+1}$. After a rotation of the solid into itself, let the vertex A_1 be replaced by the vertex A_{e_1}, and A_2 by A_{e_2}, etc. Then the rotation may be represented by the permutation $e_1 e_2 \ldots e_{n+1}$ of the numbers $1, 2, \ldots, n+1$. The resultant of two rotations is given by the product of the two corresponding permutations. The group of the simplex, then, is simply isomorphic with a sub-group of the group of permutations on $n+1$ letters. Consider three vertices e_1, e_2, e_3; these are contained in a plane, and the remaining $n-2$ vertices are contained in the $n-2$ sub-space perpendicular to this

plane. A rotation may take place in this plane, leaving the $n-2$ other vertices unaltered. The rotation corresponds to the cyclic interchange of the numbers e_1, e_2, e_3, and is thus a permissible rotation of the simplex. The group generated by all such cyclic interchanges on three numbers is the alternating group. The group of rotation of the simplex with $n+1$ vertices is the alternating group of order $(n+1)!/2$. Thus the group of the triangle is of order 3; the group of the tetrahedron is the alternating group of order 12; and the group of the pentahedroid is the icosahedral group.

It should be noted that, if the simplex be regarded as a flat figure in $n+1$ space, a further rotation is permissible, interchanging any two vertices and leaving the other vertices unchanged. The group then will be the symmetric group of order $(n+1)!$. Thus the group of the triangular dihedral is of order 6; and the group of the tetrahedral dihedroid is of order 24 and is, in fact, the octahedral group.

6. *The orthogonal and the reciprocal solid.*

We next consider the group of the orthogonal. Take orthogonal axes OX_1, OX_2, ..., OX_n. We shall consider the solid reciprocal to the orthogonal, and suppose the vertices to be at $x_i = \pm 1$, $x_j = 0$ $(j \neq i)$, giving us $2n$ vertices.

Consider the figure first as a flat solid in $n+1$ dimensions. A possible rotation will be one interchanging the vertices $x_i = 1$ and $x_i = -1$, leaving the others unchanged. Call this rotation S_i. Another possible rotation will interchange the vertices $x_i = 1$, $x_j = 1$, and simultaneously $x_i = -1$, $x_j = -1$, leaving the others unchanged. Denote this rotation by T_{ij}. These two types of rotation generate the group.

Clearly the rotations T_{ij} generate a sub-group simply isomorphic with the symmetric group of order $n!$. We have also

$$S_i^2 = 1$$

and $$S_i T_{ij} = T_{ij} S_j, \quad S_j T_{ij} = T_{ij} S_i, \quad S_i T_{jk} = T_{jk} S_i \quad (i \neq j, i \neq k).$$

These generate a group of order $2^n n!$.

If we now term every S_i negative, as well as those members of the symmetric sub-group usually termed negative, then the group of the orthogonal, considered as a solid in n dimensions only, is the positive sub-group of the above group. Its order will be $2^{n-1} n!$.

The group of the octahedron, as calculated by this means, is generated by the group of permutations on three letters, and S_1, S_2, S_3. This group,

of order 48, is the group of the octahedral dihedroid. The group of the octahedron is the positive sub-group of this. It seems, by a fortunate chance only, that its representation as a symmetric group is possible. Representing the interchange of the first two letters by (12), and a cyclic interchange of the first three by (123), the group is that generated by $S_1(12)$, $S_1(23)$, S_1S_2, S_1S_3. If we associate these with the permutations on four letters represented by (1234), (23), (13), (24), (13), we see that the group generated is the same.

No such simplification is possible with the group of the hypercube or 16-hedroid, or of any orthogonal of higher degree.

I am indebted to Professor Richardson for introducing the problem to me.

UPPER AND LOWER LEBESGUE INTEGRALS

By M. D. KENNEDY.

[Received 7 October, 1929.—Read 14 November, 1929.]

Introduction.

The definition given by Perron† of the integral of a function $f(x)$ was shown by Alexandroff‡ to be completely equivalent to the definition which yields the "special Denjoy integral". The object of this paper is to show how Perron's procedure may be modified so as to give an integral completely equivalent to that of Lebesgue.

The Perron form of definition yields naturally two functions associated with the original functions $f(x)$. If these coincide, their common value is the integral. If they differ, they are of the nature of upper and lower integrals respectively; and, in view of the fact that their coincidence would give a Lebesgue integral, they are here called the upper and the lower Lebesgue integrals. Their properties are investigated and conditions for their existence are given.

The essential modification of Perron's definition is a restriction upon the character of the auxiliary functions used. To the requirement of continuity is added that of absolute continuity in one sense. The property thus named must be defined and briefly investigated before the modified Perron integral can be defined.

Functions absolutely continuous in one sense.

1. In what follows the range of the variable x is a finite fundamental interval (a, b), taken closed. It is convenient to consider elementary systems of intervals within (a, b). An *elementary system* I consists

† 6.
‡ 1.

of a finite number of non-overlapping intervals,

$$(a_1, b_1),\ (a_2, b_2),\ \ldots,\ (a_k, b_k), \tag{1.1}$$

whose end-points belong to closed (a, b). The *complement* of I is the system obtained when these intervals, taken open, are subtracted from (a, b), degenerate intervals consisting of a single point being omitted.

The well known condition that a function $F(x)$ should be absolutely continuous in (a, b) may be stated as follows.

Let τI be the function of elementary systems defined by

$$\tau I = \sum_{n=1}^{k} |F(b_n)-F(a_n)|.$$

The condition that $F(x)$ should be absolutely continuous in (a, b) is

$$\lim_{mI\to 0} \tau I = 0, \tag{1.2}$$

where mI, the measure of the system I, is the sum of the lengths of its intervals.

This condition may, however, be broken up into two parts which, taken together, are equivalent to (1.2). For let

$$\sigma I = \sum_{n=1}^{k} \Big(F(b_n)-F(a_n)\Big).$$

The statements

$$\overline{\lim_{mI\to 0}}\ \sigma I \leqslant 0, \tag{1.3}$$

and

$$\underline{\lim_{mI\to 0}}\ \sigma I \geqslant 0, \tag{1.4}$$

are both true of any function $F(x)$ that satisfies (1.2), and, further, any function that satisfies both (1.3) and (1.4) also satisfies (1.2).

A function $F(x)$ which satisfies (1.3) will be said to be *absolutely continuous above*. A function $F(x)$ which satisfies (1.4) will be said to be *absolutely continuous below*.

Let

$$\pi I = \tfrac{1}{2} \sum_{n=1}^{k} \Big\{|F(b_n)-F(a_n)|+\Big(F(b_n)-F(a_n)\Big)\Big\} = \tfrac{1}{2}\{\tau I+\sigma I\},$$

$$\nu I = \tfrac{1}{2} \sum_{n=1}^{k} \Big\{|F(b_n)-F(a_n)|-\Big(F(b_n)-F(a_n)\Big)\Big\} = \tfrac{1}{2}\{\tau I-\sigma I\}.$$

Then (1.3) and (1.4) are equivalent to

$$\lim_{mI \to 0} \pi I = 0 \tag{1.3*}$$

and

$$\lim_{mI \to 0} \nu I = 0, \tag{1.4*}$$

respectively.

It may be pointed out that πI is the sum of the positive terms in the sum σI, while νI is the absolute value of the sum of the negative terms in σI.

A function absolutely continuous in one sense in a given interval is clearly absolutely continuous in that sense in every sub-interval of that interval.

A bounded monotone function is absolutely continuous below if it is non-decreasing, and absolutely continuous above if it is non-increasing.

It is evident from the definitions that if $F(x)$ is absolutely continuous in one sense, then $-F(x)$ is absolutely continuous in the other sense.

Again, the sum of a finite number of functions, all absolutely continuous in the same sense, or a linear combination of such functions with positive coefficients, is absolutely continuous in the same sense.

A finite function, absolutely continuous in one sense, is of bounded variation.

Consider the function $F(x)$, absolutely continuous above in (a, b), and suppose that it is not of bounded positive variation in the same interval. From the absolute continuity above it follows that, to any given number $\epsilon > 0$, there corresponds a number $\eta > 0$ such that, if I is any system of intervals (1.1),

$$\pi I < \epsilon$$

follows from

$$mI \leqslant \eta.$$

Let (a, b) be divided by the points of division

$$a < x_1 < x_2 < \ldots < x_n < b \tag{1.5}$$

into $n+1$ intervals each of length less than or equal to η. It is always possible to take† $n = [(b-a)/\eta]$, and it will be assumed that this is, in fact, the value of n.

Again, let (a, b) be divided by the points of division

$$a < y_1 < y_2 < \ldots < y_r < b \tag{1.6}$$

† $[p]$ denoting the integral part of p.

into a system of intervals J such that

$$\pi J > N = \left\{\left[\frac{b-a}{\eta}\right]+1\right\}\epsilon.$$

This is possible in view of the assumption that $F(x)$ is not of bounded positive variation.

Those intervals of the system determined by (1.6) which provide contributions to πJ are now taken. If they are subdivided by means of those points of (1.5) which fall within them, the result is an elementary system of intervals K such that

$$\pi K > N.$$

But K may be exhibited as the sum of at most $n+1$ non-overlapping elementary systems each of measure less than or equal to η. Thus

$$\pi K < (n+1)\epsilon$$

$$= N.$$

The contradiction shows that $F(x)$ is of bounded positive variation; it follows that it is also of bounded negative and total variation.

The proof in the case of a function absolutely continuous below can be reduced to that just given by the consideration of the function $-F(x)$.

LEMMA. *If the continuous function $F(x)$ is absolutely continuous below in the interval (a, b), and has almost everywhere in this interval a regular derivative*

$$DF(x) \geqslant h, \tag{1.7}$$

then

$$\frac{F(b)-F(a)}{b-a} \geqslant h. \tag{1.8}$$

The corresponding statement for functions absolutely continuous above is obtained by reversing the inequalities (1.7) *and* (1.8).

These are special cases of a result proved in a previous paper†, where it was shown that if a function of intervals $F(c, d)$, determinate below in a given interval (a, b), has, almost everywhere in (a, b), a regular

† **3**, 73.

derivative

$$DF(x) \geqslant h,$$

then

$$\frac{F(a, b)}{b-a} \geqslant h,$$

with a corresponding result for functions determinate above.

For the present purpose, then, it is only necessary to show that the function of intervals

$$F(c, d) = F(d)-F(c)$$

is determinate below if the continuous function $F(x)$ is absolutely continuous below, and determinate above if it is absolutely continuous above.

That $F(c, d)$ is continuous follows at once from the continuity of $F(x)$. The function $F(c, d)$ is determinate below in (a, b) if the following statement can be proved.

In the interval (a, b) there is no perfect set P of measure zero, such that

$$\frac{F(d)-F(c)}{d-c} = \lambda$$

whenever (c, d) is a complementary interval of P, while

$$\frac{F(d)-F(c)}{d-c} < \lambda \tag{1.9}$$

whenever c and d are points of P, not end-points of the same complementary interval, λ being any finite number.

Suppose that $F(x)$ is absolutely continuous below in the interval (a, b), and that P is a perfect set with the properties set out above, whose complementary intervals are

$$(c_1, d_1), \quad (c_2, d_2), \quad \ldots, \quad (c_n, d_n), \quad \ldots,$$

and γ and δ its extreme points. Then, by (1.9),

$$\frac{F(\delta)-F(\gamma)}{\delta-\gamma} = \mu < \lambda,$$

Let the elementary system I_r be the complement, with respect to (γ, δ), of the system

$$(c_1, d_1), \quad (c_2, d_2), \quad \ldots, \quad (c_r, d_r).$$

The measure of I_r can be made arbitrarily small by suitable choice of r. Also

$$\sigma I_r + \sum_{n=1}^{r} \left(F(d_n) - F(c_n)\right) = F(\delta) - F(\gamma)$$

$$= \mu(\delta - \gamma);$$

$$\sigma I_r = \mu(\delta - \gamma) - \lambda \sum_{n=1}^{r} (d_n - c_n)$$

$$= \mu(\delta - \gamma) - \lambda(\delta - \gamma - mI_r)$$

$$= (\mu - \lambda)(\delta - \gamma) + \lambda \cdot mI_r;$$

$$\lim_{mI_r \to 0} \sigma I_r = (\mu - \lambda)(\delta - \gamma) < 0,$$

which contradicts the assumption that $F(x)$ is absolutely continuous below. This establishes the lemma for functions absolutely continuous below, and the corresponding result for functions absolutely continuous above is obtained at once if $F(x)$ is replaced by $-F(x)$.

Functions absolutely continuous in one sense have been studied by W. H. Young†, who named them *semi-integrals*. He showed, in particular, that a function absolutely continuous below in (a, b) is the sum of an absolutely continuous function and a non-decreasing function, which vanishes when $x = a$ and has zero as unique derivative almost everywhere in (a, b), while a function absolutely continuous above is the sum of an absolutely continuous function and a non-increasing function with the same special properties. This result is also implicitly contained in Lebesgue's discussion‡ of the *fonction des singularités* of a non-absolutely continuous function. It is possible to prove the lemma just given by the use of the property in question. No applications of it occur in what follows.

Upper and lower functions.

2. The following notation will be used.

The symbol $\sim$ between two functions represents the words "is metrically equivalent to" or "is almost everywhere equal to".

The symbols $\gtrsim$ and $\lesssim$ represent "is almost everywhere greater than or equal to" and "is almost everywhere less than or equal to" respectively.

† 7.

‡ 5, 160 *ff.*

Let $F(x)$ be a given function. By *any derivative* of $F(x)$ and by the symbol $DF(x)$ will be meant any single-valued function of x whose value at each point x of (a, b) is a limit as $h \to 0$ of the ratio $\{F(x+h)-F(x)\}/h$, h being such that $x+h$ is also a point of (a, b).

Let a function $f(x)$, not necessarily finite, be defined in (a, b). A function $\phi(x)$ such that

(α_*) $\phi(x)$ is continuous and absolutely continuous above in (a, b),

(β_*) $\phi(a) = 0$,

(γ_*) $D\phi(x) \lesssim f(x)$,

is a *lower function* associated with $f(x)$.

An *upper function* $\psi(x)$ associated with $f(x)$ is such that

(α^*) $\psi(x)$ is continuous and absolutely continuous below in (a, b),

(β^*) $\psi(a) = 0$,

(γ^*) $D\psi(x) \gtrsim f(x)$.

Since functions absolutely continuous in one sense are of bounded variation, the derivatives $D\phi(x)$ and $D\psi(x)$ of lower and upper functions are unique derivatives for almost all values of x.

It is seen at once from the definitions that, if $\phi(x)$ and $\psi(x)$ are a lower and an upper function associated with $f(x)$, then $-\phi(x)$ and $-\psi(x)$ are an upper and a lower function associated with $-f(x)$.

Let $\phi_1(x)$ and $\phi_2(x)$ be two lower functions associated with $f(x)$, and let ξ be any point $a < \xi < b$. Then the function $\phi_*(x)$, such that

$$\phi_*(x) = \phi_1(x) \qquad (a \leqslant x \leqslant \xi),$$

$$\phi_*(x) = \phi_1(\xi) + \phi_2(x) - \phi_2(\xi) \qquad (\xi \leqslant x \leqslant b),$$

is also a lower function associated with $f(x)$. For it can be seen without difficulty that $\phi_*(x)$ satisfies the three conditions (α_*), (β_*), and (γ_*).

Thus, geometrically, if PQ is an arc of a curve representing a lower function, or a lower curve, P and Q having abscissae ξ and b respectively, then PQ remains an arc of a lower curve if it is displaced parallel to itself in the direction perpendicular to the x-axis, provided only that the new position P' of P is a point of some lower curve.

The corresponding result for upper functions is easy to obtain.

THEOREM I. *If $\phi(x)$ is a lower function and $\psi(x)$ an upper function associated with $f(x)$, the difference*

$$\psi(x)-\phi(x)$$

is non-negative and non-decreasing.

The functions $\phi(x)$ and $\psi(x)$, being of bounded variation, have finite unique derivatives almost everywhere in (a, b). Thus, by (γ_*) and (γ^*),

$$D\big(\psi(x)-\phi(x)\big) \sim D\psi(x)-D\phi(x)$$

$$\gtrsim 0.$$

The continuous function $\psi(x)-\phi(x)$ is absolutely continuous below as the sum of the two functions $\psi(x)$ and $-\phi(x)$ which are absolutely continuous below. The lemma of § 1 shows at once that, if x_1 and x_2 are any two points of (a, b), $x_1 < x_2$, then

$$\psi(x_2)-\phi(x_2) \geqslant \psi(x_1)-\phi(x_1).$$

Since, by (β_*) and (β^*),

$$\psi(a)-\phi(a) = 0,$$

$$\psi(x)-\phi(x) \geqslant 0,$$

and the result is established.

Upper and lower integral functions.

3. Let the function $f(x)$ possess lower functions. The upper bounding function, $\Phi(x)$, of the lower functions is the *lower integral function* associated with $f(x)$. The *upper integral function*, $\Psi(x)$, is the lower bounding function of the upper functions of $f(x)$, if such exist.

If $f(x)$ has a lower function $\phi(x)$ and an upper function $\psi(x)$, it has also both upper and lower integral functions, and these are bounded functions. For they both lie between the two continuous functions $\phi(x)$ and $\psi(x)$.

Throughout the present paragraph it is assumed that both upper and lower functions exist.

It should be noticed that, if $\Phi(x)$ and $\Psi(x)$ are the lower and the upper integral functions of $f(x)$, then $-\Phi(x)$ and $-\Psi(x)$ are the upper and the lower integral functions of $-f(x)$.

THEOREM IIA. *If $\phi(x)$ is any lower function associated with $f(x)$, and $\Phi(x)$ the lower integral function, then*

$$\Phi(x)-\phi(x)$$

is non-negative and non-decreasing.

That $\Phi(x)-\phi(x)$ is non-negative follows at once from the definition of $\Phi(x)$.

Suppose that, for some pair of values x_1 and x_2 of x, with $a \leqslant x_1 < x_2 \leqslant b$,

$$\{\Phi(x_1)-\phi(x_1)\}-\{\Phi(x_2)-\phi(x_2)\} = h > 0.$$

There is a lower function $\phi_1(x)$ such that

$$\Phi(x_1)-\phi_1(x_1) < h.$$

The function $\phi_*(x)$, where

$$\phi_*(x) = \phi_1(x) \qquad (a \leqslant x \leqslant x_1),$$

$$\phi_*(x) = \phi_1(x_1)+\phi(x)-\phi(x_1) \qquad (x_1 \leqslant x \leqslant b),$$

is a lower function associated with $f(x)$. But

$$\phi_*(x_2) = \phi_1(x_1)-\phi(x_1)+\phi(x_2)$$

$$= \phi_1(x_1)+h-\Phi(x_1)+\Phi(x_2).$$

$$\Phi(x_2)-\phi_*(x_2) = \Phi(x_1)-\phi_1(x_1)-h$$

$$< 0.$$

This contradicts the definition of $\Phi(x)$, and the result is established.

THEOREM IIB. *If $\psi(x)$ is any upper function associated with $f(x)$, and $\Psi(x)$ is the upper integral function, then*

$$\psi(x)-\Psi(x)$$

is non-negative and non-decreasing.

This is reduced to the previous result if $-f(x)$ is written for $f(x)$; for $-\psi(x)$ is a lower function, and $-\Psi(x)$ is the lower integral function associated with $-f(x)$.

THEOREM III. *If $\Phi(x)$ and $\Psi(x)$ are the lower and the upper integral functions of some function $f(x)$ in (a, b), then*

$$\Psi(x)-\Phi(x)$$

is non-negative and non-decreasing in (a, b).

It is clear that

$$\Psi(a)-\Phi(a) = 0.$$

Also, from I, it is seen without difficulty that, for all values of x, $a \leqslant x \leqslant b$,

$$\Psi(x)-\Phi(x) \geqslant 0.$$

Suppose that, for some pair of values x_1 and x_2 of x, with $a \leqslant x_1 < x_2 \leqslant b$,

$$\{\Psi(x_1)-\Phi(x_1)\} - \{\Psi(x_2)-\Phi(x_2)\} = h > 0. \tag{3.1}$$

It is possible to find a lower function $\phi(x)$ and an upper function $\psi(x)$ such that

$$\Phi(x_2)-\phi(x_2) < \tfrac{1}{2}h, \quad \psi(x_2)-\Psi(x_2) < \tfrac{1}{2}h.$$

Thus, from (3.1),

$$\begin{aligned}\Psi(x_1)-\Phi(x_1) &= \{\tfrac{1}{2}h+\Psi(x_2)\} + \{\tfrac{1}{2}h-\Phi(x_2)\} \\ &> \psi(x_2)-\phi(x_2) \\ &\geqslant \psi(x_1)-\phi(x_1).\end{aligned}$$

This is impossible in view of the definitions of $\Phi(x)$ and $\Psi(x)$. Thus (3.1) is contradicted; the required result follows.

THEOREM IV. *A lower integral function is absolutely continuous above and an upper integral function is absolutely continuous below*†.

Suppose that the lower integral function $\Phi(x)$ of the function $f(x)$ does not satisfy the condition (1.3*). Thus, if

$$\pi\Phi(I) = \tfrac{1}{2}\sum_{n=1}^{k}\left\{|\Phi(b_n)-\Phi(a_n)|+\Big(\Phi(b_n)-\Phi(a_n)\Big)\right\},$$

$$\overline{\lim_{ml\to 0}}\ \pi\Phi(I) = h > 0, \tag{3.2}$$

where I is a system (1.1) and h may be $+\infty$.

Let $\phi(x)$ be a lower function such that

$$\Phi(b)-\phi(b) < \epsilon,$$

where $0 < \epsilon \leqslant \frac{1}{3}h$ if h is finite, and where ϵ is any positive number if h is $+\infty$. The function $\phi(x)$ is absolutely continuous above, so that

† It is proved below that these functions are absolutely continuous without restriction. See p. 37.

it is possible to find a number $\eta > 0$ such that, for $mI < \eta$,

$$\pi\phi(I) < \epsilon. \tag{3.3}$$

By (3.2) there must be a system I, $mI < \eta$, for which

$$\pi\Phi(I) > 2\epsilon,$$

but, for this same system,

$$\pi\phi(I) \geqslant \pi\Phi(I) - \sum_{n=1}^{k} \left\{ \Big(\Phi(b_n) - \phi(b_n)\Big) - \Big(\Phi(a_n) - \phi(a_n)\Big) \right\}.$$

The sum on the right cannot exceed ϵ, since the function $\Phi(x) - \phi(x)$ is non-decreasing and is always less than ϵ. Thus

$$\pi\phi(I) > \epsilon,$$

and (3.3) is contradicted.

The proof relating to upper integral functions is reduced to that just given if $-f(x)$ is considered in place of $f(x)$.

It should be noticed that, if $f(x)$ is bounded, it has upper and lower integral functions, and these are absolutely continuous without restriction. For suppose that $H \leqslant f(x) \leqslant K$. Then $H(x-a)$ and $K(x-a)$ are respectively a lower and an upper function of $f(x)$. If $\Phi(x)$ is the lower integral function of $f(x)$, then $\Phi(x)$ exceeds the absolutely continuous function $H(x-a)$ by a non-decreasing function. Thus $\Phi(x)$, as the sum of two functions absolutely continuous below, is absolutely continuous below as well as above. In a similar way it may be shown that the upper integral function is absolutely continuous.

THEOREM V. *The lower integral function is a lower function, and the upper integral function is an upper function.*

Only the lower integral function will be considered in detail, the proof for the upper integral function being similar.

The function $\Phi(x)$ is continuous, for it can be exhibited as the limit of a sequence of continuous functions which tends uniformly to $\Phi(x)$. It has just been proved that $\Phi(x)$ is absolutely continuous above. Thus $\Phi(x)$ satisfies condition (α_*), and obviously condition (β_*) is also fulfilled. It only remains to show that $\Phi(x)$ satisfies condition (γ_*).

Let $D\Phi(x)$ be any derivative of $\Phi(x)$, and suppose that the set E of points x in (a, b) such that

$$D\Phi(x) - f(x) > 0$$

has outer measure $m^*E > 0$. There is a number $\epsilon > 0$ such that

$$D\Phi(x) - f(x) \geqslant \epsilon \tag{3.4}$$

at the points of a set E^*, where $m^*E^* = \eta > 0$.

If $\phi(x)$ is a lower function, then the function

$$G(x) = \Phi(x) - \phi(x)$$

is continuous and of bounded variation, and has a unique derivative almost everywhere, so that

$$DG(x) \sim D\Phi(x) - D\phi(x).$$

Let $\phi(x)$ be so chosen that

$$G(b) < \tfrac{1}{2}\epsilon\eta. \tag{3.5}$$

The set $\bar{E}$ of points at which $G(x)$ has a right-hand derivative not less than ϵ is measurable, and since $\phi(x)$ satisfies condition (γ_*), and in view of (3.4), $\bar{E}$ contains E^*, with the possible exception of a set of measure zero, so that $m\bar{E} \geqslant \eta$. Let Q be a closed set of points contained in $\bar{E}$ and such that $mQ > \frac{1}{2}\eta$, and let the complementary intervals of Q be

$$(a_1, b_1), \quad (a_2, b_2), \quad \ldots, \quad (a_n, b_n), \quad \ldots,$$

and its extreme points α and β, where $a \leqslant \alpha < \beta \leqslant b$. Then, by a result proved elsewhere†,

$$G(b) - G(a) \geqslant \frac{\epsilon\eta}{2} + \big(G(\alpha) - G(a)\big) + \big(G(b) - G(\beta)\big) + \sum_{n=1}^{\infty} \big(G(b_n) - G(a_n)\big).$$

Hence, since $G(x)$ is non-decreasing,

$$G(b) - G(a) = \Phi(b) - \phi(b) \geqslant \tfrac{1}{2}\epsilon\eta.$$

The contradiction of (3.5) shows that the assumption $m^*E > 0$ is false. It follows that $\Phi(x)$ satisfies condition (γ_*).

The modified Perron integral and the Lebesgue integral.

4. If the function $f(x)$ has upper and lower integral functions which coincide, the common value

$$\Phi(b) = \Psi(b)$$

† 3, 75.

will be called the *modified Perron integral* of $f(x)$ in (a, b). It may be written $I_a^b f(x)\,dx$.

The function

$$\Phi(x) = \Psi(x)$$

is an *indefinite modified Perron integral* of $f(x)$. It may be written $I_a^x f(t)\,dt$.

A necessary and sufficient condition that $f(x)$ should be integrable in (a, b) according to the modified Perron definition is that $f(x)$ should possess a lower function $\phi(x)$ and an upper function $\psi(x)$ such that

$$\psi(b) - \phi(b) < \epsilon, \tag{4.1}$$

where ϵ is an arbitrary positive number given in advance.

If the modified Perron integral exists in (a, b), it evidently exists in every sub-interval of (a, b).

It is clear at once that an indefinite modified Perron integral is absolutely continuous.

It is shown below that the modified Perron integral coincides exactly with the Lebesgue integral; that is to say that, if $f(x)$ is integrable in accordance with the above definition, it is also summable in the sense of Lebesgue, and if it is summable it has a modified Perron integral, and the values coincide.

THEOREM VIA. *If $f(x)$ is summable in (a, b), then it is integrable in the same interval according to the modified Perron definition, and*

$$\int_a^b f(x)\,dx = I_a^b f(x)\,dx. \tag{4.2}$$

The indefinite integral

$$F(x) = \int_a^x f(t)\,dt \quad (a \leqslant x \leqslant b)$$

is itself both an upper and a lower function associated with $f(x)$. For $F(x)$ is absolutely continuous, and so fulfils the conditions (α_*) and (α^*). Since $F(a)$ vanishes, the conditions (β_*) and (β^*) are satisfied. Also, if $DF(x)$ is any derivative of $F(x)$,

$$DF(x) \sim f(x),$$

so that (γ_*) and (γ^*) are fulfilled. The condition (4.1) is clearly satisfied, and the function $f(x)$ is seen to be integrable according to the modified Perron definition.

Since $F(x)$ is both a lower and an upper function, it is also the lower and the upper integral function associated with $f(x)$, and (4.2) follows

THEOREM VIb. *If $f(x)$ is integrable in (a, b) according to the modified Perron definition, then $f(x)$ is summable in the same interval, and (4.2) holds.*

The upper and lower integral functions associated with $f(x)$ coincide, giving the indefinite modified Perron integral, $I_a^x f(t)\,dt$, which will be denoted, for convenience, by $I(x)$.

It is clear that $I(x)$ is absolutely continuous. Thus, if $DI(x)$ is any derivative of $I(x)$, since $I(a) = 0$,

$$I(x) = \int_a^x DI(t)\,dt.$$

From V,

$$DI(x) \lesssim f(x), \quad DI(x) \gtrsim f(x).$$

Thus

$$DI(x) \sim f(x).$$

It follows immediately that $f(x)$ is summable, and that

$$I_a^b f(x)\,dx = \int_a^b f(x)\,dx.$$

The modified Perron definition provides a ready means of establishing the well known properties of the Lebesgue integral. A number of results of this kind were given by Perron† for the integral which he defined. The same methods applied to the modified Perron integral at once give the corresponding results, as can be seen without difficulty. It is therefore unnecessary to carry out any of the work here.

Upper and lower Lebesgue integrals.

5. If the function $f(x)$ has associated upper and lower functions it is natural, in view of the results of § 4, to describe the upper and lower integral functions $\Psi(x)$ and $\Phi(x)$ as the upper and lower Lebesgue integrals of $f(x)$ in the interval (a, x), with the notation

$$\Psi(x) = \overline{\int_a^x} f(t)\,dt, \qquad \Phi(x) = \underline{\int_a^x} f(t)\,dt.$$

For a function $f(x)$ which has associated lower functions but not upper functions, or upper functions but not lower functions, neither upper nor lower Lebesgue integrals are defined.

† 6.

THEOREM VII. *If the function $f(x)$ has upper and lower Lebesgue integrals in (a, b), then the function $|f(x)|$ also has upper and lower Lebesgue integrals.*

It is sufficient to show that the function $|f(x)|$ has upper and lower functions.

Let $\Psi(x)$ and $\Phi(x)$ be the upper and lower integral functions of $f(x)$ in (a, b). Then $\Phi(x)$ is a lower function. It is continuous and of bounded variation, and may be expressed as usual in the form

$$\Phi(x) = \underline{P}(x) - \underline{N}(x),$$

where $\underline{P}(x)$ and $\underline{N}(x)$, the positive and negative variations of $\Phi(x)$, are continuous non-decreasing functions, and

$$\underline{P}(a) = \underline{N}(a) = 0.$$

The function

$$-\underline{N}(x) = \Phi(x) - \underline{P}(x)$$

is absolutely continuous above. Since $\Phi(x)$ satisfies the condition (γ_*), so, *a fortiori*, does $-\underline{N}(x)$, and it is seen that $-\underline{N}(x)$ is a lower function of $f(x)$.

In the same way, if

$$\Psi(x) = \bar{P}(x) - \bar{N}(x),$$

$\bar{P}(x)$ is an upper function of $f(x)$.

The function $\bar{P}(x) + \underline{N}(x)$ is continuous and absolutely continuous below, and it vanishes when $x = a$. Further

$$D\bar{P}(x) \gtrsim f(x), \qquad D\big(-\underline{N}(x)\big) \lesssim f(x),$$

so that

$$D\underline{N}(x) \gtrsim |f(x)|$$

at the set of points x such that $f(x) < 0$. It follows that

$$D\big(\bar{P}(x) + \underline{N}(x)\big) \gtrsim |f(x)|.$$

Thus an upper function of $|f(x)|$ has been found.

The function $\phi(x) = 0$ is a lower function of $|f(x)|$, which therefore has upper and lower Lebesgue integrals.

In addition to the properties of upper and lower integrals which have been established in § 3, the following property, where the integrand is the sum of two functions, is of importance.

THEOREM VIII. *If*

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x)$ and $f_2(x)$ both possess upper and lower Lebesgue integrals, then $f(x)$ has upper and lower Lebesgue integrals, and

$$\overline{\int_a^b} f(x)\,dx \leqslant \overline{\int_a^b} f_1(x)\,dx + \overline{\int_a^b} f_2(x)\,dx, \tag{5.01}$$

$$\underline{\int_a^b} f(x)\,dx \geqslant \underline{\int_a^b} f_1(x)\,dx + \underline{\int_a^b} f_2(x)\,dx. \tag{5.02}$$

If one of the functions $f_1(x)$, $f_2(x)$ is integrable, the signs of inequality in (5.01) and (5.02) may be replaced by signs of equality.

Let $\Psi_1(x)$ and $\Psi_2(x)$ be the upper integral functions of $f_1(x)$ and $f_2(x)$ respectively. The function $\Psi_1(x)+\Psi_2(x)$ is continuous, and absolutely continuous below, as the sum of two functions having these properties. It vanishes when $x = a$.

$$D\Psi_1(x) \gtrsim f_1(x), \qquad D\Psi_2(x) \gtrsim f_2(x);$$

so that

$$D\Psi_1(x) + D\Psi_2(x) \gtrsim f(x).$$

But, since $\Psi_1(x)$ and $\Psi_2(x)$ have unique derivatives almost everywhere, their sum has the same property, and

$$D\Big(\Psi_1(x) + \Psi_2(x)\Big) \gtrsim f(x).$$

Thus $\Psi_1(x)+\Psi_2(x)$ is an upper function of $f(x)$. In the same way, it may be seen that $\Phi_1(x)+\Phi_2(x)$ is a lower function of $f(x)$. It follows at once that $f(x)$ has upper and lower Lebesgue integrals, and that these satisfy the inequalities (5.01) and (5.02) respectively.

Let $f_1(x)$ be integrable, so that

$$\Phi_1(x) = \Psi_1(x),$$

and let $\Psi(x)$ and $\Phi(x)$ be the upper and the lower integral functions of $f(x)$. The difference $\Psi(x)-\Psi_1(x)$ is absolutely continuous below, since $\Psi_1(x)$ is absolutely continuous without restriction. It is continuous, and vanishes when $x = a$. Also, since

$$D\Psi_1(x) \sim f_1(x),$$

$$D\Big(\Psi(x) - \Psi_1(x)\Big) \sim D\Psi(x) - D\Psi_1(x)$$

$$\gtrsim f_1(x) + f_2(x) - f_1(x).$$

It follows that $\Psi(x)-\Psi_1(x)$ is an upper function of $f_2(x)$, so that

$$\Psi(b)-\Psi_1(b) \geqslant \Psi_2(b),$$

$$\overline{\int_a^b} f(x)\,dx \geqslant \int_a^b f_1(x)\,dx + \overline{\int_a^b} f_2(x)\,dx.$$

Thus in (5.01) the sign of equality alone is possible, when $f_1(x)$ is integrable. The proof in the case of lower integrals is similar.

In what follows, the notation introduced by de la Vallée-Poussin,

$$f_n(x) = \min\{f(x),\ n\} \qquad (n = 1,\ 2,\ \ldots),$$

is used. Also

$$f_{-n}(x) = \max\{f(x),\ -n\} \quad (n = 1,\ 2,\ \ldots);$$

and if m and n are any two positive integers

$$f_{m,n}(x) = \max\{f_m(x),\ -n\}.$$

Further, the symbols $\Psi_n(x)$, $\Phi_n(x)$ and $\Psi_{-n}(x)$, $\Phi_{-n}(x)$ denote the upper and lower integral functions of $f_n(x)$ and $f_{-n}(x)$ respectively.

THEOREM IX. *If the function $f(x)$ has upper and lower Lebesgue integrals in $(a,\ b)$, then*

$$\overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \overline{\int_a^b} f_n(x)\,dx = \lim_{n\to\infty} \overline{\int_a^b} f_{-n}(x)\,dx = \lim_{m,\,n\to\infty} \overline{\int_a^b} f_{m,n}(x)\,dx$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \overline{\int_a^b} f_{m,n}(x)\,dx = \lim_{n\to\infty} \lim_{m\to\infty} \overline{\int_a^b} f_{m,n}(x)\,dx,$$

$$\underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \underline{\int_a^b} f_n(x)\,dx = \lim_{n\to\infty} \underline{\int_a^b} f_{-n}(x)\,dx = \lim_{m,\,n\to\infty} \underline{\int_a^b} f_{m,n}(x)\,dx$$

$$= \lim_{m\to\infty} \lim_{n\to\infty} \underline{\int_a^b} f_{m,n}(x)\,dx = \lim_{n\to\infty} \lim_{m\to\infty} \underline{\int_a^b} f_{m,n}(x)\,dx.$$

In the course of the proof of IX the following result is established.

THEOREM IV*. *The upper and the lower integral functions of $f(x)$ are absolutely continuous.*

Assume first that $f(x)$ is bounded below. The bounded function $f_n(x)$ has upper and lower integral functions, $\Psi_n(x)$ and $\Phi_n(x)$, and these are absolutely continuous without restriction. Consider the functions of the sequence $\{\Phi_n(x)\}$.

If n and p are any two positive integers, the function $\Phi_{n+p}(x)-\Phi_n(x)$ is non-negative and non-decreasing, for $\Phi_n(x)$ is a lower function of $f_n(x)$,

and so, *a fortiori*, of $f_{n+p}(x)$. Also, for all values of n, $\Phi_n(x)$ is a lower function of $f(x)$, so that $\Phi(x)-\Phi_n(x)$ is non-negative. Let

$$\lim_{n\to\infty} \Phi_n(x) = \mathrm{X}(x).$$

Then

$$\mathrm{X}(x) \leqslant \Phi(x). \tag{5.04}$$

The difference $\mathrm{X}(x)-\Phi_n(x)$ is non-decreasing. For suppose that for two values x_1 and x_2 of x, where $a \leqslant x_1 < x_2 \leqslant b$,

$$\{\Phi_n(x_2)-\Phi_n(x_1)\} - \{\mathrm{X}(x_2)-\mathrm{X}(x_1)\} = \epsilon > 0, \tag{5.05}$$

and let p be such that

$$\mathrm{X}(x_1)-\Phi_{n+p}(x_1) < \epsilon.$$

Then, since $\Phi_{n+p}(x)-\Phi_n(x)$ is non-decreasing,

$$\Phi_n(x_2)-\Phi_n(x_1) \leqslant \Phi_{n+p}(x_2)-\Phi_{n+p}(x_1);$$

(5.05) gives

$$\Phi_{n+p}(x_2)-\Phi_{n+p}(x_1)-\mathrm{X}(x_2)+\mathrm{X}(x_1) \geqslant \epsilon,$$

$$\Phi_{n+p}(x_2)-\mathrm{X}(x_2) > \epsilon-\epsilon = 0,$$

and the definition of $\mathrm{X}(x)$ is contradicted.

The argument used in the proof of IV can be applied to show that the function $\mathrm{X}(x)$ is absolutely continuous above. As the sum of a non-decreasing function and the absolutely continuous function $\Phi_n(x)$, it is also absolutely continuous below.

Suppose now that

$$\mathrm{X}(b) < \Phi(b).$$

Since $\Phi(x)-\Phi_n(x)$ is non-decreasing, it is easy to see that $\Phi(x)-\mathrm{X}(x)$ is non-decreasing, so that

$$D\Phi(x) \gtrsim D\mathrm{X}(x).$$

The difference $\Phi(x)-\mathrm{X}(x)$ is absolutely continuous above, and

$$\Phi(b)-\mathrm{X}(b) > 0,$$

$$\Phi(a)-\mathrm{X}(a) = 0.$$

The lemma of § 1 shows that, if ϵ is a fixed number such that

$$0 < \epsilon < \frac{\Phi(b)-\mathrm{X}(b)}{b-a},$$

then the function $\Phi(x)-X(x)$ has a derivative greater than ϵ at the points of a set of non-zero measure. But $\Phi(x)$ and $X(x)$ have unique derivatives almost everywhere; it follows that the set E of points at which

$$D\Phi(x)-DX(x) \geqslant \epsilon \tag{5.06}$$

is measurable, $mE > 0$.

Since $\Phi(x)$ has a finite unique derivative almost everywhere, there is a positive integer n such that the subset E_n of E at whose points

$$D\Phi(x) \leqslant n \tag{5.07}$$

is not of measure zero.

Since

$$DX(x) \gtrsim D\Phi_n(x),$$

(5.06) shows that, at points of E,

$$D\Phi(x)-D\Phi_n(x) \gtrsim \epsilon.$$

At points of E_n,

$$D\Phi(x) \lesssim f_n(x),$$

for if $f_n(x) < n$, then $f_n(x) = f(x)$, and if $f_n(x) = n$, then by (5.07) $D\Phi(x) \leqslant n$. Thus, at points of E_n,

$$D\Phi_n(x) \lesssim f_n(x)-\epsilon. \tag{5.08}$$

Let $g(x)$ be the characteristic function of E_n. Then, if

$$G(x) = \int_a^x g(t)\,dt,$$

the function $\epsilon G(x)$ is absolutely continuous, and has ϵ as unique derivative at almost all points of E_n, and zero as unique derivative at almost all points of the complement of E_n. Thus, in view of (5.08), the function

$$\Phi_n(x)+\epsilon G(x) \tag{5.09}$$

is a lower function of $f_n(x)$. But, since $mE_n > 0$, $G(b) > 0$, and (5.09) is a lower function of $f_n(x)$ which exceeds $\Phi_n(x)$ when $x = b$. This is impossible; the contradiction shows that in (5.04) the sign of equality alone is admissible, and

$$\lim_{n\to\infty} \Phi_n(x) = \Phi(x).$$

Since $X(x)$ is absolutely continuous, $\Phi(x)$ is absolutely continuous.

The sequence $\{\Psi_n(x)\}$ must now be considered.

It is clear that $\Psi(x)$ is an upper function of $f_n(x)$ for each value of n. Thus

$$\Psi(x) \geqslant \Psi_n(x). \tag{5.10}$$

The sequence $\{\Psi_n(x)\}$ increases, for $\Psi_{n+p}(x)$ is an upper function of $f_n(x)$ when p is any positive integer. Since, by (5.10), the sequence is bounded, $\lim_{n\to\infty} \Psi_n(x)$ exists, and, if it is denoted by $\Theta(x)$, obviously

$$\Theta(x) \leqslant \Psi(x). \tag{5.11}$$

The difference $\Theta(x)-\Psi_n(x)$ is non-decreasing for each value of n, for, if it were not, it would be possible, n being fixed, to find a positive integer p such that $\Psi_{n+p}(x)-\Psi(x)$ decreases, and this gives a contradiction.

The function $\Theta(x)$ is absolutely continuous below as the sum of the absolutely continuous function $\Psi_n(x)$ and a non-decreasing function. By an argument similar to that given above in the case of $X(x)$, it may be shown that $\Theta(x)$ is also absolutely continuous above. Further, for all values of n,

$$D\Theta(x) \gtrsim D\Psi_n(x)$$

$$\gtrsim f_n(x),$$

so that

$$D\Theta(x) \gtrsim f(x),$$

and $\Theta(x)$ is an upper function of $f(x)$. In (5.11) the sign of equality alone is admissible, and

$$\lim_{n\to\infty} \Psi_n(x) = \Psi(x).$$

It is now seen that the upper and the lower integral functions of any function bounded below are absolutely continuous without restriction. If $f(x)$ is replaced by $-f(x)$, the properties just proved may be shown to hold for functions bounded above.

If $f(x)$ is unbounded but has upper and lower integral functions, the function $f_n(x)$ has upper and lower integral functions $\Psi_n(x)$ and $\Phi_n(x)$; for $f_n(x)$ has $\Psi(x)$ as an upper function, and the upper integral function of $|f(x)|$ is an upper function of $-f_n(x)$. Also, since $f_n(x)$ is bounded above, $\Psi_n(x)$ and $\Phi_n(x)$ are absolutely continuous, as has just been shown.

The proofs given above for the case when $f(x)$ is bounded below apply without any alteration to show that, if $f(x)$ is unbounded, $\Psi(x)$ and $\Phi(x)$ are absolutely continuous and

$$\lim_{n\to\infty} \Phi_n(x) = \Phi(x),$$

$$\lim_{n\to\infty} \Psi_n(x) = \Psi(x).$$

If $-f(x)$ is considered in place of $f(x)$, it is easy to see that

$$\lim_{n\to\infty} \Phi_{-n}(x) = \Phi(x),$$

$$\lim_{n\to\infty} \Psi_{-n}(x) = \Psi(x).$$

It has thus been established that if $f(x)$ is any function which possesses an upper and a lower Lebesgue integral, then its upper and its lower integral functions are absolutely continuous, and

$$(5.12) \qquad \begin{cases} \overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \overline{\int_a^b} f_n(x)\,dx, \\ \underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \underline{\int_a^b} f_n(x)\,dx, \end{cases}$$

$$(5.13) \qquad \begin{cases} \overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \overline{\int_a^b} f_{-n}(x)\,dx, \\ \underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} \underline{\int_a^b} f_{-n}(x)\,dx. \end{cases}$$

Now, from (5.13),

$$\lim_{n\to\infty} \overline{\int_a^b} f_{m,n}(x)\,dx = \overline{\int_a^b} f_m(x)\,dx,$$

$$\lim_{n\to\infty} \underline{\int_a^b} f_{m,n}(x)\,dx = \underline{\int_a^b} f_m(x)\,dx,$$

and hence, by (5.12),

$$\overline{\int_a^b} f(x)\,dx = \lim_{m\to\infty} \lim_{n\to\infty} \overline{\int_a^b} f(x)\,dx,$$

$$\underline{\int_a^b} f(x)\,dx = \lim_{m\to\infty} \lim_{n\to\infty} \underline{\int_a^b} f(x)\,dx.$$

The order of the limit operations is clearly immaterial.

In view of the fact that the sequence $\left\{\overline{\int}_a^b f_{m,n}(x)\,dx\right\}$ decreases as n increases, if m is fixed, and increases as m increases, if n is fixed, it can be proved by elementary methods that

$$\overline{\int}_a^b f(x)\,dx = \lim_{m,n\to\infty} \overline{\int}_a^b f_{m,n}(x)\,dx,$$

$$\underline{\int}_a^b f(x)\,dx = \lim_{m,n\to\infty} \underline{\int}_a^b f_{m,n}(x)\,dx.$$

The function $f(x)$ is the difference $f_+(x)-f_-(x)$ of the two non-negative functions

$$f_+(x) = \max\{f(x), 0\},$$

$$f_-(x) = \max\{-f(x), 0\}.$$

A result more precise than that stated in VIII holds for this representation of $f(x)$.

THEOREM X. *If $f_+(x)$ and $-f_-(x)$ are the positive and negative parts of the function $f(x)$, which has an upper and a lower Lebesgue integral, then $f_+(x)$ and $f_-(x)$ have upper and lower Lebesgue integrals and*

$$\overline{\int}_a^b f(x)\,dx = \overline{\int}_a^b f_+(x)\,dx - \underline{\int}_a^b f_-(x)\,dx, \tag{5.14}$$

$$\underline{\int}_a^b f(x)\,dx = \underline{\int}_a^b f_+(x)\,dx - \overline{\int}_a^b f_-(x)\,dx. \tag{5.15}$$

The function $|f(x)|$ has upper and lower Lebesgue integrals. The upper integral function of $|f(x)|$ is an upper function of both $f_+(x)$ and $f_-(x)$, and $\phi(x)=0$ is a lower function of both; it follows that both functions have upper and lower Lebesgue integrals.

With the notation introduced in the proof of VII, let

$$\Phi(x) = \underline{P}(x) - \underline{N}(x),$$

$$\Psi(x) = \overline{P}(x) - \overline{N}(x).$$

It will be shown that

$$\begin{cases} \underline{P}(x) = \underline{\int}_a^x f_+(t)\,dt, & \overline{P}(x) = \overline{\int}_a^x f_+(t)\,dt, \\ \underline{N}(x) = \overline{\int}_a^x f_-(t)\,dt, & \overline{N}(x) = \underline{\int}_a^x f_-(t)\,dt. \end{cases} \tag{5.16}$$

The function $\Phi(x)$ is absolutely continuous without restriction, and $\underline{P}(x)$ and $\underline{N}(x)$ necessarily have the same property. There is an integrable function $g(x)$ such that

$$\int_a^x g(t)\,dt = \Phi(x),$$

and, if $g_+(x)$ and $-g_-(x)$ are the positive and negative parts of $g(x)$,

$$\int_a^x g_+(t)\,dt = \underline{P}(x), \qquad \int_a^x g_-(t)\,dt = \underline{N}(x). \tag{5.17}$$

Now

$$f(x) \gtrsim D\Phi(x) \sim g(x),$$

so that, from (5.17),

$$f_+(x) \gtrsim g_+(x) \sim D\underline{P}(x), \qquad f_-(x) \lesssim g_-(x) \sim D\underline{N}(x).$$

Evidently $\underline{P}(x)$ satisfies the conditions (α_*), (β_*), and (γ_*) with respect to $f_+(x)$. Thus

$$\underline{P}(b) \leqslant \int_a^b f_+(x)\,dx, \tag{5.18}$$

and in the same way it is seen that

$$\underline{N}(b) \geqslant \int_a^b f_-(x)\,dx.$$

Suppose that, in (5.18), the sign of equality does not hold, and let

$$\Phi^*(x) = \int_a^x f_+(t)\,dt - \underline{N}(x).$$

Then

$$\Phi^*(b) > \Phi(b). \tag{5.19}$$

Evidently $\Phi^*(x)$ satisfies conditions (α_*) and (β_*), and further

$$\begin{aligned} D\Phi^*(x) &\sim f_+(x) - D\underline{N}(x) \\ &\lesssim f_+(x) - f_-(x) \\ &= f(x). \end{aligned}$$

This shows that $\Phi^*(x)$ is a lower function of $f(x)$, and (5.19) is contradicted. In (5.18) the sign of equality alone is admissible.

The remaining statements (5.16) can be established by a procedure similar to that just carried out. The required results (5.14) and (5.15) follow at once.

Upper and lower Lebesgue integrals and the measure of sets of ordinates.

6. THEOREM XI. *If the upper and the lower Lebesgue integrals of a function $f(x)$ exist in (a, b) but do not coincide, then $f(x)$ is not measurable in (a, b).*

The result will first be proved for a non-negative function $f(x)$.

It has been shown that $f(x)$ is not integrable. Suppose that it is measurable in (a, b). The function $f_n(x)$ is bounded and measurable and therefore integrable. Thus $\int_a^b f_n(x)\,dx$ exists for all values of n, but

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = +\infty.$$

Any lower function of $f_n(x)$ is a lower function of $f(x)$. In particular, $\int_a^x f_n(t)\,dt$ is a lower function of $f(x)$, so that, for all values of n,

$$\int_a^b f_n(x)\,dx \leqslant \int_a^b f(x)\,dx.$$

This is impossible, for the lower integral must be a finite number; it follows that $f(x)$ is not measurable.

Let the restriction on the sign of $f(x)$ be removed, and let $f_+(x)$ and $f_-(x)$ be the positive and negative parts of $f(x)$. From X it is seen that at least one of the functions $f_+(x)$ and $f_-(x)$ has an upper and a lower Lebesgue integral which do not coincide. Since both these functions are non-negative, one at least of them is not measurable; it follows that $f(x)$ is not measurable.

Let $f(x)$ be a given non-negative function defined in (a, b). Consider the plane set $\underline{O}$ of points (x, y) defined by

$$a \leqslant x \leqslant b, \quad 0 \leqslant y < f(x).$$

This set $\underline{O}$ is called the *least set of ordinates* of $f(x)$.

The set $\overline{O}$, defined by

$$a \leqslant x \leqslant b, \quad 0 \leqslant y \leqslant f(x),$$

is the *greatest set of ordinates* of $f(x)$.

Any set O such that

$$\underline{O} < O < \overline{O}$$

is a *set of ordinates* of $f(x)$

Any two sets of ordinates of the same function have equal inner measures and equal outer measures†

It is known that if $f(x)$ is measurable its sets of ordinates are also measurable, and they all have the same measure. Their measure is $\int_a^b f(x)\,dx$, if $f(x)$ is integrable, and is infinite if $f(x)$ is not integrable in (a, b). Conversely, if a set O of ordinates is measurable, the function $f(x)$ is measurable, and if mO is finite,

$$mO = \int_a^b f(x)\,dx,$$

while, if $mO = +\infty$, the function $f(x)$ is not integrable.

If $f(x)$ takes negative values, and if $f_+(x)$ and $-f_-(x)$ are its positive and negative parts, O_+ a set of ordinates of $f_+(x)$, and O_- a set of ordinates of $f_-(x)$, then, if $f(x)$ is integrable,

$$\int_a^b f(x)\,dx = mO_+ - mO_-.$$

The corresponding connection will now be established between the upper and lower Lebesgue integrals whose analytical definition has been given, and the geometrical definition by means of the inner and outer measure of sets of ordinates, originally given, for a bounded integrand, by Lebesgue‡.

LEMMA. *If O is any set of ordinates of a non-negative function $f(x)$ in (a, b), and if m^*O is finite, then there is a non-negative integrable function $q(x)$ defined in (a, b), such that, if Q is any set of ordinates of $q(x)$, Q contains O, and*

$$mQ \leqslant m^*O + \epsilon, \tag{6.01}$$

where $\epsilon > 0$ is given in advance; also $\int_a^x q(t)\,dt$ is an upper function of $f(x)$.

For $n = 1, 2, \ldots$, let P_n be the set of points x such that $f(x) \geqslant n\eta$, where $\eta = \epsilon/(b-a)$. Then

$$(a, b) > P_1 > P_2 > \ldots > P_n > \ldots.$$

† **2**, 419.

‡ **4**, 250.

For $n = 1, 2, \ldots$, let $\bar{P}_n$ be a measurable set of points such that

$$m\bar{P}_n = m^* P_n,$$

$$\bar{P}_n > P_n,$$

$$\bar{P}_n < \bar{P}_{n-1},$$

where $\bar{P}_0$ is the closed interval (a, b). It is easy to see that sets $\bar{P}_1, \bar{P}_2, \ldots$ fulfilling these conditions can be found. Consider the function $q(x)$ defined as follows:

$$\begin{aligned}
q(x) &= \eta \quad \text{if } x \text{ is a point of } \bar{P}_0 - \bar{P}_1,\\
q(x) &= 2\eta \quad ,, \quad ,, \quad \bar{P}_1 - \bar{P}_2,\\
&\vdots \qquad\qquad\qquad \vdots\\
q(x) &= n\eta \quad ,, \quad ,, \quad \bar{P}_{n-1} - \bar{P}_n,\\
&\vdots \qquad\qquad\qquad \vdots\\
q(x) &= +\infty \quad ,, \quad ,, \quad \lim_{n\to\infty} \bar{P}_n.
\end{aligned}$$

The function $q(x)$ is defined for all values of x, $a \leqslant x \leqslant b$, it is measurable, and

$$q(x) > f(x),$$

so that, if Q is a set of ordinates of $q(x)$,

$$Q > O.$$

Now

$$mQ = \sum_{n=0}^{\infty} \eta m\bar{P}_n,$$

$$mQ = \epsilon + \eta \sum_{n=1}^{\infty} m\bar{P}_n. \tag{6.02}$$

Let O_n be the common part of O and the set R_n of points (x, y) such that $a \leqslant x \leqslant b$, $0 \leqslant y < n\eta$. Then

$$O = O_1 + \sum_{n=1}^{\infty} (O_{n+1} - O_n).$$

For $n = 1, 2, \ldots$, the sets $O_{n+1} - O_n$ are subsets of the distinct measurable sets $R_{n+1} - R_n$, and O_1 is a subset of R_1 which is distinct from each of these. Hence

$$m^* O = m^* O_1 + \sum_{n=1}^{\infty} m^*(O_{n+1} - O_n).$$

Now

$$m^*(O_{n+1}-O_n) \geqslant \eta \,.\, m^* P_{n+1},$$

$$m^* O_1 \geqslant \eta \,.\, m^* P_1,$$

so that

$$m^* O \geqslant \eta \sum_{n=1}^{\infty} m \bar{P}_n. \tag{6.03}$$

The relations (6.02) and (6.03) yield (6.01) immediately.

Since m^*O is finite, Q is of finite measure. It follows that $q(x)$ is integrable and

$$\int_a^b q(x)\,dx = mQ.$$

It is clear that $\int_a^x q(t)\,dt$ satisfies the conditions (α^*) and (β^*). Further

$$D\int_a^x q(t)\,dt \sim q(x)$$
$$> f(x).$$

Thus $\int_a^x q(t)\,dt$ is an upper function of $f(x)$.

THEOREM XII. *If $f(x)$ is a non-negative function defined in (a, b), and having upper and lower Lebesgue integrals, and if O is any set of ordinates of $f(x)$, then*

$$m^* O = \overline{\int_a^b} f(x)\,dx, \tag{6.04}$$

$$m_* O = \underline{\int_a^b} f(x)\,dx. \tag{6.05}$$

It is only necessary to consider the case where $f(x)$ is not measurable. It will first be supposed that $f(x)$ is bounded,

$$f(x) \leqslant H$$

in (a, b).

Consider the upper integral function $\Psi(x)$ of $f(x)$, and let $O(a, x)$ be any set of ordinates of $f(x)$ in the interval (a, x).

Let $D\Psi(x)$ be any derivative of $\Psi(x)$. Since $H(x-a)$ is an upper function of $f(x)$, the function $H(x-a)-\Psi(x)$ is non-decreasing, and it is clear that $D\Psi(x) \leqslant H$ in (a, x).

Let $P(a, x)$ be a set of ordinates of $D\Psi(x)$ in (a, x). Then, since $D\Psi(x) \gtrsim f(x)$, either $P(a, x)$ contains $O(a, x)$ or it can be increased by the addition of a set of points of plane measure zero to a set $P^*(a, x)$

such that

$$P^*(a, x) > O(a, x).$$

$$mP(a, x) = mP^*(a, x)$$

$$\geqslant m^* O(a, x).$$

Further, since $D\Psi(x)$ is measurable and bounded,

$$mP(a, x) = \int_a^x D\Psi(x)\,dx = \Psi(x),$$

for $\Psi(x)$ is absolutely continuous and $\Psi(a) = O$. Thus

$$\Psi(x) \geqslant m^* O(a, x). \tag{6.06}$$

Let $q(x)$ be a function having the properties described in the lemma, and let $Q(a, x)$ be a set of its ordinates in the interval (a, x). Then

$$mQ(a, x) \leqslant m^* O(a, x) + \epsilon, \tag{6.07}$$

and, since $\int_a^x q(x)\,dx$ exists and is an upper function of $f(x)$,

$$mQ(a, x) \geqslant \Psi(x). \tag{6.08}$$

By (6.07) and (6.08),

$$\Psi(x) \leqslant m^* O(a, x) + \epsilon, \tag{6.09}$$

for all values of ϵ greater than zero. From (6.06) and (6.09)

$$\Psi(x) = m^* O(a, x),$$

which gives (6.04) in the case where $f(x)$ is bounded.

If

$$g(x) = H - f(x),$$

then $g(x)$ is a non-negative bounded function.

Let $\bar{O}$ be the greatest set of ordinates of $f(x)$ in (a, b), and let $\underline{S}$ be the least set of ordinates of $g(x)$ in (a, b). Then $\underline{S}$ is congruent to the complement of $\bar{O}$ with respect to the rectangle $(a \leqslant x \leqslant b\,;\ 0 \leqslant y \leqslant H)$, so that

$$m_* \bar{O} + m^* \underline{S} = H(b - a),$$

$$m_* \bar{O} = H(b - a) - \overline{\int_a^b} g(x)\,dx.$$

By VIII,

$$\underline{\int_a^b} f(x)\,dx = \int_a^b H dx + \underline{\int_a^b} \big(-g(x)\big)\,dx$$

$$= H(b-a) - \overline{\int_a^b} g(x)\,dx\,;$$

and it follows that, if $f(x)$ is bounded,

$$\underline{\int_a^b} f(x)\,dx = m_* \bar{O} = m_* O.$$

Suppose that $f(x)$ is non-negative and unbounded above. Let O be a set of ordinates of $f(x)$, and O_n the common part of O and the rectangle R_n, $(a \leqslant x \leqslant b\,;\ 0 \leqslant y \leqslant n)$. Then O_n is a set of ordinates of $f_n(x)$, so that

$$m^* O_n = \overline{\int_a^b} f_n(x)\,dx,$$

$$m_* O_n = \underline{\int_a^b} f_n(x)\,dx.$$

From IX,

$$\overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} m^* O_n, \tag{6.10}$$

$$\underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} m_* O_n. \tag{6.11}$$

The sets O and O_n may be exhibited in the forms

$$O = O_1 + \sum_{r=1}^{\infty} (O_{r+1} - O_r), \qquad O_n = O_1 + \sum_{r=1}^{n-1} (O_{r+1} - O_r).$$

For $n = 1, 2, \ldots$, the sets $O_{n+1} - O_n$ are subsets of the distinct measurable sets $R_{n+1} - R_n$, and O_1 is a subset of R_1, which is distinct from each of these; hence

$$m^* O = m^* O_1 + \sum_{r=1}^{\infty} m^*(O_{r+1} - O_r), \qquad m^* O_n = m^* O_1 + \sum_{r=1}^{n-1} m^*(O_{r+1} - O_r),$$

$$m_* O = m_* O_1 + \sum_{r=1}^{\infty} m_*(O_{r+1} - O_r), \qquad m_* O_n = m_* O_1 + \sum_{r=1}^{n-1} m_*(O_{r+1} - O_r)\,;$$

so that

$$m^* O = \lim_{n\to\infty} m^* O_n,$$

$$m_* O = \lim_{n\to\infty} m_* O_n.$$

From (6.10) and (6.11),

$$\overline{\int_a^b} f(x)\,dx = m^* O,$$

$$\underline{\int_a^b} f(x)\,dx = m_* O,$$

provided that $f(x)$ is a non-negative function which has an upper and a lower Lebesgue integral. This completes the proof of XII.

COROLLARY. *A necessary and sufficient condition that the non-negative function $f(x)$ should have an upper and a lower Lebesgue integral is that a set of its ordinates should be of finite outer measure.*

That the condition is necessary is shown by XII. That it is sufficient is shown by the lemma above.

Theorem X may now be stated in the following form.

If $f_+(x)$ and $-f_-(x)$ are the positive and negative parts of the function $f(x)$, and if O_+ and O_- are sets of ordinates of $f_+(x)$ and $f_-(x)$ respectively in (a, b), then a necessary and sufficient condition that $f(x)$ should have upper and lower Lebesgue integrals in (a, b) is that O_+ and O_- should be of finite outer measure; if this condition is satisfied

$$\overline{\int_a^b} f(x)\,dx = m^* O_+ - m_* O_-,$$

$$\underline{\int_a^b} f(x)\,dx = m_* O_+ - m^* O_-.$$

References.

1. P. Alexandroff, *Math. Zeitschrift*, 20 (1924), 213–222.
2. C. Carathéodory, *Vorlesungen über reele Funktionen*, 2nd ed. (Leipzig, 1927).
3. M. D. Kennedy, *Proc. London Math. Soc.* (2), 30 (1930), 58–80.
4. H. Lebesgue, *Annali di Mat.* (3), 7 (1902), 231-359.
5. H. Lebesgue, *Leçons sur l'intégration et la recherche des fonctions primitives*, 2nd ed. (Paris, 1928).
6. O. Perron, *Sitzungsberichte der Heidelberger Akad.*, 5A (1914), No. 14.
7. W. H. Young, *Proc. London Math. Soc.* (2), 9 (1911), 286–324.

ON THE MAXIMUM MODULUS PRINCIPLE FOR FUNCTIONS WITH ZEROS AND POLES

By MARY L. CARTWRIGHT.

[Received 15 November, 1929.—Read 12 December, 1929.]

1. Phragmén and Lindelöf* have proved the following theorem :

THEOREM A. *If the function* $f(z) = f(re^{i\theta})$ *is regular for* $\alpha_1 \leqslant \theta \leqslant \alpha_2$, $r > r_0$, *and*

$$|f(re^{i\theta})| = O\{e^{(A\cos\theta\rho + B\sin\theta\rho)r^\rho}\} \tag{1.1}$$

for $\theta = \alpha_1$, $\theta = \alpha_2$, *where* $\alpha_2 - \alpha_1 = 2\alpha < \mathrm{Min}\,(\pi/\rho\,;\,2\pi)$; *if, further,*

$$|f(re^{i\theta})| = O\{e^{\eta r^{\frac{1}{2}\pi/\alpha}}\} \tag{1.2}$$

for every $\eta > 0$, *on a sequence of arcs* $r = R_1, R_2, \ldots, R_n \to \infty$, $\alpha_1 \leqslant \theta \leqslant \alpha_2$, *then* (1.1) *holds uniformly for* $\alpha_1 \leqslant \theta \leqslant \alpha_2$.

This theorem expresses the fact that, in general terms, a regular function cannot be larger than the largest simple exponential which will satisfy the boundary conditions. The theorems which I shall prove here show that a function cannot be much larger than the largest function composed of exponentials, which satisfies the boundary conditions, and also has the same number of poles and zeros.

Consider, for simplicity, functions of order 1 in the angle $-\alpha \leqslant \theta \leqslant \alpha < \frac{1}{2}\pi$. A regular function which satisfies

$$|f(re^{\pm i\alpha})| = O\{e^{r\cos\alpha}\}, \tag{1.3}$$

and the appropriate general condition throughout the angle, may be as

* E. Phragmén and E. Lindelöf, *Acta Math.*, 31 (1908), 386. For a short account of theorems of this nature see G. Pólya and G. Szegö, *Aufgaben und Lehrsätze*, 1 (1925), Abschn. III, Nr. 332, *et seq.* I shall refer to this book in future as P.S.

large as e^z. If the function has as many zeros on the real axis as $\sin z$, it may be only as large, approximately, as

$$e^{Az} \sin z,$$

where $A \cos \alpha + \sin \alpha = \cos \alpha$, *i.e.* $A = 1 - \tan \alpha$. Obviously

$$e^{(1-\tan\alpha)z} \sin z \tag{1.4}$$

is much smaller than e^z inside the angle $-\alpha \leqslant \theta \leqslant \alpha$.

Theorem I gives a result of this type. The argument, when applied to $1/f(z)$, shows that if a function without zeros is as large as e^z on the boundary, and has as many poles as $\operatorname{cosec} z$ on the real axis, then it is as large as

$$e^{Bz} \operatorname{cosec} z,$$

where $B \cos \alpha - \sin \alpha = \cos \alpha$, *i.e.* $B = 1 + \tan \alpha$, and

$$e^{(1+\tan\alpha)z} \operatorname{cosec} z \tag{1.5}$$

is much larger than e^z inside the angle. Theorem II gives a result in the opposite direction, and shows, roughly speaking, that if a function has not more poles than $\operatorname{cosec} z$, it is not larger than (1.5), except very near the poles. Applying this argument to $1/f(z)$, we see that a function which is larger than e^z on the boundary, and which has not more zeros than $\sin z$, cannot be much smaller than (1.4). Thus each theorem is shown to be a best possible theorem, within the limits of its accuracy, by using the argument of the other. I suppose, of course, in all cases that the general condition equivalent to (1.2) is satisfied.

Theorems III, IV, V, VI, and VII are generalizations of Theorems I and II. In Theorem VIII I use these results to show that, roughly speaking, if a function behaves like a simple exponential, say e^z, throughout the angle, then it has very few zeros inside it.

Up to this point we have been comparing the behaviour of a function inside and on the boundary of an angle which is less than what we may call the "critical angle" of the exponentials concerned. The general conclusion is that a function is smaller throughout the part of the angle containing the zeros and larger throughout the part containing the poles than elsewhere. Consider again functions of order 1. Inside any angle less than π we can reduce $|f(z)|$ to 1 by multiplying by the appropriate exponential, *e.g.* e^{-z} in (1.3), and then we can apply the maximum modulus principle. If the angle is equal to π, this process fails unless $|f(z)| < K$ on the boundary, in which case we can apply the maximum modulus principle at once.

The behaviour of a function in an angle equal to the critical angle may be better understood by considering the Nevanlinna* principle for the average value of a function throughout the whole circle. Consider the functions e^z, $\sinh z$, $\operatorname{cosech} z$, which have approximately the same value on the imaginary axis, except near the poles and zeros. The function e^z, which has neither poles nor zeros, is large in one half-plane and small in the other; $\sinh z$, which has zeros only, is large in both half-planes; and $\operatorname{cosech} z$, which has poles only, is small in both. The general conclusion expressed by the Nevanlinna principle is that a function with zeros is large except near the zeros, and a function with poles is small except near the poles, while a function without either is large in some places and small in others.

Theorem IX is a generalization of a theorem due to Carlson†. It shows that if a function is fairly small on the boundary of the critical angle, and has a large number of zeros inside the angle, then the function vanishes identically. My proof follows the method used by Hardy‡, which is similar to that of Carlson. I reduce the function to one which tends to zero along the boundary, and then apply the maximum modulus principle to show that the function must be so small inside the angle that it vanishes altogether. R. and F. Nevanlinna§ have proved a still more general form of this theorem. They use a formula which works in the opposite direction, and shows that a function with zeros inside the critical sector must be large on the boundary. Hence they obtain a contradiction, unless the function vanishes identically. My method can be extended so as to cover more general cases, but my chief reason for proving the theorem is to show its relation to the earlier ones.

In all cases it is necessary for the zeros or poles to be more or less evenly distributed. The zeros must be arranged so as to include an evenly distributed sequence of numbers, and it is with this sequence that we work. Similarly the poles must be included by an evenly distributed sequence of numbers. A zero or pole of the k-th order must, of course, be counted k times.

Finally, I should like to express my thanks to Prof. Hardy and Prof. Titchmarsh for their valuable advice.

* R. Nevanlinna, *Acta Soc. Sci. Fennicae*, 50 (vi), (1924), 6.

† F. Carlson, "Sur une classe de séries de Taylor", Thèse (Upsal, 1914), 58–61. See also P.S. I, Ab. III, Nr. 328.

‡ G. H. Hardy, *Acta Math.*, 42 (1920), 328–330.

§ R. and F. Nevanlinna, *Acta Soc. Sci. Fennicae*, 50 (v), (1922). See also R. Nevanlinna, *loc. cit.*, 50 (xii), (1925), and G. Pólya, *Math. Annalen*, 89 (1923), 182.

2. Let $z_1, z_2, \ldots, z_n = a_n e^{i\theta_n}, \ldots$ denote a sequence of numbers such that $0 < a_1 \leqq a_2 \leqq \ldots$, and let $n(r)$ be the number of numbers $a_n \leqq r$.

THEOREM I. *Suppose that $f(z)$ satisfies the hypotheses of Theorem* A, *and that the zeros of $f(z)$ include a sequence of numbers $z_1, z_2, \ldots, z_n = a_n e^{i\theta_n} \ldots$, for which $\theta_1 = \theta_2 = \ldots = \alpha_0 = \frac{1}{2}(\alpha_1 + \alpha_2)$, and*

$$n(r) \sim \lambda r^\rho, \tag{2.1}$$

as $r \to \infty$. Then

$$\log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho + \epsilon - \pi\lambda\sin(\alpha - |\theta - \alpha_0|)\rho\sec\alpha\rho\}r^\rho \tag{2.2}$$

for every $\epsilon > 0$ and every $r \geqq r_0(\epsilon)$ in the angle $\alpha_1 \leqq \theta \leqq \alpha_2$.

Proof. By (2.1) we can construct the function

$$s_p(z) = \prod_1^\infty \left(1 - \frac{z^p}{z_n^p}\right),$$

where $\rho < p$.

Pólya* and Titchmarsh† have shown that, if, as $r \to \infty$,

$$n(r) \sim \lambda r^\rho,$$

where $0 < \rho < 1$, then

$$\log|s_1(re^{i\theta})| \sim \pi\lambda\cos(\pi - |\theta - \alpha_0|)\rho\operatorname{cosec}\pi\rho \cdot r^\rho$$

for $0 < |\theta - \alpha_0| < 2\pi$. Hence it is easy to see that

$$\log|s_1(re^{i\theta})| < \{\pi\lambda\cos(\pi - |\theta - \alpha_0|)\rho\operatorname{cosec}\pi\rho + \epsilon\}r^\rho \tag{2.3}$$

for every $\epsilon > 0$, and every $r > r_0(\epsilon)$, for all values of θ, provided that $0 < \rho < 1$. The general case amounts to considering the function

$$s_1(z^p),$$

when

$$n(r) \sim \lambda r^{\rho/p},$$

where $0 < \rho < p$. We have therefore

$$\log|s_p(re^{i\theta})| < \{\pi\lambda\cos(\pi/p - |\theta - \alpha_0|)\rho\operatorname{cosec}\pi\rho/p + \epsilon\}r^\rho \tag{2.4}$$

for every $\epsilon > 0$, every $r \geqq r_0(\epsilon)$, and for $0 \leqq |\theta - \alpha_0| \leqq 2\pi/p$. Also, given

* P.S. II, Ab. IV, Nr. 61.

† E. C. Titchmarsh, *Proc. London Math. Soc.* (2), 26 (1927), 185–200. Titchmarsh obtains this result as a corollary to a deeper theorem (191–194), which includes (2.3), but Pólya's result is sufficient here.

any $\epsilon > 0$ and $\delta > 0$, we can choose $r_0 = r_0(\epsilon, \delta)$, so that

$$\log|s_p(re^{i\theta})| > \{\pi\lambda \cos(\pi/p - |\theta - a_0|)\rho \operatorname{cosec} \pi\rho/p + \epsilon\} r^\rho \tag{2.5}$$

for every $r \geqq r_0(\epsilon, \delta)$, provided that $\delta \leqq |\theta - a_0| \leqq 2\pi/p - \delta$.

Choose p so that $-2\pi/p + a_0 < a_1$, and $2\pi/p + a_0 > a_2$, *i.e.* so that $\alpha < 2\pi/p$. Since $\alpha < \frac{1}{2}\pi/\rho$, it is easy to satisfy this condition and yet choose $p > \rho$. Let

$$F(z) = \frac{f(z)}{s_p(z)}, \tag{2.6}$$

then $F(z)$ is regular for $a_1 \leqq \theta \leqq a_2$. Since $f(z)$ satisfies (1.1) and $s_p(z)$ satisfies (2.5) for $\theta = a_1$, $\theta = a_2$, we have

$$\begin{aligned}\log|F(re^{ia_1})| &< \{A \cos a_1\rho + B \sin a_1\rho \\ &\qquad - \pi\lambda \cos(\pi/p - a_0 + a_1)\rho \operatorname{cosec} \pi\rho/p + \epsilon\} r^\rho \\ &= \{A + C) \cos a_1\rho + (B + D) \sin a_1\rho + \epsilon\} r^\rho \text{ (say)}, \end{aligned} \tag{2.7}$$

$$\begin{aligned}\log|F(re^{ia_2})| &< \{A \cos a_2\rho + B \sin a_2\rho \\ &\qquad - \pi\lambda \cos(\pi/p - a_2 + a_0)\rho \operatorname{cosec} \pi\rho/p + \epsilon\} r^\rho \\ &= \{(A + C) \cos a_2\rho + (B + D) \sin a_2\rho + \epsilon\} r^\rho \text{ (say)} \end{aligned} \tag{2.8}$$

for every $\epsilon > 0$, every $r \geqq r_0(\epsilon)$. Also, by Valiron's* theorem,

$$|s_p(re^{i\theta})| > e^{-Kr^\rho}$$

for some K on a sequence of circles $r = R_1, R_2, \ldots, R_n \to \infty$. Hence, on this sequence of circles,

$$|F(re^{i\theta})| < e^{K'r^\rho}$$

for some K'. It follows from Theorem A that

$$\log|F(re^{i\theta})| < \{(A + C) \cos\theta\rho + (B + D) \sin\theta\rho + \epsilon\} r^\rho.$$

From (2.7) and (2.8) we have

$$C \cos a_1\rho + D \sin a_1\rho = -\pi\lambda \cos(\pi/p - \alpha)\rho \operatorname{cosec} \pi\rho/p,$$

$$C \cos a_2\rho + D \sin a_2\rho = -\pi\lambda \cos(\pi/p - \alpha)\rho \operatorname{cosec} \pi\rho/p.$$

Hence

$$C \cos\theta\rho + D \sin\theta\rho = -\pi\lambda \cos(\pi/p - \alpha)\rho \operatorname{cosec} \pi\rho/p \sec\alpha\rho \cos(a_0 - \theta)\rho.$$

---

* G. Valiron, *Integral functions* (Toulouse, 1923), 89. Titchmarsh's theorem proves considerably more than this, see (3.3), but even Valiron's result is more than we actually need here.

It follows now, from (2.6) and (2.4), that

$$\frac{\log|f(re^{i\theta})|}{r^\rho} < A\cos\theta\rho + B\sin\theta\rho + \epsilon - \frac{\pi\lambda\{\cos(\pi/p-\alpha)\rho\cos(\alpha_0-\theta)\rho - \cos(\pi/p-|\theta-\alpha_0|)\rho\cos\alpha\rho\}}{\sin\pi\rho/p\cos\alpha\rho}$$

$$= A\cos\theta\rho + B\sin\theta\rho + \epsilon - \pi\lambda E(\theta),$$

where $$E(\theta) = \sin(\alpha - |\theta-\alpha_0|)\rho\sec\alpha\rho,$$

which is the required result.

3. As before, we denote by $\zeta_1, \zeta_2, \ldots, \xi_m = b_m e^{i\phi_m} \ldots$, a sequence of numbers such that $0 < b_1 \leqq b_2 \leqq \ldots$, and by $m(r)$ the number of numbers $b_m \leqq r$.

THEOREM II. *Suppose that $f(z)$ satisfies the hypotheses of Theorem* A, *except for the existence of a number of poles inside the angle $\alpha_1 \leqq \theta \leqq \alpha_2$. Suppose, further, that these poles form part of a sequence of numbers $\zeta_1, \zeta_2, \ldots, \xi_m = b_m e^{i\phi_m}, \ldots$, such that*

$$\phi_1 = \phi_2 = \ldots = \alpha_0 = \tfrac{1}{2}(\alpha_1+\alpha_2),$$

and

$$m(r) \sim \lambda r^\rho, \tag{3.1}$$

as $r \to \infty$. Then (i) *given any positive numbers ϵ_1 and δ, we can choose $r_0 = r_0(\epsilon_1, \delta)$ such that*

$$\log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho + \epsilon_1 + \pi\lambda\sin(\alpha-|\theta-\alpha_0|)\rho\sec\alpha\rho\}\, r^\rho \tag{3.2}$$

uniformly for $\delta \leqq |\theta-\alpha_0| \leqq \alpha$, $r \geqq r_0$;

(ii) *given any positive numbers ϵ_1, ϵ_2, we can choose $R_0 = R_0(\epsilon_1, \epsilon_2)$, so that* (3.2) *is satisfied for $\alpha_1 \leqq \theta \leqq \alpha_2$, $0 < r < R$, $R > R_0$, except in a set of measure less than $\epsilon_2 R$.*

Proof. We construct the function

$$\sigma_p(z) = \prod_1^\infty \left(1 - \frac{z^p}{\xi_m^p}\right),$$

whose properties are exactly similar to those of $s_p(z)$. In addition to (2.4) and (2.5) Titchmarsh* has proved that, if

$$m(r) \sim \lambda r^\rho \quad (0 < \rho < p),$$

---

* *Loc. cit.*, 191. Here we use the full result.

and if ϵ_1, ϵ_2 are any positive numbers, then

$$\log|\sigma_p(re^{i\alpha_0})| > (\pi\lambda \cot \pi\rho/p - \epsilon_1) r^\rho$$

for $0 < r < R$, $R > R_0(\epsilon_1, \epsilon_2)$, except in a set of measure less than $\epsilon_2 R$. And

$$\log|\sigma_p(re^{i(\alpha_0+\delta)})|^2 = \sum_1^\infty \log\left|1-\frac{re^{i\delta}}{b_m}\right|^2 \geqslant \sum_1^\infty \log\left|1-\frac{r}{b_m}\right|^2 = \log|\sigma_p(re^{i\alpha_0})|^2$$

for all values of δ. Hence

$$\log|\sigma_p(re^{i\theta})| > \{\pi\lambda \cos(\pi/p - |\theta-\alpha_0|)\rho \operatorname{cosec} \pi\rho/p - \epsilon_1\} r^\rho \tag{3.3}$$

for all values of θ, and for $0 < r < R$, except possibly in a set of measure less than $\epsilon_2 R$.

Choose p as in Theorem I, *i.e.* so that $\alpha < 2\pi/p$, $\rho < p$, and let

$$\Phi(z) = f(z)\sigma_p(z). \tag{3.4}$$

Then $\Phi(z)$ is regular for $\alpha_1 \leqslant \theta \leqslant \alpha_2$. Since $f(z)$ satisfies (1.1), and $s_p(z)$ satisfies (2.4) for $\theta = \alpha_1$, $\theta = \alpha_2$,

$$\log|\Phi(re^{i\alpha_1})| < \{A\cos\alpha_1\rho + B\sin\alpha_1\rho + \pi\lambda\cos(\pi/p - \alpha_0 + \alpha_1)\rho \operatorname{cosec}\pi\rho/p + \epsilon\} r^\rho$$

$$= \{(A+C)\cos\alpha_1\rho + (B+D)\sin\alpha_1\rho + \epsilon\} r^\rho \text{ (say)}, \tag{3.5}$$

$$\log|\Phi(re^{i\alpha_2})| < \{A\cos\alpha_2\rho + B\sin\alpha_2\rho + \pi\lambda\cos(\pi/p - \alpha_2 + \alpha_0)\rho \operatorname{cosec}\pi p/\rho + \epsilon\} r^\rho$$

$$= \{(A+C)\cos\alpha_2\rho + (B+D)\sin\alpha_2\rho + \epsilon\} r^\rho \text{ (say)}. \tag{3.6}$$

Now $f(z)$ satisfies (1.2) on a sequence of arcs $r = R_1, R_2, \ldots, R_n \to \infty$, $\alpha_1 \leqslant \theta \leqslant \alpha_2$, and $\sigma_p(z)$ satisfies (2.4) uniformly for $r \geqslant r_0(\epsilon)$. It follows that $\Phi(z)$ satisfies (1.2) on that sequence of arcs, and we can apply Theorem A to $\Phi(z)$. Hence

$$\log|\Phi(re^{i\theta})| < \{(A+C)\cos\theta\rho + (B+D)\sin\theta\rho + \epsilon\} r^\rho.$$

From (3.5) and (3.6) we deduce, as before, that

$$C\cos\theta\rho + D\sin\theta\rho = \pi\lambda\cos(\pi/p - \alpha_0)\rho \operatorname{cosec}\pi\rho/p \sec\alpha\rho \cos(\alpha_0 - \theta)\rho. \tag{3.7}$$

The required result follows from (3.4) by using (3.7) with (2.5) for $|\theta-\alpha_0| > \delta$, and with (3.3) for $0 \leqslant |\theta-\alpha_0| \leqslant \delta$. The working is precisely the same as for Theorem I with the sign of λ changed.

4. I shall now show that these are best possible results. Consider the function

$$f(z) = e^{(A+C-iB-iD)z^\rho} s_p(z),$$

where A, B, C, and D have the same values as in Theorem I. Then $f(z)$ satisfies the hypotheses of Theorem I, and $1/f(z)$ satisfies the hypotheses of Theorem II with A and B negative. Hence we have a *function $f(z)$ satisfying the hypotheses of Theorem* I *such that,*

(i) *given any positive numbers ϵ_1, δ, we can choose $r_0 = r_0(\epsilon_1, \delta)$ so that*

$$\log|f(re^{i\theta})| > \{A\cos\theta\rho + B\sin\theta\rho - \epsilon_1 - \pi\lambda\sin(\alpha - |\theta - a_0|)\rho\sec\alpha\rho\}r^\rho \tag{4.1}$$

uniformly for $\delta \leqslant |\theta - a_0| \leqslant a_1$, $r \geqslant r_0(\epsilon_1, \delta)$;

(ii) *given any positive numbers ϵ_1, ϵ_2, we can choose $R_0 = R_0(\epsilon_1, \epsilon_2)$ so that* (4.1) *is satisfied for $0 < r < R$, $R > R_0(\epsilon_1, \epsilon_2)$, except in a set of measure less than $\epsilon_2 R$.*

To obtain the corresponding result for Theorem II we consider the function

$$f(z) = e^{(A+C-iB-iD)z^\rho}/\sigma_p(z),$$

where A, B, C, and D have the same values as in Theorem II. We then apply Theorem I to $1/f(z)$, or we can obtain the result by using the inequalities for $\sigma_p(z)$ directly. The same arguments can be applied to Theorems III and IV.

5. I shall now proceed to generalize these theorems in three directions. First, the zeros or poles need not lie exactly on a straight line. Provided that $z_n = a_n e^{i\theta_n}$, where $|a_0 - \theta_n| < \delta$ for every $\delta > 0$ and every $n > n_0(\delta)$, this makes no difference to the result. A trivial argument will show that the auxiliary function still behaves in the same way. Next, the line, near which the zeros lie, need not be the bisector of the angle of the theorem. This naturally makes the result lop-sided, but the difference is purely formal.

Lastly, the rate of increase of the function, and of the number of zeros or poles along any line, may be slightly larger or smaller than r^ρ, in which case everything varies accordingly. In fact, we may replace r^ρ by $r^\rho L(r)$ where $L(z)$ satisfies the following conditions:

(5.1) $L(re^{i\theta})$ is regular for $a_1 \leqslant \theta \leqslant a_2$, where

$$a_2 - a_1 = 2\alpha < \mathrm{Min}(\pi/\rho;\ 2\pi),$$

provided that $r > r_0 > 0$.

(5.2) $L(r) > 0$ for $r > r_0$.

(5.3) As $r \to \infty$, $\dfrac{L(r)}{r^\epsilon} \to 0$ uniformly for every $\epsilon > 0$, while $\overline{\lim}\, r^\epsilon L(r) \to \infty$ for every $\epsilon > 0$.

(5.4) As $r \to \infty$, $\dfrac{L(re^{i\theta})}{L(r)} \to 1$ uniformly for $\alpha_1 \leqslant \theta \leqslant \alpha_2$.

These conditions are the ones imposed in the following theorem of Phragmén and Lindelöf, which is a straightforward generalization of Theorem A.

THEOREM B. *If $f(z)$ is regular and of the form*

$$|f(re^{i\theta})| = O\{e^{A\cos\theta\rho + B\sin\theta\rho)r^\rho L(r)}\} \tag{5.5}$$

for $\theta = \alpha_1$, $\theta = \alpha_2$, where $\alpha_2 - \alpha_1 = 2\alpha < \pi/\rho$; if, further,

$$|f(re^{i\theta})| = O\{e^{\eta r^{\frac{1}{2}\pi/\alpha}}\} \tag{5.6}$$

for every $\eta > 0$ on a sequence of arcs $r = R_1, R_2, \ldots, R_n \to \infty$, $\alpha_1 \leqslant \theta \leqslant \alpha_2$, then (5.5) holds uniformly for $\alpha_1 \leqslant \theta \leqslant \alpha_2$.

In dealing with the auxiliary function we require a more stringent condition in place of (5.3):

(5.7) $L(r)$ is monotonic for $r > r_0$, and, as $r \to \infty$,

$$\frac{L(2r)}{L(r)} \to 1.$$

The last condition implies that*, as $r \to \infty$,

$$\frac{L(cr)}{L(r)} \to 1$$

for every finite $c > 0$. The chief point of the definition is that it covers all such functions as

$$L(r) = (\log r)^{\rho_1}(\log\log r)^{\rho_2} \ldots (\log_k r)^{\rho_k},$$

where $\rho_1, \rho_2, \ldots, \rho_k$ are all real, and $\log_k r = \log(\log_{k-1} r)$.

6. We can now prove

THEOREM III. *Suppose that $f(z)$ satisfies the hypotheses of Theorem B, and that the zeros of $f(z)$ include a sequence of numbers*

* See P.S. I, Ab. II, Nr. 150, 152.

$z_1, z_2, \ldots, z_n = a_n e^{i\theta_n}, \ldots$. *Suppose, further, that* $|\alpha_3 - \theta_n| \leqslant \delta$, *where* $\alpha_1 < \alpha_3 < \alpha_2$, *for every* $\delta > 0$ *and every* $n > n_0(\delta)$, *and that*

$$n(r) \sim \lambda r^\rho L(r). \tag{6.1}$$

Then, given any $\epsilon > 0$, *we can choose* $r_0 = r_0(\epsilon)$ *so that*

$$\log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho + \epsilon - \pi\lambda E(\theta)\} r^\rho L(r), \tag{6.2}$$

where

$$E(\theta) = 2\sin(\alpha_2 - \alpha_3)\rho \sin(\theta - \alpha_1)\rho \operatorname{cosec} 2\alpha\rho \quad (\alpha_1 \leqslant \theta \leqslant \alpha_3)$$
$$= 2\sin(\alpha_3 - \alpha_1)\rho \sin(\alpha_2 - \theta)\rho \operatorname{cosec} 2\alpha\rho \quad (\alpha_3 \leqslant \theta \leqslant \alpha_2),$$

for $r \geqslant r_0$, $\alpha_1 \leqslant \theta \leqslant \alpha_2$.

The corresponding theorem for poles is

THEOREM IV. *Suppose that* $f(z)$ *satisfies the hypotheses of Theorem* B, *except for the existence of certain poles which form part of a sequence of numbers* $\xi_1, \xi_2, \ldots, \xi_m = b_m e^{i\phi_m}, \ldots$. *Suppose, further, that* $|\alpha_3 - \phi_m| \leqslant \delta$, *where* $\alpha_1 < \alpha_3 < \alpha_2$, *for every* $\delta > 0$ *and every* $m > m_0(\delta)$, *and that*

$$m(r) \sim \lambda r^\rho L(r). \tag{6.3}$$

Then, (i) *given any positive numbers* ϵ_1, δ, *we can choose* $r_0 = r_0(\epsilon_1, \delta)$ *so that*

$$\log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho + \epsilon_1 + \pi\lambda E(\theta)\} r^\rho L(r), \tag{6.4}$$

where $E(\theta)$ *is defined as in Theorem* III, *uniformly for* $r \geqslant r_0$, $\alpha_1 \leqslant \theta \leqslant \alpha_3 - \delta$, *and* $\alpha_3 + \delta \leqslant \theta \leqslant \alpha_2$;

(ii) *given any positive numbers* ϵ_1, ϵ_2, *we can choose* $R_0 = R_0(\epsilon_1\ \epsilon_2)$ *so that* (6.4) *is satisfied for* $\alpha_1 \leqslant \theta \leqslant \alpha_2$, $0 < r < R$, $R > R_0$, *except in a set of measure less than* $\epsilon_2 R$.

7.1. *The auxiliary function for Theorem* III We can construct the function $s_p(z)$ as before. We can also construct the allied function

$$g_p(z) = \prod_1^\infty \left(1 - \frac{z^p}{a_n^p}\right).$$

Pólya's theorem* shows that if (6.1) is satisfied, then

$$\log|g_p(re^{i\theta})| \sim \pi\lambda\cos(\pi/p - |\theta|)\rho \operatorname{cosec}\pi\rho/p \,.\, r^\rho L(r)$$

* P.S. II, Ab. IV, Nr. 61.

for $0<|\theta-\alpha_3|<2\pi/p$. Also

$$\log|s_p(re^{i\theta})|^2=\sum_1^{n_0}+\sum_{n_0+1}^{\infty}\log\left|1-\frac{r^p}{a_n^p}e^{ip(\theta-\theta_n)}\right|^2$$

$$<K\log r+\sum_{n_0+1}^{\infty}\log\left|1-\frac{2r^p}{a_n^p}\cos p(\theta-\theta_n)+\frac{r^{2p}}{a_n^{2p}}\right|$$

$$<K'\log r+\sum_1^{\infty}\log\left|1-\frac{2r^p}{a_n^p}\cos p(|\theta-\alpha_3|+\delta)+\frac{r^{2p}}{a_n^{2p}}\right|$$

$$=K'\log r+\log|g_p(re^{i(|\theta-\alpha_3|+\delta)})|^2$$

$$<\{2\pi\lambda\cos(\pi/p-|\theta-\alpha_3|+\delta)\,\rho\,\operatorname{cosec}\pi\rho/p+\epsilon\}\,r^\rho L(r)$$

for every $\epsilon>0$, every $\delta>0$, and every $r>r_0(\epsilon,\ \delta)$, provided that $|\theta-\alpha_3|\leqq\pi/p-\delta$. For $\pi/p-\delta\leqq|\theta-\alpha_3|\leqq\pi/p+\delta$, it is easy to see that

$$\log|s_p(re^{i\theta})|^2\leqq\sum_1^{\infty}\log\left(1+\frac{r^p}{a_n^p}\right)^2$$

$$<(2\pi\lambda\operatorname{cosec}\pi\rho/p+\epsilon)\,r^\rho L(r).$$

Hence, by choosing δ sufficiently small, we see that

(7.12) $$\log|s_p(re^{i\theta})|<\{\pi\lambda\cos(\pi/p-|\theta-\alpha_3|)\rho\operatorname{cosec}\pi\rho/p+\epsilon\}\,r^\rho L(r)$$

for every $\epsilon>0$, every $r>r_0(\epsilon)$, and all values of θ. Similarly, we can show that

(7.13) $$\log|s_p(re^{i\theta})|>\{\pi\lambda\cos(\pi/p-|\theta-\alpha_3|)\rho\operatorname{cosec}\pi\rho/p-\epsilon\}\,r^\rho L(r)$$

for every $\epsilon>0$ and every $r>r_0(\epsilon,\ \delta)$, provided that $\delta\leqq|\theta-\alpha_3|\leqq2\pi/p-\delta$.

7.2. *Proof of Theorem* III. Choose $p>\rho$, so that

$$-2\pi/p+\alpha_3<\alpha_1\quad\text{and}\quad\alpha_2<2\pi/p+\alpha_3,$$

i.e. so that $$2\pi/p>\operatorname{Max}(\alpha_3-\alpha_1;\ \alpha_2-\alpha_3).$$

It is easy to do this for

$$\operatorname{Max}(\alpha_3-\alpha_1;\ \alpha_2-\alpha_3)<2\alpha<\operatorname{Min}(\pi/\rho\,;\ 2\pi),$$

so that it is sufficient to make

$$\tfrac{1}{2}p\leqq\operatorname{Max}(\rho;\ \tfrac{1}{2}).$$

Let

(7.21) $$F(z)=\frac{f(z)}{s_p(z)},$$

then $F(z)$ is regular for $\alpha_1 \leqslant \theta \leqslant \alpha_2$. By (5.5) and (7.13)

$$\log|F(re^{i\alpha_1})| < \{A\cos\alpha_1\rho + B\sin\alpha_1\rho + \epsilon - \pi\lambda\cos(\pi/p - \alpha_3 + \alpha_1)\rho\ \operatorname{cosec}\pi\rho/p\}\, r^\rho L(r)$$

$$= \{(A+C)\cos\alpha_1\rho + (B+D)\sin\alpha_1\rho + \epsilon\}\, r_\rho L(r) \quad \text{(say)}, \tag{7.22}$$

$$\log|F(re^{i\alpha_2})| < \{A\cos\alpha_2\rho + B\sin\alpha_2\rho + \epsilon - \pi\lambda\cos(\pi/p - \alpha_2 + \alpha_3)\rho\ \operatorname{cosec}\pi\rho/p\}\, r^\rho L(r)$$

$$= \{(A+C)\cos\alpha_2\rho + (B+D)\sin\alpha_2\rho + \epsilon\}\, r^\rho L(r) \quad \text{(say)} \tag{7.23}$$

for every $\epsilon > 0$ and every $r > r_0(\epsilon)$. Applying Valiron's theorem to $s_p(z)$ we find, as in Theorem I, that

$$|F(re^{i\theta})| > e^{Kr^\rho L(r)}$$

for some K on a sequence of arcs $r = R_1, R_3, \ldots, R_n \to \infty$, $\alpha_1 \leqslant \theta \leqslant \alpha_2$. Hence, by Theorem B,

$$\log|F(re^{i\theta})| < \{(A+C)\cos\theta\rho + (B+D)\sin\theta\rho + \epsilon\}\, r^\rho L(r).$$

From (7.22) and (7.23) it follows that

$$C\cos\alpha_1\rho + D\sin\alpha_1\rho = -\pi\lambda\cos(\pi/p - \alpha_3 + \alpha_1)\rho\ \operatorname{cosec}\pi\rho/p,$$

$$C\cos\alpha_2\rho + D\sin\alpha_2\rho = -\pi\lambda\cos(\pi/p - \alpha_2 + \alpha_3)\rho\ \operatorname{cosec}\pi\rho/p.$$

Therefore

$$\sin 2\alpha\rho\{C\cos\theta\rho + D\sin\theta\rho\} = -\pi\lambda\cos(\alpha_2 + \alpha_1 - \alpha_3 - \theta)\rho - \pi\lambda\sin(\pi/p - \alpha_2 + \alpha_1)\rho\cos(\alpha_3 - \theta)\rho\ \operatorname{cosec}\pi\rho/p.$$

It follows, from (7.21) and (7.12), that

$$\log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho + \epsilon - \pi\lambda E(\theta)\}\, r^\rho L(r),$$

where

$$E(\theta) = \frac{\cos(\alpha_2 + \alpha_1 - \alpha_3 - \theta)\rho}{\sin 2\alpha\rho} - \frac{\sin(\pi/p - \alpha_2 + \alpha_1)\rho\cos(\alpha_3 - \theta)\rho - \sin 2\alpha\rho\cos(\pi/p - |\theta - \alpha_3|)\rho}{\sin\pi\rho/p\sin 2\alpha\rho}$$

$$= \frac{1}{\sin 2\alpha\rho}\{\cos(\alpha_2 + \alpha_1 - \alpha_3 - \theta)\rho - \cos(\alpha_2 - \alpha_1 - |\theta - \alpha_3|)\rho\}$$

$$= 2\sin(\alpha_2 - \alpha_3)\rho\sin(\theta - \alpha_1)\rho\ \operatorname{cosec} 2\alpha\rho \quad (\alpha_1 \leqslant \theta \leqslant \alpha_3)$$

$$2\sin(\alpha_3 - \alpha_1)\rho\sin(\alpha_2 - \theta)\rho\ \operatorname{cosec} 2\alpha\rho \quad (\alpha_3 \leqslant \theta \leqslant \alpha_2),$$

which is the required result.

8.1. *The auxiliary function for Theorem* IV. We construct the function $\sigma_p(z)$, whose properties are similar to those of the function considered in (7.1). We also construct the function

$$\gamma_p(z) \prod_1^\infty \left(1-\frac{z^p}{b_m^p}\right)$$

which behaves like $g_p(z)$. We require a result corresponding to (3.3) when (6.3) is satisfied.

We first prove

LEMMA 8.2. *If, as* $r\to\infty$,

$$m(r) \sim r^\rho L(r) \quad (0<\rho<1), \tag{8.21}$$

then, given any positive numbers ϵ_1, ϵ_2, *we can choose* R_0 *so that*

$$\log|\gamma_1(r)| > (\pi\lambda \cot \pi\rho - \epsilon_1) r^\rho L(r)$$

for $0<r<R$, $R>R_0$ (ϵ_1, ϵ_2), *except in a set of measure less than* $\epsilon_2 R$.

Proof. Let $c>1$, and let q be an integer depending on r such that

$$c^q \leqslant r < c^{q+1}.$$

Let $$M = m(c^{q-1}), \quad N = m(c^{q+2}).$$

Pólya* has shown that if (8.21) is satisfied, then

$$\lim_{r\to\infty} \frac{1}{m(r)} \sum_{b_m \leqslant r/c} \log\left(\frac{r}{b_m}-1\right) = \int_0^{c^{-\rho}} \log(x^{-1/\rho}-1)\,dx.$$

It follows that

$$\sum_1^M \log\left(\frac{r}{b_m}-1\right) > \sum_1^M \log\left(\frac{c^q}{b_m}-1\right)$$

$$> (1-\delta)\, m(c^q) \int_0^{c^{-\rho}} \log(x^{-1/\rho}-1)\,dx$$

for every $\delta>0$ and every $r>r_0(\delta)$. Now

$$m(c^q) > (1-\delta)\lambda c^{q\rho} L(c^q)$$

$$> (1-\delta)^2 \lambda c^{-\rho} r^\rho L(r).$$

Hence $$\sum_1^M \log\left(\frac{r}{b_m}-1\right) > (1-\delta)^3 c^{-\rho} \lambda r^\rho L(r) \int_0^{c^{-\rho}} \log(x^{-1/\rho}-1)\,dx$$

* P.S. I, Ab. II, Nr. 155, 156.

for every $\delta>0$, every $c>1$, and every $r>r_0(\delta, c)$. Similarly

$$\sum_{N+1}^{\infty}\log\left(1-\frac{r}{b_m}\right)>(1-\delta)^3c^\rho\lambda r^\rho L(r)\int_{c\rho}^{\infty}\log(1-x^{-1/\rho})dx.$$

Hence

$$\log|\gamma_1(r)|=\sum_1^M+\sum_{M+1}^N+\sum_{N+1}^{\infty}\log\left|1-\frac{r}{b_m}\right|$$

$$>(1-\delta)^3c^{-\rho}\lambda r^\rho L(r)\int_0^{c^{-\rho}}\log(x^{-1/\rho}-1)dx+\log|P(r)|$$

$$+(1-\delta)^3\lambda r^\rho L(r)\int_{c\rho}^{\infty}\log(1-x^{-1/\rho})dx$$

$$>\lambda r^\rho L(r)\{1-O(\delta)-O(c^{-\rho}-1)\log|c-1|\}\int_0^{\infty}\log|1-x^{-1/\rho}|\,dx$$

$$+\log|P(r)|$$

$$=\pi\lambda r^\rho L(r)\cot\pi\rho\{1-O(\delta)-O(c^{-\rho}-1)\log(c-1)\}+\log|P(r)|$$

for every $\delta>0$, every $c>1$, and every $r>r_0(\delta, c)$.

Putting* $r=c^{q-1}+(c^{q+2}-c^{q-1})\xi$, we have

$$\left|\prod_{M+1}^N(r-b_m)\right|=(c^{q+2}-c^{q-1})^{N-M}\left|\prod_{M+1}^N\left\{\xi-\frac{b_m-c^{q-1}}{c^{q+2}-c^{q-1}}\right\}\right|.$$

The product involving ξ is a polynomial of degree $N-M$, whose zeros lie between 0 and 1. Hence†

$$\left|\prod_{M+1}^N\left\{\xi-\frac{b_m-c^{q-1}}{c^{q+2}-c^{q-1}}\right\}\right|>(2e)^{-H(N-M)}$$

for $0<\xi<1$, except in a set of measure less than $1/H$. Hence

$$\prod_{M+1}^N|r-b_m|>\left\{\frac{c^{q-1}(c^3-1)}{2^He^H}\right\}^{N-M}$$

for $c^{q-1}<r<c^{q+2}$, and so *a fortiori* for $c^q\leqslant r<c^{q+1}$, except in a set of measure less than $(c^{q+2}-c^{q-1})/H$. It follows that outside this set

$$|P(r)|=\prod_{M+1}^N\left|1-\frac{r}{b_m}\right|>\frac{1}{b_{M+1}\dots b_N}\left\{\frac{3c^{q-1}(c-1)}{2^He^H}\right\}^{N-M}.$$

---

* Compare E. C. Titchmarsh, *loc. cit.*, 193, 194.

† See G. Valiron, *loc. cit.*, 79.

Also $b_{M+1} \dots b_N < (c^{q+2})^{N-M}$, so that outside the exceptional set

$$|P(r)| > \left\{\frac{3(c-1)}{2^H e^H c^3}\right\}^{N-M}.$$

Now
$$\begin{aligned} N-M &= m(c^{q+2}) - m(c^{q-1}) \\ &< (1+\delta)c^{(q+2)\rho} L(c^{q+2}) - (1-\delta)c^{(q-1)\rho} L(c^{q-1}) \\ &< (1+\delta)^2 c^{\rho} r^{\rho} L(r) - (1-\delta)^2 c^{-\rho} r^{\rho} L(r) \\ &< r^{\rho} L(r)\{c^{\rho} - c^{-\rho} + O(\delta)\}. \end{aligned}$$

Hence

$$\begin{aligned} \log|P(r)| &> r^{\rho} L(r)\{c^{\rho} - c^{-\rho} + O(\delta)\}\{\log(c-1) - H\log 2e + O(1)\} \\ &> -\epsilon r^{\rho} L(r) \end{aligned}$$

for every $\epsilon > 0$ and every $r > r_0(\epsilon)$. The result is obtained by first fixing H, then choosing c sufficiently near 1, then δ sufficiently small, and then r_0 sufficiently large.

The measure of the subset of the interval (c^q, c^{q+1}) which is exceptional is less than

$$\frac{c^{q+2} - c^{q-1}}{H} = \frac{c^{q-1}(c^3-1)}{H} \leqslant \frac{3c^{q+1}(c-1)}{H} = \frac{3c(c^{q+1} - c^q)}{H}.$$

Hence the part of the interval $(0, c^q)$ which is exceptional is less than $3c^{q+1}/H$. This proves the lemma.

8.3. We can now show that, if (6.3) is satisfied, *then, given any positive numbers* ϵ_1, ϵ_2, *we can choose* R_0 *so that for all values of* θ

(8.31) $$\log|\sigma_p(re^{i\theta})| > \{\pi\lambda\cos(\pi/p - |\theta - a_3|)\rho\,\mathrm{cosec}\,\pi\rho/p - \epsilon_1\} r^{\rho} L(r)$$

for $0 < r < R$, $R > R_0(\epsilon_1, \epsilon_2)$, *except in a set of measure less than* $\epsilon_2 R$. For, if $|\theta - a_3| \leqslant \delta$, $0 < r < R$,

$$\begin{aligned} \log|\sigma_p(re^{i\theta})|^2 &\geqslant \sum_1^{\infty} \log\left|1 - \frac{2r^p}{b_m^p}\cos p(|\theta - a_3| - \delta) + \frac{r^{2p}}{b_m^{2p}}\right| \\ &= \log|\gamma_p(re^{i(|\theta - a_3| - \delta)})|^2 \\ &> \{\pi\lambda\cot\pi\rho/p - \epsilon_1\} r^{\rho} L(r), \end{aligned}$$

provided that $\delta < \delta_0(\epsilon_1)$, except in a set of measure less than $\epsilon_2 R$. If $|\theta - a_3| \geqslant \delta > 0$, (8.31) holds uniformly for $r > r_0(\epsilon_1)$ by Pólya's* theorem.

---

* P.S. II, Ab. IV, Nr. 61.

8.4. *Proof of Theorem* IV. Let $\Phi(z)$ be defined as in (3.4). We follow the method of Theorem II exactly, and the formal details are the same as for Theorem III with the sign of λ changed. The required result follows without any difficulty.

9.1. I shall now consider cases in which the zeros or poles are distributed all over a certain angle greater than $\delta > 0$. I shall suppose that this angle is placed symmetrically with regard to the angle of the theorem in order to reduce the formal complications.

THEOREM V. *Suppose that $f(z)$ satisfies the hypotheses of Theorem* B, *and that the zeros of $f(z)$ include a sequence of numbers $z_1, z_2, \ldots, z_n = a_n e^{i\theta_n}, \ldots$. Suppose that $|a_0 - \theta_n| \leqslant \beta < \alpha$, where $a_0 = \frac{1}{2}(a_1 + a_2)$, for all n, and that* (6.1) *is satisfied. Then, given any $\epsilon > 0$, we can choose $r_0 = r_0(\epsilon)$ so that*

$$\log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho + \epsilon - \pi\lambda E(\theta)\}r^\rho L(r),$$

where

$$E(\theta) = 2\sin(\alpha-\beta)\rho\,\sin(\alpha-|\theta-a_0|)\rho\,\operatorname{cosec} 2\alpha\rho,$$

for $r \geqslant r_0$, $|\theta - a_0| \leqslant \alpha - \beta$.

$E(\theta)$ is evidently positive for the values of θ concerned, so that the result is a definite improvement on the result of Theorem B near the bisector of the angle. The value of $E(\theta)$ for $\theta \leqslant a_0$ is the same as if all the zeros were on the line $\theta = a_0 + \beta$, and for $\theta \geqslant a_0$, it is the same as if all the zeros were on the line $\theta = a_0 - \beta$. Hence this result is also the best possible.

THEOREM VI. *Suppose that $f(z)$ satisfies the hypotheses of Theorem* B, *except for poles which form part of a sequence of numbers $\xi_1, \xi_2, \ldots, \xi_m = b_m e^{i\phi_m}, \ldots$. Suppose that $|a_0 - \phi_m| \leqslant \beta < \alpha$ for all m, and that* (6.3) *is satisfied. Then,* (i) *given any positive numbers ϵ_1, δ, we can choose $r_0 = r_0(\epsilon_1, \delta)$ so that*

$$\log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho + \epsilon_1 + \pi\lambda E(\theta)\}r^\rho L(r), \tag{9.11}$$

where

$$E(\theta) = 2\{\cos(\theta - a_0)\rho - \cos^2\alpha\rho\}\operatorname{cosec} 2\alpha\rho,$$

uniformly for $r \geqslant r_0$, $\beta + \delta \leqslant |\theta - a_0| \leqslant \alpha$;

(ii) *given any positive numbers ϵ_1, ϵ_2, we can choose $R_0 = R_0(\epsilon_1, \epsilon_2)$ so that* (9.11) *is satisfied for $a_1 \leqslant \theta \leqslant a_2$, $0 < r < R$, $R > R_0$, except in a set of measure less than $\epsilon_2 R$.*

I have aimed here at establishing an upper limit for $|f(re^{i\theta})|$ throughout the angle; the maximum of $E(\theta)$ occurs at $\theta = a_0$, where its value

is the same as if all the poles were on the line $\theta = \alpha_0$, and is therefore the best possible. I actually obtain a slightly better value of $E(\theta)$ for $\beta+\delta \leqslant |\theta-\alpha_0| \leqslant \alpha$; but I think that the best possible result for $\theta = \theta_0 \neq \alpha_0$ can only be obtained by taking as auxiliary function $s_p(z)$, where p depends on θ_0.

9.2. *The auxiliary function for Theorems* V *and* VI. The following lemma sums up all the results which we use in this paper about the auxiliary function.

LEMMA 9.2. *Let $s_p(z)$ be the function formed from the sequence $z_1, z_2, \ldots, z_n = a_n e^{i\theta_n}, \ldots$ as usual. Suppose that $|\theta_n| \leqslant \beta < \pi/p$, and that* (6.1) *is satisfied. Then the following results hold:*

(i) *Given any $\epsilon_1 > 0$, we can choose $r_0 = r_0(\epsilon_1)$ so that*

$$\log|s_p(re^{i\theta})| < \{\pi\lambda \operatorname{cosec} \pi\rho/p + \epsilon_1\} r^\rho L(r) \tag{9.21}$$

for all values of θ; and in particular for $0 \leqslant |\theta| \leqslant \pi/p-\beta$ and $\pi/p+\beta \leqslant |\theta| \leqslant 3\pi/p-\beta$,

$$\log|s_p(re^{i\theta})| < \{\pi\lambda \cos(|\pi/p-|\theta||-\beta)\rho \operatorname{cosec} \pi\rho/p+\epsilon_1\} r^\rho L(r), \tag{9.22}$$

provided that $r \geqslant r_0$.

(ii) *Given any positive numbers ϵ_1, δ, we can choose $r_0 = r_0(\epsilon_1, \delta)$ so that*

$$\log|s_p(re^{i\theta})| > \{\pi\lambda \cos(|\pi/p-|\theta||+\beta)\rho \operatorname{cosec} \pi\rho/p-\epsilon_1\} r^\rho L(r) \tag{9.23}$$

uniformly for $\beta+\delta \leqslant |\theta| \leqslant 2\pi/p-\beta-\delta$, $r \geqslant r_0$.

(iii) *Given any positive numbers ϵ_1, ϵ_2, we can choose $R_0 = R_0(\epsilon_1, \epsilon_2)$ so that*

$$\log|s_p(re^{i\theta})| > (\pi\lambda \cot \pi\rho/p-\epsilon_1) r^\rho L(r) \tag{9.24}$$

for all values of θ and for $0 < r < R$, $R \geqslant R_0$, except in a set of measure less than $\epsilon_2 R$.

Proof. As in (7.1), we construct the function $g_1(z)$. Then, for $0 < \rho < 1$,

$$\begin{aligned}\log|s_1(re^{i\theta})| &< \log|g(re^{i(|\theta|+\beta)})| \\ &< \{\pi\lambda \cos(\pi-\beta-|\theta|)\rho \operatorname{cosec} \pi\rho+\epsilon\} r^\rho L(r),\end{aligned}$$

provided that $|\theta| \leqslant \pi-\beta$. Also, for all values of θ,

$$\begin{aligned}\log|s_1(re^{i\theta})| &\leqslant \sum_1^\infty \log|1+r/a_n| = \log g_1(re^{i\pi}) \\ &< (\pi\lambda \operatorname{cosec} \pi\rho+\epsilon) r^\rho L(r).\end{aligned}$$

These results can be extended to $s_p(z)$ and in $g_p(z)$, as in (2.4), and then they give (9.22) and (9.21). (9.23) and (9.24) follow by the same method.

9.3. *Proof of Theorem* V. As usual, we construct the auxiliary function $s_p(z)$, where $p>\rho$, and from it $F(z)$. Since $\beta<\alpha<\frac{1}{2}\pi/\rho$, we can choose p so that

$$\beta < \pi/p = \alpha. \tag{9.31}$$

Then we can use (9.23) for $\theta=\alpha_1$, $\theta=\alpha_2$, remembering that the axes have been turned through an angle α_0. We have

$$\log|F(re^{i\alpha_1})| < \{A\cos\alpha_1\rho+B\sin\alpha_1\rho+\epsilon-\pi\lambda\cos\beta\rho\operatorname{cosec}\alpha\rho\}r^\rho L(r)$$

$$= \{(A+C)\cos\alpha_1\rho+(B+D)\sin\alpha_1\rho+\epsilon\}r^\rho L(r), \tag{9.32}$$

$$\log|F(re^{i\alpha_2})| < \{A\cos\alpha_2\rho+B\sin\alpha_2\rho+\epsilon-\pi\lambda\cos\beta\rho\operatorname{cosec}\alpha\rho\}r^\rho L(r)$$

$$= \{(A+C)\cos\alpha_2\rho+(B+D)\sin\alpha_2\rho+\epsilon\}r^\rho L(r). \tag{9.33}$$

We can now apply Theorem B, and we have

$$\log|F(re^{i\theta})| < \{(A+C)\cos\theta\rho+(B+D)\sin\theta\rho+\epsilon\}r^\rho L(r)$$

for $\alpha_1\leqslant\theta\leqslant\alpha_2$. By (9.32) and (9.33) we have

$$C\cos\theta\rho+D\sin\theta\rho = -\pi\lambda\cos\beta\rho\cos(\alpha_0-\theta)\rho\operatorname{cosec}\alpha\rho\sec\alpha\rho.$$

It follows from (9.22) that, if $0\leqslant|\theta-\alpha_0|\leqslant\alpha-\beta$,

$$\log|f(re^{i\theta})| < \{A\cos\theta\rho+B\sin\theta\rho+\epsilon-\pi\lambda E(\theta)\}r^\rho L(r),$$

where

$$E(\theta) = \frac{1}{\sin\alpha\rho\cos\alpha\rho}\{\cos\beta\rho\cos(\alpha_0-\theta)\rho-\cos(\alpha-\beta-|\theta-\alpha_0|)\rho\cos\alpha\rho\}$$

$$= 2\sin(\alpha-|\theta-\alpha_0|)\rho\sin(\alpha-\beta)\rho\operatorname{cosec}2\alpha\rho.$$

This is the required result.

9.4. *Proof of Theorem* VI. We define $\sigma_p(z)$ and $\Phi(z)$ as usual. We can again take $\pi/p=\alpha$, then, by (9.21),

$$\log|\Phi(re^{i\alpha_1})| < \{A\cos\alpha_1\rho+B\sin\alpha_1\rho+\epsilon+\pi\lambda\operatorname{cosec}\alpha\rho\}r^\rho L(r),$$

$$\log|\Phi(re^{i\alpha_2})| < \{A\cos\alpha_2\rho+B\sin\alpha_2\rho+\epsilon+\pi\lambda\operatorname{cosec}\alpha\rho\}r^\rho L(r).$$

It follows, as before, that

$$\log|\Phi(re^{i\theta})| < \left\{A\cos\theta\rho+B\sin\theta\rho+\epsilon+\frac{\pi\lambda\cos(\theta-\alpha_0)\rho}{\sin\alpha\rho\cos\alpha\rho}\right\}r^\rho L(r).$$

By (9.23), given any positive numbers ϵ_1, δ, we can choose $r_0 = r_0(\epsilon_1, \delta)$ so that

$$(9.41) \quad \log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho + \epsilon + \pi\lambda E(\theta)\} r^\rho L(r),$$

where

$$E(\theta) = 2\{\cos(\theta-\alpha_0)\rho - \cos\alpha\rho\cos(\alpha+\beta-|\theta-\alpha_0|)\rho\}\operatorname{cosec} 2\alpha\rho,$$

for $\beta+\delta \leqslant |\theta-\alpha_0| \leqslant \alpha$, $r \geqslant r_0$. By (9.24), given any positive numbers ϵ_1, ϵ_2, we can choose $R_0 = R_0(\epsilon_1, \epsilon_2)$ so that (9.41) holds where now

$$E(\theta) = 2\{\cos(\theta-\alpha_0)\rho - \cos^2\alpha\rho\}\operatorname{cosec} 2\alpha\rho,$$

for $\alpha_1 \leqslant \theta \leqslant \alpha_2$, $0 < r < R$, $R > R_0$ except in a set of measure less than $\epsilon_2 R$. This completes the proof.

10. We can, of course, combine the theorems for poles and zeros, as, for instance, in

THEOREM VII. *Suppose that $f(z)$ satisfies the hypotheses of Theorem* III, *except for certain poles which form part of a sequence of numbers $\zeta_1, \zeta_2, \ldots, \zeta_m = b_m e^{i\phi_m}$. Suppose that $|\theta-\alpha_4| \leqslant \delta$ for every $\delta > 0$ and every $m > m_0(\delta)$, and that*

$$m(r) \sim \mu r^\rho L(r).$$

Then, (i) *given any positive numbers ϵ_1, δ, we can choose $r_0 = r_0(\epsilon_1, \delta)$ so that*

$$(10.1) \quad \log|f(re^{i\theta})| < \{A\cos\theta\rho + B\sin\theta\rho - \pi\lambda E(\theta, \alpha_3) + \pi\mu E(\theta, \alpha_4) + \epsilon_1\} r^\rho L(r),$$

where

$$E(\theta, \alpha_3) = 2\sin(\alpha_2-\alpha_3)\rho\sin(\theta-\alpha_1)\rho\operatorname{cosec} 2\alpha\rho \quad (\alpha_1 \leqslant \theta \leqslant \alpha_3)$$

$$= 2\sin(\alpha_3-\alpha_1)\rho\sin(\alpha_2-\theta)\rho\operatorname{cosec} 2\alpha\rho \quad (\alpha_3 \leqslant \theta \leqslant \alpha_2)$$

uniformly for $r \geqslant r_0$, $\alpha_1 \leqslant \theta \leqslant \alpha_4-\delta$ and $\alpha_4+\delta \leqslant \theta \leqslant \alpha_2$.

(ii) *Given any positive numbers ϵ_1, ϵ_2, we can choose $R_0 = R_0(\epsilon_1, \epsilon_2)$ so that* (10.1) *is satisfied for $\alpha_1 \leqslant \theta \leqslant \alpha_2$, $0 < r < R$, $R > R_0$, except in a set of measure less than $\epsilon_2 R$.*

Proof. We first construct the function

$$\Phi(z) = f(z)\,\sigma_p(z),$$

and apply Theorem B to it. We then construct the function

$$\Psi(z) = \Phi(z)/s_p(z),$$

and again apply Theorem B. The formal working is a repetition of Theorems III and IV.

11. From Theorem V we can deduce the following general theorem.

THEOREM VIII. *Suppose that $f(z)$ satisfies the hypotheses of Theorem* B, *and that*

$$|f(re^{i\theta})| \neq O\{e^{(A\cos\theta\rho+B\sin\theta\rho-\delta)r^\rho L(r)}\} \tag{11.1}$$

for any $\delta>0$, or any θ such that $\alpha_1<\theta<\alpha_2$. Suppose, further, that the zeros of $f(z)$ include a sequence of numbers $z_1, z_2, \ldots, z_n=a_n e^{i\theta_n}, \ldots$, such that $|\alpha_0-\theta_n|\leqslant\beta<\alpha$, and that

$$n(r)\sim\lambda r^\rho L(r).$$

Then

$$\lambda=0.$$

Proof. Suppose that the conclusion is false. Then we can apply Theorem V to $f(z)$. We have

$$\log|f(re^{i\theta})|<\{A\cos\theta\rho+B\sin\theta\rho+\epsilon-\pi\lambda E(\theta)\}r^\rho L(r),$$

where $\lambda>0$ and $E(\theta)>0$ for $|\theta-\alpha_0|\leqslant\alpha-\beta$, for every $\epsilon>0$ and every $r\geqslant r_0(\epsilon)$. By choosing ϵ sufficiently small we can obtain a contradiction to (11.1). Hence the conclusion of the theorem is true.

12. We shall now consider the function $f(z)$ in an angle equal to the critical angle π/ρ.

THEOREM IX. *Suppose that $f(z)$ is regular, and of the form*

$$O\{e^{(A\cos\theta\rho+B|\sin\theta\rho|)r^\rho L(r)}\},$$

where $B<\pi\lambda\cos\beta\rho$, $0<\beta<\frac{1}{2}\pi/\rho$, for $|\theta|\leqslant\frac{1}{2}\pi/\rho$. Suppose, further, that the zeros of $f(z)$ include a sequence of numbers $z_1, z_2, \ldots, z_n=a_n e^{i\theta_n}, \ldots$, for which $|\theta_n|\leqslant\beta$, and $n(r)\sim\lambda r^\rho L(r)$. Then $f(z)$ vanishes identically.

Proof. Suppose that $f(z)$ does not vanish identically. Construct $s_p(z)$ and $F(z)$ as usual, and make $p=2\rho$. Also let

$$H(\theta)=\overline{\lim_{r\to\infty}}\frac{\log|F(re^{i\theta})|}{r^\rho L(r)}.$$

Then $F(z)$ is regular for $|\theta|\leqslant\frac{1}{2}\pi/\rho$, and it is easy to show that

$$H(\theta)<K$$

for some K and for $|\theta|\leqslant\frac{1}{2}\pi/\rho$.

If $H(\theta) = -\infty$* for any θ such that $|\theta| < \frac{1}{2}\pi/\rho$, then $F(z)$ vanishes identically, and therefore $f(z)$ vanishes identically.

If this is not so, $H(\theta)$ is finite and continuous† for $|\theta| \leqslant \frac{1}{2}\pi/\rho$. But

$$H(e^{-i\frac{1}{2}\pi/\rho}) < B - \pi\lambda \cos\beta\rho + \epsilon$$

for every $\epsilon > 0$, so that

$$H(e^{-i\frac{1}{2}\pi/\rho}) < 0. \tag{12.1}$$

Similarly

$$H(e^{i\frac{1}{2}\pi/\rho}) < 0. \tag{12.2}$$

Now either $H(\theta)$ is negative for $|\theta| \leqslant \frac{1}{2}\pi/\rho$, or else there is an interval $-\frac{1}{2}\pi/\rho < \alpha_1 \leqslant \theta \leqslant \alpha_2 < \frac{1}{2}\pi/\rho$ in which $H(\theta)$ is positive. But Phragmén and Lindelöf have shown that every interval in which $H(\theta)$ is positive is greater than or equal to π/ρ. Therefore $H(\theta)$ must be negative throughout the interval $|\theta| \leqslant \frac{1}{2}\pi/\rho$. On the other hand, a negative interval is not greater than π/ρ, and is limited by values of (θ) for which $H(\theta) = 0$. This contradicts (12.1) and (12.2). Hence $f(z)$ vanishes identically.

---

* See P.S. I, Ab. III, Nr. 326.

† See E. Phragmén and E. Lindelöf, *loc. cit.*, 404–5.

LINE CONGRUENCES IN THREE DIMENSIONS

By L. Roth.

[Received 9 November, 1929.—Read 14 November, 1929.]

Introduction.

1. The representations of a scroll by the points of a curve and of a congruence by the points of a surface are immediate consequences of the work of Klein*; until recently, however, very little use had been made of this fact, save by Schumacher†, whose representation of a congruence derives essentially from Klein's. In 1925, the well known functional method for curves was in this manner applied by James to investigate the enumerative properties of a scroll in three dimensions, and the process was later extended to scrolls in higher space.

The object of the present paper is to apply a similar method to the study of line congruences which possess a finite number of singular points and planes; the theoretical basis, which in this case means the justification of the functional process for surfaces, has been established in a previous paper‡. Apart from any interest which the method may have, the work is necessary because some of Schumacher's formulae are of incomplete generality, a fact which was first pointed out by James§. It appears that a general congruence is determined by *five* independent characters—not four, as one would be led to expect from its representation on a surface; and Schumacher's work is invalidated by this assumption, besides restricting the discussion to congruences without singularities.

2. *Representation of a congruence.* The six coordinates p_{rs} of a line l in three dimensions satisfy a quadratic relation

$$\Omega \equiv p_{12}p_{34}+p_{23}p_{14}+p_{31}p_{24} = 0,$$

* See also Segre, *Memorie di Torino* (2), 36 (1886).

† Schumacher, *Math. Ann.*, 37 (1900), 100; some use of the representation is also made by Fano, *Memorie di Torino* (2), 51 (1901), 1, in his discussion of cubic congruences.

‡ *Proc. Camb. Phil. Soc.*, 25 (1929), 390.

§ James, *Messenger of Math.*, 55 (1925), 9. But see § 9.

and are such that the condition that two lines l, l' should intersect is

$$p_{12}p'_{34}+p'_{12}p_{34}\ldots+p'_{31}p_{24}=0.$$

It follows that the aggregate of lines l may be represented by the points L of a quadric fourfold Ω in [5], and that the points L, L' corresponding to two intersecting lines are conjugate with regard to Ω. Hence the line LL' is a generator of Ω; the ∞^1 lines of the pencil (l, l') correspond to the points of this generator.

2.1. Corresponding to three coplanar lines are three points, whose joins in pairs lie on Ω, and whose plane therefore lies on Ω. Given a pencil formed by two intersecting lines l, l', there are ∞^2 lines through its vertex and ∞^2 lines in its plane; these correspond to the points of two planes of Ω, of the "first" and "second" system respectively, which pass through LL'. Hence through any generator of Ω there passes one plane of each system. Since any two points define a line, it follows that any two planes of the first (and, similarly, of the second) system meet in a point; but since an arbitrary point does not lie in a given plane, two planes of opposite systems do not in general intersect. If they do, they meet in a line corresponding to a pencil in the original space*. There are ∞^5 generators of Ω, corresponding to the pencils of [3].

2.2. Corresponding to a prime section of Ω is a linear complex, since this is given by a linear relation between the coordinates. If the prime touches Ω, its section is a quadric cone in [4], the joins of whose points to the vertex are generators of Ω. Hence all the lines of the complex meet a fixed line corresponding to the point of contact of the prime.

A [3]-section of Ω corresponds to a linear congruence, the intersection of two linear complexes. If the given [3] touches Ω at a point, the rays of the congruence all meet a fixed directrix; if it touches Ω along a *line*, its section with Ω consists of two planes of opposite systems, and the congruence is then degenerate.

A plane section of Ω represents a regulus; but if the plane touches Ω, its section is a pair of intersecting generators. Hence the regulus consists of two pencils with a common ray.

2.3. The aggregate of ∞^2 rays forming a congruence Γ correspond to the points of a surface F on Ω. If F meets planes of the first and second systems in m and n points respectively, it may be shown that m and n are constant, so that F is of order $m+n$. Thus m rays of Γ pass through

* See Baker, *Principles of geometry*, 4 (1925), 40.

an arbitrary point, and n rays lie in an arbitrary plane; m and n are the order and class of Γ. It may be shown that two surfaces F, F' of characters (m, n) and (m', n') meet in $mm'+nn'$ points; whence we derive Halphen's theorem, that the congruences Γ, Γ' have $mm'+nn'$ common rays.

2.4. *Schumacher's representation.* If we project Ω from a point P of itself on to an arbitrary plane ω, the lines of [3] correspond to the points of ω, save that one line p corresponds to the projection of all the generators through P, forming a quadric surface Φ in ω. Generators which do not pass through P project into secants of Φ; the two systems of planes through P project into the generators of Φ, and all other planes of Ω into planes through these generators. Hence a pencil of [3] corresponds to a secant of Φ, unless it contains p, when it corresponds to a point of Φ; a linear complex, congruence, and regulus containing p correspond to a space, a plane, and a line in ω; if they do not contain p, they correspond to quadric loci of appropriate dimension. The projection of F, which is assumed not to contain P, is a surface f_{m+n} to which the two systems of generators of Φ are m- and n-secant respectively. We thus arrive at Schumacher's correlation* between Γ and f_{m+n}.

3. *Characters of a congruence.* Returning to the representation on Ω, we note that F may be such that one or more planes of Ω meet it in *curves*; these will correspond to singular points and planes of Γ. Through *a singular point of order h* there passes a cone of rays of the same order; and similarly with a singular plane. The corresponding plane of Ω will in general meet F in points not lying on the singular curve. Thus an h-ple point or plane of Γ will in general contain a finite number of residual rays.

Besides m and n, F has three additive characters which are defined as follows†:

a, the rank of a prime section;

j, the number of apparent pinch-points; and

c, the class.

That these five characters are independent may be shown as follows. Suppose that they are connected by some relation $f(m, n, a, j, c) = 0$.

* Schumacher, *loc. cit.*; the necessary corrections for the account given by Jessop, *Treatise on the line complex* (1903), are to be found in the above-mentioned paper by James.

† For details of this and the functional method (§ 4), see the writer's paper mentioned in § 1.

For a second surface F' we shall have $f(m', n', a', j', c') = 0$. Now the surface of which F and F' represent a degeneration has characters $m+m'$, ..., etc., so that we have the relation

$$f(m+m', n+n', a+a', j+j', c+c') = 0.$$

Hence $$f(m+m', \ldots)-f(m, \ldots)-f(m', \ldots) = 0.$$

The solution of this functional equation is therefore

$$f \equiv Aa+Jj+Cc+Mm+Nn,$$

where the capitals denote constants. Since the identity $f = 0$ must hold for all congruences, it must be true in particular for the complete intersection of two complexes. Substituting the characters of such a congruence (§ 4) we obtain at once $A = J = C = 0$, $M+N = 0$; and, from the last result, it follows that $M = N = 0$. Hence the characters are independent.

It is clear that the character a is the rank of the scroll common to Γ and an arbitrary linear complex; in order to interpret j and c in terms of Γ we must consider the focal surface.

3.1. *The focal surface.* The tangent plane to F at a point L is also tangent to Ω, and therefore meets it in a pair of generators. Hence every ray l is met by two consecutive rays, the points of intersection being the "foci" of l, and the two planes corresponding to the generators being the focal planes; the locus of focal points is the surface ϕ, and this is identical with the envelope of focal planes, provided that Γ has at most a finite number of singularities*.

Thus there are ∞^2 generators of Ω which touch F. Now it is known that the tangents to F meet an arbitrary plane in a curve of order j (this being an immediate deduction from the definition of j); hence the tangent generators meet any plane π of Ω (of the second system, say) in a curve of order j. To the points of this curve correspond ∞^1 focal planes whose foci lie on a given plane π' corresponding to π. *Hence j is the class of the developable of these planes.*

3.2. We shall now show that c represents the *rank of* ϕ. In the first place, it is easily proved† that if F_1, F_2 are the foci of a ray l, then the focal plane at F_1 touches ϕ at F_2, and *vice versa*. Now consider a [3] containing two planes ϖ, ϖ' of Ω, meeting in a line g. Any prime S through these must touch Ω at a point P of g, and therefore corresponds

* See § 10.

† *E.g.* Jessop, *loc. cit.*, 261.

to a special linear complex with a directrix p in the plane of the pencil G corresponding to g. There is a pencil of primes S corresponding to a pencil of lines p, and c is the number of such primes which touch F. If S touches F at Q, the tangent plane to F at Q meets ϖ, ϖ', and hence the two generators in this plane meet ϖ, ϖ' at R, R', say. Thus in [3] the corresponding ray q and its consecutive rays r, r' must all meet p; and, further, the focus of the plane (q, r) must lie on p, or else the pencil (q, r) would lie in G, which is not the case. It follows that the plane (q, r') contains p, and that p passes through the focus of (q, r). Hence p touches ϕ at this focus and is tangent to the section of ϕ by G; and, since the lines p form a pencil, the result is established.

4. *The functional method.* We thus have the result that a general congruence is determined by five independent characters, m, n, a, j, c, of which the last three are self-dual. These characters are additive, so that to obtain the various properties of Γ we merely associate two arbitrary congruences Γ, Γ' (or their corresponding surfaces) and form the functional equations, in the manner described elsewhere. The solution of an equation of the first type will contain five linear terms whose coefficients must be determined from five known special cases; equations of the second type do not occur in the present work.

A considerable number of formulae have been obtained by Voss* for congruences which are the complete intersection of two complexes of orders k and k'. The characters† of such a congruence (kk', kk') are $m = n = kk'$, $a = 2kk'(k+k'-1)$, $j = 2kk'(kk'-1)$, $c = 2kk'\{(k+k'-1)^2-kk'+1\}$, and a solution of this type provides four relations between the unknown coefficients (three only if $k' = 1$). Further solutions must be obtained by means of the principle of redundancy.

4.1. The congruences which we shall consider have a finite number of singular points (and planes). These will not affect a formula for the order of a certain curve, but they will occur redundantly in formulae for the order of a scroll or for a finite number of points. It is clear that an h-ple point of the type mentioned in § 3 is additive for two arbitrarily associated congruences, and it may therefore be shown (as in previous work) that an h-ple point counts λh times‡ in a given formula, *where* λ *is independent of the congruence and depends only on the problem.* This

* Voss, *Math. Ann.*, 9 (1876), 55.

† These are found from Severi's formulae for the intersection of three forms in [5].

‡ That is, for the order of a scroll, the number of its rays which meet a given line includes h generators of each h-cone, counted λ times.

principle holds for formulae concerned with rays of the congruence; but in the case of formulae for the *focal surface* ϕ considered apart from the congruence, it is necessary to examine the nature of the h-ple points in relation to ϕ. In general, an h-ple point is a conic node of ϕ, or is equivalent to a number of such nodes (we shall confine the discussion to this case); if, then, ϕ possesses C conic nodes, each of these counts λ times in the formulae, where λ is a constant for the problem in question*.

4.2. The congruences to be used as special cases are given below; the numbers a_h of h-ple points, if not quoted, may be found in the original papers. The quadratic congruences are described at length by Jessop and Sturm†; these correspond to del Pezzo surfaces‡ (or their projections) on Ω. So do the congruences $(3, n)_1$, which were enumerated by Segre in his discussion of cubic forms§; these and the congruences $(3, n)_2$, for which the prime section of F is of genus 2, are to be found in Fano's work already referred to. The remaining examples are the complete intersections of complexes—(3, 3) of a linear and a cubic, (4, 4) of two

(m, n)	a	j	c	a_h	C
(2, n)	$2(n+2)$	$4(n-1)$	12	(Jessop)	$18-n$
$(3, n)_1$	$2(n+3)$	$4n$	12	(Fano, Segre)	$12-n$
$(3, n)_2$	$2(n+4)$	$4n$	20	$a_2 = n-3$ $a_1 = 18-2n$	$15-n$
(3, 3)	18	12	42	—	—
(4, 4)	24	24	48	—	—
(4, 4)′	32	24	104	—	—
(1, 1)	2	0	2	—	—
(1, 2)	4	2	3	—	—
(1, 3)	6	6	3	—	—

quadratic, (4, 4)′ of a linear and a quartic, and (1, 1) of two linear complexes, and two linear congruences which are required for a few problems; (1, 3) corresponds to a Veronese surface on Ω.

* In what follows, the dual results may be obtained by precisely similar methods.

† Jessop, *loc. cit.*; Sturm, *Liniengeometrie*, 2 (1893).

‡ del Pezzo, *Rend. Palermo.* 1 (1887), 241.

§ Segre, *Memorie di Torino* (2), 39 (1889), 1.

Elementary formulae.

5. Our first application of the method is to obtain the number of double rays of Γ; these correspond to d improper nodes of F. Replacing Γ by congruences Γ, Γ', we have

$$d = \int mm' + nn' \text{ by } \S\,2.3,$$

whence, from special cases,

$$d = \binom{m}{2} + \binom{n}{2} - \tfrac{1}{2}j. \tag{1}$$

The existence of these nodes explains why *five* characters are necessary for the specification of F and therefore of Γ; for a surface immersed in Ω possesses in general a finite number of improper nodes.

5.1. There is a finite number r of chords of F passing through a given point L of Ω; these must be generators of Ω, and therefore represent the number of pairs of rays which lie in a pencil with the given line l; this number is the so-called "rank" of Γ. We obviously have

$$r = \int (m+n)(m'+n') - (mm'+nn') = \int mn' + m'n,$$

and, with special cases,

$$r = mn - \tfrac{1}{2}a. \tag{2}$$

5.2. The third elementary character corresponds to the number t of apparent triple points of F, *i.e.* the number of trisecant planes passing through an arbitrary line g. Since the three points of intersection will lie on a plane section of Ω which contains the two intersections of g with Ω, we see that t is *the number of triads of rays which form a regulus with two given lines.* This number can be written down from a known formula for F, giving

$$t = \binom{m+n}{3} - \tfrac{1}{2}a(m+n) + \tfrac{1}{3}(2a+2j+c). \tag{3}$$

5.3. *The scroll* (l). The rays which meet a given line l generate a scroll (l), which is obviously of order $m+n$, and which contains l as an m-ple line*. Hence its double curve is of order

$$\delta = \int (m+n)(m'+n') - mm' = mn + \binom{n}{2} - \tfrac{1}{2}a = r + \binom{n}{2}, \tag{4}$$

* The rays of the scroll obviously correspond to the intersection of F with the cone of secant generators passing through L.

in which form the result is obvious; the curve meets l in r points. Hence the number of triple points* on the double curve is

$$\tau_3 = \int (m+n)\,\delta' + (m'+n')\,\delta,$$

i.e.

$$\tau_3 = \binom{n}{3} + m\binom{n}{2} - \tfrac{1}{2}an + a - m, \qquad (5)$$

vanishing for (1, 1), (1, 2), (2, 1), (1, 3), (3, 1).

The scroll common to Γ and a complex of order k can be dealt with by the Severi formulae: it is of order $k(m+n)$ and rank $(m+n)k(k-1)+ak$, whence the rest of its properties may be deduced from the ordinary formulae.

The focal surface.

6. We return now to the consideration of the focal surface ϕ; its order μ and class ν are evidently the numbers of generators tangent to F which meet two planes of the second and first systems respectively. For a composite congruence Γ, Γ', the focal surface consists of the surfaces ϕ and ϕ'; hence μ must be linear in the characters of Γ and, since it must also be independent of j and c, which are two characters of ϕ, it is of the form $Aa+Mm+Nn$. This must give the values† 4 for (2, n) and 6 for (3, n)$_1$, whence we have

$$\mu = a - 2n. \qquad (6)$$

In the same manner we obtain

$$\nu = a - 2m. \qquad (7)$$

6.1. There is a scroll $S(\pi)$ of rays, one of whose foci lies in a given plane π; the number of its generators meeting a line l may be found by taking l to be in π, whence the scroll is of order $\mu+2n = a$; it has for double generators the n rays in π.

It follows that the foci of generators of (l) lie on a curve of order a, and that the scroll touches ϕ along this curve.

6.2. There is a scroll $S_{2:}$ of rays with coincident‡ foci—these are flecnodal tangents to ϕ; its order $\rho_{2:}$ may be found by applying the general correspondence theorem to the involution defined by the points of contact

* The modification for the case of singularities is given in § 12.

† Jessop and Segre, *loc. cit.*

‡ These correspond to tangent planes of F which touch Ω along a line.

with ϕ of the generators of (l), which lie on a curve of order a. The only coincidences arise from the $\rho_{2:}$ rays which meet l, so that we have

$$\rho_{2:} = 2a - 2(m+n) \tag{8}$$

The order $c_{2:}$ of the curve of contact of $S_{2:}$ and ϕ must be linear in the characters; from special cases and Voss's formula* we obtain

$$c_{2:} = -a+j+c-2m+2n. \tag{9}$$

The rank of $S_{2:}$ is similarly found to be

$$r_{2:} = -18a+8j+8c+4m+16n. \tag{10}$$

6.3. We may now find the order of the curve of residual foci for the generators of $S(\pi)$. For if we associate pairs of foci of the same rays, the only coincidences are the $c_{2:}$ points in π, whence the order x of the required curve is given by

$$x+\mu-a = c_{2:}. \tag{11}$$

This gives the number x of *rays whose foci lie on two given planes.*

7. *Singularities of* ϕ. There are two singular curves on ϕ : the bifocal curve f_2, in points of which concur two pairs of consecutive rays—this is a double curve; and the trifocal curve f_3, in points of which concur three consecutive rays, giving a cuspidal curve on ϕ.

The order μ_2 of f_2 is obviously given by $\int \mu\mu'$, and it must vanish for all congruences of order less than four which do not contain singular curves. We thus obtain

$$\mu_2 = \binom{a}{2} - 2an + 4\binom{n}{2} + \frac{3a}{2} - 3j - 2c + 12n, \tag{12}$$

which satisfies all the required conditions.

The order μ_3 of f_3 is linear in the characters; but, since it does not vanish for cubic congruences, there is a lack of special cases, so that it is best determined by the relation

$$c = \mu(\mu-1) - 2\mu_2 - 2\mu_3,$$

which gives

$$\mu_3 = -a+2j+c-6n. \tag{13}$$

This vanishes for $(2, n)$, and gives a sextic† for $(3, n)_1$.

---

* Voss, *loc. cit.*

† Segre, *loc. cit.*

7.1. The rank of f_2 is given by

$$r_2 = \int \mu\mu'(\mu+\mu'-2) - 2\mu_2\mu' - 3\mu_3\mu' \dots,$$

which reduces, as it should, to $\int c\mu' + c'\mu$. This must agree with the Voss formula, and give the value λC for $(2, n)$, $(3, n)_1$ and $(3, n)_2$, where C is the number of nodes of ϕ. We thus derive the formula

$$r_2 = c\mu + 41a - 17j - 22c - 26m - 12n + 2C, \tag{14}$$

which fulfils the required conditions.

The rank r_3 of f_3 must give λC for $(2, n)$ and $18+\lambda C$ for $(3, n)_1$, which possesses a cuspidal sextic of genus 4. Hence

$$r_3 = -26a + 10j + 12c + 18m + 10n - 2C. \tag{15}$$

7.2. The number j_0 of pinch-points on f_2 is linear in the characters, and is found to be

$$j_0 = 2(-a + j + 4m - 2n). \tag{16}$$

7.3. The *inflexional tangents* to ϕ form a congruence of class

$$\iota = -4a + 2j + 4c. \tag{17}$$

Since the result is independent of m and n, it also represents the order of the congruence.

7.4. The *class of immersion* ρ of f_2 must be given by $\int c\mu' + c'\mu$, whence from special cases we find that

$$\rho = c\mu + 14a - 6j - 16c - 4m + 4C. \tag{18}$$

The class of immersion of f_3 is found to be

$$\sigma = 2m - 5a + 2j + 5c + 2C. \tag{19}$$

7.5. Finally, we consider the singular points on f_2 and f_3. There is a number μ_4 of stationary points on f_3 through which pass four consecutive rays. Since μ_4 must be linear in the characters and must vanish for $(2, n)$, $(3, n)_1$, $(3, n)_2$, and $(3, 3)$, we have

$$\mu_4 = -54a + 22j + 12c + 44m + 24n + 4C. \tag{20}$$

There is a number μ_{31} of stationary points on f_2, given by $\int \mu_3\mu' + \mu_3'\mu$, for which one pair is replaced by three consecutive rays. This must vanish for the congruences stated above, giving

$$\mu_{31} = \mu\mu_3 + 228a - 96j - 60c - 180m - 84n - 12C. \tag{21}$$

The number of triple points on f_2, through which pass three pairs of consecutive rays, is

$$\begin{aligned}\mu_{21} &= \int \mu_2\mu' + \mu_2'\mu \\ &= \binom{a}{3} - 2n\binom{a}{2} + 4a\binom{n}{2} - 8\binom{n}{3} + \tfrac{3}{2}a^2 + 9an - 24n^2 \\ &\quad -(3j+2c)(a-2n) - 198a + 85j + 59c + \tfrac{1}{3}(a+j-c) \\ &\quad +152m+64n+8C. \end{aligned} \tag{22}$$

The verifications are as before.

8. *The bifocal and trifocal scrolls.* We consider now the scrolls generated by rays which define f_2 and f_3. The order ρ_3 of the latter must be linear in the characters and agree with Voss's formula*; further, when we substitute the characters of $(2, n)$, the result will involve redundantly the singularities of the congruence and must vanish identically in virtue of some relation between them. Now there is one such relation linear in n, due to Masoni†, which gives $\Sigma a_h h = 4(n+2)$. Hence we obtain

$$\rho_3 = 2(3a-4m-4n) - \Sigma a_h h. \tag{23}$$

8.1. The order ρ_2 of f_2 is (§ 6.1) given by $\int a\mu' + a'\mu$, but we may find it by considering the correspondence between the contacts and the residual intersection of ϕ with the generators of (l). Since l touches ϕ along a curve of order a, we have

$$2\{(m+n)\mu - 2a\} + (\mu-4)a - 2(\mu-4)(m+n) = \rho_2 + 2\rho_3,$$

where the coefficient of ρ_3 is determined by one special case. Thus

$$\rho_2 = a\mu - 20a + 24(m+n) + 2\Sigma a_h h. \tag{24}$$

Since this must vanish for $(3, n)_1$ and $(3, n)_2$, we derive *the following relations*‡ *between the singularities* in the two cases:

$$\left.\begin{aligned} \Sigma a_h h &= 2(n+3), \\ \Sigma a_h h &= 12. \end{aligned}\right\} \tag{25}$$

8.2. The orders of the scrolls of residual rays through points of f_2 and f_3 are found immediately from the values of ρ_2 and ρ_3. Thus, for the

---

* Voss, *loc. cit.*

† Masoni, *Rend. Napoli*, 22 (1883), 145.

‡ These were previously discovered by Fano, *op. cit.*

former,

$$\sigma_2 = (m+n)\mu_2 - 2\rho_2 = (m+n)\mu_2 - 2a\mu + 40a - 48(m+n) - 4\Sigma a_h h \quad (26)$$

and for the latter,

$$\sigma_3 = (m+n)\mu_3 - 3\rho_3 = (m+n)\mu_3 - 18a + 24(m+n) + 3\Sigma a_h h. \quad (27)$$

As a verification, we find that σ_2 vanishes for (3, 3), (4, 4), and (4, 4)′.

9. *Historical note.* In view of the extent to which the work of previous writers overlaps, it is desirable to indicate the contributions made to the subject by Voss, Schumacher, and James, and the errors into which the two latter have fallen. The formulae given by Voss are for the *complete intersection* of two complexes (k), (k'); the results for $\rho_{2:}$, $c_{2:}$, $r_{2:}$, ρ_3; μ, ν, c, ι, r_3, ρ, and σ are applicable for all values of k and k'; the remaining formulae for the singularities of ϕ apply only when k' is unity. Schumacher considers congruences without singularities and, unfortunately, without double rays either; for he establishes the formula $j = m(m-1)+n(n-1)$, equivalent to the assumption* that $d = 0$. His formula for c is therefore erroneous, and that for ρ_2 is wrong for other reasons; in particular, the result is unsymmetrical in m and n. Nevertheless, he obtains values for μ, ν, μ_2, μ_3, ρ_3, and μ_4, which happen to be true generally, provided that the congruence has no singularities. James first pointed out the assumption in Schumacher's work, and obtained the correct values of d, μ_2, and μ_3 by other methods. But he mistakenly supposes that f_3 is a triple, instead of a cuspidal, curve on ϕ. He also attempts to find ρ_2 and ρ_3 by correspondence methods, in the case where the congruence possesses singular points, but the formulae obtained are of the wrong dimensions in the characters. The formula for j, which is given by Schumacher, is also to be found in Jessop's treatise, in which t_0 is mistaken for t of the present paper (§ 12 . 2); the confusion is due to the fact that Schumacher writes t for t_0, despite its generally accepted significance in the theory of reciprocal surfaces.

10. *The focal variety.* We have seen that the points and tangent planes of ϕ correspond to generators of Ω which touch F; these generate a variety V_3 of order $2j$, which in general meets every plane of Ω in a curve of order j. If, however, Γ contains a singular curve or developable, this corresponds to ∞^1 planes of the same system which meet F in curves of the same order, and the ∞^2 tangents to these curves form part of V_3. It

* Zindler, *Encykl. Math. Wiss.*, III C 8, 1176, quoting from Schumacher, gives only the four characters m, n, r, t_0. There is no mention of the consequent lack of generality, but at the same time there are no incriminating formulae.

follows that the locus of focal points is identical with the envelope of focal planes, except when Γ contains a singular curve or developable.

The variety V contains F as a double surface, since every point of F is a point of concurrence of two "focal generators"; and the curve representing the scroll $S_{2,1}$ (§ 6.2) is a curve of pinch-points of V which lies on F. But, in addition to F, V has a double surface f, the locus of all other intersections of focal generators; a point P of f corresponds to a line p containing two foci for which the focal planes pass through p, so that it is bitangent to ϕ. Thus f represents the residual congruence γ of bitangents, which is confocal with Γ. From the known value of ι (§ 7.3) and Plücker's formulae we find the congruence of bitangents to be of order

$$\alpha = \binom{c}{2} + \frac{11}{2}a - 3j - 6c + m - C,$$

and class

$$\beta = \binom{c}{2} + \frac{11}{2}a - 3j - 6c + n - C',$$

where C' is the dual of C. Removing Γ, we obtain the order and class of γ, which are therefore equal, provided that ϕ is without accidental singularities. We thus have the order of f. Further, f contains a pinch-curve corresponding to the flecnodal tangents to ϕ which are residual to $S_{2,1}$; it has a triple curve corresponding to the tritangents to ϕ, and a number of triple points for this curve which are sextuple for f, representing the quadritangents to ϕ. All these characters may be obtained quite easily by the method of a previous paper*, in which, however, the surface ϕ had no cuspidal curve; the introduction of extra characters makes no essential difference to the work.

11. *Ruled congruences*†. If F is a *scroll*, then Γ contains ∞^1 pencils of rays, corresponding to the generators of F; the vertices P of the pencils describe a singular curve C and their planes p a singular developable D, and it is evident that C is of order n and D of class m. A tangent plane to F at L contains a generator of F and a residual generator of Ω; hence one focal plane of l is the plane p of its pencil. Since the tangent plane to F contains a consecutive point not on the generator, the second focal plane passes through the tangent to C at the vertex P. The two foci of l are P and the intersection of the second focal plane with the tangent in p to the cuspidal edge of D. Since C and D are in $(1, 1)$ correspondence, they have the same genus, so that their ranks r, r' are connected by the

* "Enumerative formulae for surfaces", *Proc. Camb. Phil. Soc.*, 26 (1930), 43.

† For a different treatment, see Baldus, *Math. Annalen*, 71 (1912), 275.

relation
$$r-2n = r'-2m. \tag{28}$$

The locus of focal points consists of C and the generators of D, which describe a surface of order r'. We thus have

$$\mu = a-2n = r'.$$

In the same way, the envelope of focal planes gives

$$\nu = a-2m = r.$$

These two values of a are consistent in virtue of (28).

Now the scroll F is determined by *three* characters m, n, a; the remaining characters are given by

$$j = 2(a-m-n), \quad \text{and} \quad c = m+n.$$

The latter relation is equivalent to the fact that the tangents to the focal surface from any point consist of the cones of lines standing on C and D. The former gives $j = \rho_{2:} = r+r'$. Thus the scroll $S_{2:}$ is generated by the tangents to C and to the cuspidal edge of D. The definition of j (§ 3.1) gives another interpretation of this result. For the curve of foci lying in a given plane π is of order r', and the developable of focal planes passing through a given point P is of class r. Hence the number of such planes through P, which have their foci in π, is $r+r'$.

The triadic surface.

12. In the same way that generators tangent to F give rise to the focal surface, the trisecants of F give rise to a certain locus and an envelope. To a trisecant of F corresponds a plane pencil containing three rays of Γ. The vertex of the pencil describes the "triadic surface" and the plane touches the triadic envelope. It is clear that the class of the latter is τ_3 (§ 5.3), provided that we take into account the effect of singular points. A plane through an h-ple point ($h > 2$) and an arbitrary line will count $\binom{h}{3}$-ply as a tangent plane to the envelope, so that we have

$$\tau_3 = \binom{n}{3} + m\binom{n}{2} - \tfrac{1}{2}an + a - m - \Sigma\alpha_h\binom{h}{3}. \tag{29}$$

Similarly, the order of the triadic surface is found to be

$$t_3 = \binom{m}{3} + n\binom{m}{2} - \tfrac{1}{2}am + a - n - \Sigma\beta_h\binom{h}{3}, \tag{29'}$$

β_h being the number of h-ple planes of Γ.

12.1. The number τ of pairs of rays lying in a pencil with a given ray is represented by the number of trisecants through the corresponding point of F. By projection from this point, $\tau+d$ will be the number of improper nodes for the projected surface, whence

$$\tau+d=\binom{m+n-1}{2}-\tfrac{1}{2}(a-2)-\tfrac{1}{2}(j-4),$$

so that (§ 5)

$$\tau=r-(m+n)+4. \tag{30}$$

By projecting from an improper node of F, we find the number of pairs of rays lying in a pencil with a double ray to be

$$d_3=r-2(m+n)+9. \tag{31}$$

In this formula it is essential that the double ray should not pass through an h-ple point ($h>1$), else the number is indeterminate.

12.2. Given a plane π of Ω (of the second system, say) there are ∞^1 trisecants of F meeting it in a curve of order t_0. By projecting from any point of π we see that

$$t_0=t-\binom{m}{3}-\binom{n}{3}, \tag{32}$$

since a trisecant plane of F which meets π in a line must meet F in three collinear points. Corresponding to the curve of order t_0 is a developable formed by the triadic planes whose vertices lie in a given plane. Since t_0 is self-dual it follows that the developable of triadic planes, whose vertices lie in a given plane, is of class t_0, and that the curve of vertices for which the corresponding planes pass through a given point is also of order t_0. It is this number, instead of t, which is made fundamental by Schumacher in his discussion of congruences.

The singularities of the triadic surface are not easily obtained by the functional process. Schumacher derives some of them by lengthy correspondence methods which, since they are independent of his erroneous assumption, are probably valid for general congruences. For these results reference may be made to his work.

TAYLOR'S THEOREM IN THE TENSOR CALCULUS

By H. S. RUSE.

[Received 2 December, 1929.—Read 12 December, 1929.]

In a previous paper† an invariant formula was found which expressed the value of a scalar at any point of a Euclidean space in terms of its value at another point; the formula being, in fact, a tensor form of Taylor's series. We propose now to extend the theorem so that it shall be applicable to tensors in the most general Riemannian space. In so doing we have occasion to employ the normal coordinates of Lipschitz‡, for which, incidentally, an explicit definition is obtained in place of the implicit definition usually given.

1. *Notation.*

The summation convention is observed throughout. Let

$$(x^i) \equiv (x^1, x^2, \ldots, x^n)$$

be generalized coordinates in a Riemannian space of n dimensions characterized by the metric

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu. \tag{1}$$

Let (x^i) and $(\bar{x}^i)$ be two non-singular points of the space. To avoid an unduly complicated notation, we sometimes employ $x^1, x^2, \ldots, x^n$ as variables (current coordinates), and sometimes as the coordinates of a fixed point. It will be seen that no ambiguity results.

The components of any tensor $T^{p\ldots}_{q\ldots}$ will, of course, be functions of the variables (x^i); their value at $(\bar{x}^i)$ will be denoted by $\bar{T}^{p\ldots}_{q\ldots}$.

---

† *Proc. London Math. Soc.*, 31 (1930), 225.

‡ An account of these is given by Veblen, "Invariants of quadratic differential forms", *Camb. Math. Tracts*, No. 24 (1927), Ch. 6.

2. *Normal coordinates.*

It has been shown† that the equations of the geodesics which pass through the point $(\bar{x}^i)$ may be written in the form

$$x^i = \bar{x}^i + a^i s - \frac{1}{2!}\gamma^i_{\lambda\mu} a^\lambda a^\mu s^2 - \frac{1}{3!}\gamma^i_{\lambda\mu\nu} a^\lambda a^\mu a^\nu s^3 - \ldots$$
$$(i = 1, 2, \ldots, n), \quad (2)$$

where s denotes the length measured from $(\bar{x}^i)$ to (x^i) of the arc of the geodesic; where $\gamma^i_{\lambda\mu}$, $\gamma^i_{\lambda\mu\nu}$, ... are the values at $(\bar{x}^i)$ of certain functions of the Christoffel symbols and their derivatives; and where the parameters a^i specify the geodesic under consideration. In fact,

$$a^i = \frac{d\bar{x}^i}{ds}, \quad (3)$$

the right-hand side being the value at $(\bar{x}^i)$ of dx^i/ds for the geodesic in question.

If now we introduce the normal coordinates (y^i) defined by the equation

$$x^i = \bar{x}^i + y^i - \frac{1}{2!}\gamma^i_{\lambda\mu} y^\lambda y^\mu - \frac{1}{3!}\gamma^i_{\lambda\mu\nu} y^\lambda y^\mu y^\nu - \ldots$$
$$(i = 1, 2, \ldots, n), \quad (4)$$

then the equations of the geodesics passing through the point $(\bar{x}^i)$, which is the origin of the new coordinates (y^i), are simply

$$y^i = a^i s. \quad (5)$$

From (4) it at once follows that

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 = \delta^i_j, \quad (6)$$

the suffix 0 implying (as always hereafter) evaluation at the point $(y^i) = 0$ [*i.e.* at the point $(\bar{x}^i)$], and δ^i_j being equal to zero or unity according as $i \neq j$ or $i = j$.

3. *The function* Ω.

Let Ω denote one half‡ of the square of the geodesic distance between the points $(\bar{x}^i)$ and (x^i). Thus Ω is a scalar defined by the equation

$$(2\Omega)^{\frac{1}{2}} = \int ds, \quad (7)$$

† Veblen, *loc. cit.*, § 2.

‡ The function here denoted by Ω is one half of that denoted by Ω in the earlier paper referred to above.

the integral being taken along the arc of the geodesic from $(\bar{x}^i)$ to (x^i). That is,

$$\Omega = \tfrac{1}{2}s^2, \tag{7 a}$$

where s is as defined above.

It should be noticed that Ω is a function of both the x^i and the $\bar{x}^i$. Thus, for the three-dimensional Euclidean case, with rectangular Cartesian coordinates x^1, x^2, x^3, we have

$$ds^2 = (dx^1)^2+(dx^2)^2+(dx^3)^2, \tag{8}$$

and

$$\Omega = \tfrac{1}{2}\{(x^1-\bar{x}^1)^2+(x^2-\bar{x}^2)^2+(x^3-\bar{x}^3)^2\}. \tag{9}$$

To find the derivatives of Ω with respect to the x^i and the $\bar{x}^i$, consider the variation of the integral in (7) obtained by integrating along a neighbouring geodesic with end-points $(x^i+\delta x^i)$, $(\bar{x}^i+\delta\bar{x}^i)$. We get†, after integrating by parts,

$$\delta(2\Omega)^{\frac{1}{2}} = \left[g_{\mu\sigma}\frac{dx^\mu}{ds}\,\delta x^\sigma\right]-\int g_{\mu\sigma}\left(\frac{d^2x^\mu}{ds^2}+\left\{\begin{matrix}\alpha\beta,\\ \mu\end{matrix}\right\}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right)\delta x^\sigma\,ds$$

$$= g_{\mu\sigma}\frac{dx^\mu}{ds}\,\delta x^\sigma-\bar{g}_{\mu\sigma}\frac{d\bar{x}^\mu}{ds}\,\delta\bar{x}^\sigma,$$

the integral vanishing since the integrand is zero along a geodesic.

Hence, using equation (3), we get

$$\frac{\partial}{\partial x^\sigma}(2\Omega)^{\frac{1}{2}} = g_{\mu\sigma}\frac{dx^\mu}{ds}, \tag{10}$$

and

$$\frac{\partial}{\partial\bar{x}^\sigma}(2\Omega)^{\frac{1}{2}} = -\bar{g}_{\mu\sigma}a^\mu. \tag{11}$$

It follows at once from equations (5), (7 a), and (11), that

$$\frac{\partial\Omega}{\partial\bar{x}^\sigma} = -\bar{g}_{\mu\sigma}y^\mu. \tag{12}$$

As is customary, we shall write Ω_σ, Ω^σ respectively for $\dfrac{\partial\Omega}{\partial x^\sigma}$ and $g^{\sigma\alpha}\dfrac{\partial\Omega}{\partial x^\alpha}$. Further, we shall denote $\dfrac{\partial\Omega}{\partial\bar{x}^\sigma}$ by $\Omega_{(\sigma)}$, and $\bar{g}^{\sigma\alpha}\dfrac{\partial\Omega}{\partial\bar{x}^\alpha}$ by $\Omega^{(\sigma)}$. And later we shall use $\Omega^{(\lambda)}_\mu$ to represent $\bar{g}^{\lambda\alpha}\dfrac{\partial^2\Omega}{\partial x^\mu\partial\bar{x}^\alpha}$, and $\Omega^\lambda_{(\mu)}$ to represent $g^{\lambda\alpha}\dfrac{\partial^2\Omega}{\partial x^\alpha\partial\bar{x}^\mu}$.

† *Cf.* Eddington, *Mathematical theory of relativity* (1924), § 28.

In other words, we shall use simple suffixes to denote covariant and contravariant differentiations with respect to the x^i, and bracketed suffixes to denote what may analogously be called "covariant and contravariant differentiations" with respect to the $\bar{x}^i$.

Multiplying both sides of equation (12) by $-\bar{g}^{i\sigma}$ (and summing for σ), we get

$$y^i = -\Omega^{(i)}. \tag{13}$$

We thus have the result: *The normal coordinates* (y^i) *may be obtained as functions of the original coordinates* (x^i) *by the contravariant differentiation of the scalar* Ω *with respect to the* $\bar{x}^i$.

This gives an explicit formula for the normal coordinates, in place of the implicit definition (4).

4. *Taylor's theorem.*

Consider first an arbitrary covariant vector T_p, each of whose components is a function of the x^i.

Transform to the normal coordinates (y^i), and let $^*T_p$ denote the vector in the transformed system. Then, by the ordinary transformation law for tensors,

$$^*T_p = \frac{\partial x^\lambda}{\partial y^p} T_\lambda. \tag{14}$$

Further,
$$T_p = \frac{\partial y^\lambda}{\partial x^p} {}^*T_\lambda,$$

and hence, by (13),
$$T_p = -\Omega_p^{(\lambda)} {}^*T_\lambda, \tag{15}$$

where the repetition of the suffix λ implies a summation in spite of the fact that in one case the suffix is bracketed.

The point $(\bar{x}^i)$ is now the origin $(y^i) = 0$. Hence, by (6) and (14),

$$({}^*T_p)_0 = \left(\frac{\partial x^\lambda}{\partial y^p}\right)_0 \bar{T}_\lambda,$$

$$= \delta_p^\lambda \bar{T}_\lambda,$$

i.e.,
$$({}^*T_p)_0 = \bar{T}_p. \tag{16}$$

Now each component of the vector $^*T_p$ is a function of the y^i. Hence, by the ordinary Taylor's theorem, assuming the necessary conditions of convergence to be satisfied,

$$^*T_p = ({}^*T_p)_0 + \frac{1}{1!} y^\alpha \left(\frac{\partial {}^*T_p}{\partial y^\alpha}\right)_0 + \frac{1}{2!} y^\alpha y^\beta \left(\frac{\partial^2 {}^*T_p}{\partial y^\alpha \partial y^\beta}\right)_0 + \ldots. \tag{17}$$

But the derivatives $\left(\frac{\partial {}^*T_p}{\partial y^\alpha}\right)_0$, $\left(\frac{\partial^2 {}^*T_p}{\partial y^\alpha \partial y^\beta}\right)_0$, ... evaluated at the y-origin are equal to the values at $(\bar{x}^i)$ of the "affine extensions"† $T_{p,\alpha}$, $T_{p,\alpha\beta}$, ... of the vector T_p.

Substituting in (17), and using equations (13) and (16), we get

$$ {}^*T_p = \bar{T}_p - \frac{1}{1!}\Omega^{(\alpha)}\bar{T}_{p,\alpha} + \frac{1}{2!}\Omega^{(\alpha)}\Omega^{(\beta)}\bar{T}_{p,\alpha\beta} - \frac{1}{3!}\Omega^{(\alpha)}\Omega^{(\beta)}\Omega^{(\gamma)}\bar{T}_{p,\alpha\beta\gamma} + \ldots, $$

α, β, γ, ... being summation suffixes.

Hence, by (15),

$$ T_p = -\Omega_p^{(\lambda)}\left[\bar{T}_\lambda - \frac{1}{1!}\Omega^{(\alpha)}\bar{T}_{\lambda,\alpha} + \frac{1}{2!}\Omega^{(\alpha)}\Omega^{(\beta)}\bar{T}_{\lambda,\alpha\beta} - \ldots\right], \tag{18} $$

which is a tensor formula expressing the vector T_p at (x^i) in terms of its value at $(\bar{x}^i)$.

Interchanging the x's and the $\bar{x}$'s,

$$ \bar{T}_p = -\Omega_{(p)}^{\lambda}\left[T_\lambda - \frac{1}{1!}\Omega^\alpha T_{\lambda,\alpha} + \frac{1}{2!}\Omega^\alpha\Omega^\beta T_{\lambda,\alpha\beta} - \ldots\right], \tag{19} $$

which gives the value of T_p at $(\bar{x}^i)$ in terms of its value at (x^i). And this is the formula which we sought to establish.

Multiplying both sides of (19) by $\bar{g}^{pq}$, and summing for p, we get for the *contravariant* vector T^q the equation‡

$$ \bar{T}^q = -\Omega_\lambda^{(q)}\left[T^\lambda - \frac{1}{1!}\Omega^\alpha T^\lambda{}_{,\alpha} + \frac{1}{2!}\Omega^\alpha\Omega^\beta T^\lambda{}_{,\alpha\beta} - \ldots\right], \tag{20} $$

since $\bar{g}^{pq}\Omega_{(p)}^{\lambda}T_\lambda = \Omega_\lambda^{(q)}T^\lambda$, and so on.

A similar result holds for a tensor of any rank. It was purely for the sake of simplicity that a *vector* was chosen for the purpose of establishing the formula. By an exactly similar method, therefore, the following general theorem may be proved:

Let T_p^q be a mixed tensor of rank $r+s$ (p denoting the group of suffixes $p_1 p_2 \ldots p_r$ and q the group $q_1 q_2 \ldots q_s$). Then, if the series on the right is convergent,

$$ \bar{T}_p^q = (-1)^{r+s}\prod_{l=1}^{r}\Omega_{p_l}^{(\lambda_l)}\prod_{m=1}^{s}\Omega_{(\mu_m)}^{q_m}\left[T_\lambda^\mu - \frac{1}{1!}\Omega^\alpha T_{\lambda,\alpha}^\mu + \frac{1}{2!}\Omega^\alpha\Omega^\beta T_{\lambda,\alpha\beta}^\mu - \ldots\right], \tag{21} $$

where λ, μ denote respectively the groups of suffixes $\lambda_1\lambda_2\ldots\lambda_r$, $\mu_1\mu_2\ldots\mu_s$.

† The m-th affine extension of a tensor is equal to its m-th covariant derivative plus certain terms involving the Riemann tensor; Veblen, *loc. cit.*, § 5, *et seq.*

‡ In this equation, $T^\lambda{}_{,\alpha\beta}$ stands for $g^{\lambda\kappa}T_{\kappa,\alpha\beta}$ not $(g^{\lambda\kappa}T_\kappa)_{,\alpha\beta}$, these being in general different. A similar remark applies to formula (21).

In particular, for an arbitrary *scalar* T, the theorem becomes

$$\bar{T} = T - \frac{1}{1!}\Omega^\alpha T,_\alpha + \frac{1}{2!}\Omega^\alpha\Omega^\beta T,_{\alpha\beta} - \frac{1}{3!}\Omega^\alpha\Omega^\beta\Omega^\gamma T,_{\alpha\beta\gamma} + \ldots, \qquad (22)$$

a result obtained for the special case of a Euclidean space in the earlier paper mentioned above.

5. *Conclusion.*

It is interesting to notice that the generalizations of ordinary partial derivatives which appear in the above theorem are not covariant derivatives but affine extensions; these generalizations being the same when the Riemann tensor is everywhere zero (*i.e.*, when the space is Euclidean).

The formula (21) contains the classical Taylor's theorem as a particular case. For, if the space is Euclidean and the coordinates rectangular Cartesian, we have [by equations (8) and (9) if we take three dimensions only]

$$\Omega^{(p)}_\lambda = \Omega^p_{(\lambda)} = -\delta^p_\lambda,$$

$$\Omega^\alpha = x^\alpha - \bar{x}^\alpha,$$

and it is immediately evident that (21) reduces to the ordinary Taylor's theorem applied separately to each component of the tensor.

In practice the actual calculation of the function Ω usually leads to algebra which rapidly becomes intractable. It is therefore difficult to give examples of the application of the theorems of this paper. But it may be remarked that the theorem of § 3 is readily illustrated in the case of the Euclidean metric

$$ds^2 = dr^2 + r^2 d\theta^2 \quad (x^1 = r,\ x^2 = \theta),$$

for which, of course,

$$\Omega = \tfrac{1}{2}\{r^2 + \bar{r}^2 - 2r\bar{r}\cos(\theta - \bar{\theta})\}.$$

And it is easily shown that the formula (19) reduces to a simple identity if applied to the vector whose components are $T_1 = r - \cos\theta$, $T_2 = r\sin\theta$. All covariant derivatives of this vector of orders higher than the first are zero.

ON A CLASS OF PROJECTIVELY FLAT AFFINE CONNECTIONS.

By J. H. C. Whitehead.

[Received 6 July, 1929.—Read 14 November, 1929.]

In a recent lecture to the London Mathematical Society* O. Veblen defined the classical projective geometry as the theory of an invariant. This invariant is a projective connection, subject to certain restrictions in virtue of which it is described as flat.

A projectively flat affine connection is an affine connection defined by the straight lines of a flat affine space, parameterized in some way or other. We shall, however, use the term projectively flat to describe the more restricted class of affine connections which are characterized by the following conditions: their components in any coordinate system are point functions, the Weyl tensor vanishes†, and $B_{jk} = B_{kj}$ where $B_{jk} = B^{a}_{jak}$ and B^{i}_{jkl} is the curvature tensor. There exists a coordinate system in which the components of an affine connection, belonging to this class, are

$$\delta^i_j \psi_k + \delta^i_k \psi_j,$$

where ψ_i are the components of a gradient. The fact that ψ_i is a gradient is due to the condition $B_{jk} = B_{kj}$. A projectively flat affine space will mean a space to which the geometry of one of these invariants is applicable.

In § 1 of this paper we have tried to exhibit, as clearly as possible, a flat projective connection as a family of projectively flat affine connections, related to each other in a definite manner. In § 2 and § 3 we are interested in the spaces obtained by excluding a hypersurface from classical projective space. It will be seen that these are projectively flat. In § 4 and § 5 we show how a projectively flat affine space of n dimensions may be regarded as the projection of a hyperplane on to a hypersurface, where both are imbedded in an affine space of $n+1$ dimensions.

* *Journal London Math. Soc.*, 4 (1929), 140–160.

† H. Weyl, *Göttinger Nachrichten* (1921), 99–112.

I take pleasure in thanking Prof. Veblen for his valuable criticism and advice.

1. Veblen* gave as the basis of projective geometry the following differential equations†:

$$\frac{\partial^2 Z}{\partial x^\beta \partial x^\gamma} - \frac{\partial Z}{\partial x^\alpha}\,\Pi^\alpha_{\beta\gamma} = 0, \tag{1.1}$$

where

$$\begin{cases} \dfrac{\partial \Pi^\alpha_{\beta\gamma}}{\partial x^0} = 0, \\ \Pi^\alpha_{\beta 0} = \Pi^\alpha_{0\beta} = \delta^\alpha_\beta. \end{cases} \tag{1.1a}$$

These functions‡ are the components of a projective connection, and projective geometry is the theory of this invariant under transformations of the form

$$\begin{cases} \bar{x}^0 = x^0 + \log \rho(x^1, \ldots, x^n), \\ \bar{x}^i = \bar{x}^i(x^1, \ldots, x^n). \end{cases} \tag{1.2}$$

The equations (1.1) will be transformed into

$$\frac{\partial^2 Z}{\partial \bar{x}^\beta \partial \bar{x}^\gamma} - \frac{\partial Z}{\partial \bar{x}^\alpha}\,\bar{\Pi}^\alpha_{\beta\gamma} = 0, \tag{1.3}$$

where

$$\bar{\Pi}^\alpha_{\beta\gamma} = \left(\Pi^\lambda_{\mu\nu} u^\mu_\beta u^\nu_\gamma + \frac{\partial u^\lambda_\beta}{\partial \bar{x}^\gamma}\right) v^\alpha_\lambda, \tag{1.4}$$

and

$$u^\alpha_\beta = \frac{\partial x^\alpha}{\partial \bar{x}^\beta}, \quad v^\alpha_\beta = \frac{\partial \bar{x}^\alpha}{\partial x^\beta}.$$

The conditions (1.1a) are invariant under transformations of the form (1.2), since $u^\alpha_0 = v^\alpha_0 = \delta^\alpha_0$.

If we apply a transformation of the form

$$\begin{cases} \bar{x}^0 = x^0, \\ \bar{x}^i = \bar{x}^i(x), \end{cases} \tag{1.5}$$

---

* See foot-note *, p. 93.

† Greek letters, used as suffixes, will range over 0, 1, ..., n, and Roman letters over 1, 2, ..., n.

‡ When we refer to a function without further qualification, we shall mean an analytic function, regular at all points under consideration.

then $$u^0_\alpha = v^0_\alpha = \delta^0_\alpha,$$

and so

$$(1.6)\qquad \begin{cases} (a) & \bar{\Pi}^i_{jk} = \left(\Pi^\alpha_{bc}\, u^b_j\, u^c_k + \dfrac{\partial u^\alpha_j}{\partial \bar{x}^k}\right) v^i_\alpha, \\ (b) & \bar{\Pi}^0_{jk} = \Pi^0_{bc}\, u^b_j\, u^c_k. \end{cases}$$

The relations (1.6a) and (1.6b) are respectively the transformation laws for an affine connection and an affine tensor. If we apply a transformation of the form

$$(1.7)\qquad \begin{cases} \bar{x}^0 = x^0 + \log \rho, \\ \bar{x}^i = x^i, \end{cases}$$

then $$u^i_\alpha = v^i_\alpha = \delta^i_\alpha, \quad u^0_i = -\frac{\partial \log \rho}{\partial x^i},$$

and

$$(1.8)\qquad \begin{cases} (a) & \bar{\Pi}^i_{jk} = \Pi^i_{jk} + \delta^i_j\, u^0_k + \delta^i_k\, u^0_j, \\ (b) & \bar{\Pi}^0_{jk} = \Pi^0_{jk} - \dfrac{1}{\rho}\left(\dfrac{\partial^2 \rho}{\partial x^j\, \partial x^k} - \dfrac{\partial \rho}{\partial x^i}\, \Pi^i_{jk}\right). \end{cases}$$

If we restrict ourselves to transformations of the type (1.5), Π^i_{jk} and $\bar{\Pi}^i_{jk}$ may each be taken as the components of an affine connection. From (1.8a) we see that Π^i_{jk} and $\bar{\Pi}^i_{jk}$ define the same system of paths, and that these equations define a projective change of connection, due to a change of affine parameter on the paths, given by

$$(1.9)\qquad ds = \rho^2\, d\bar{s}.$$

Veblen, in the paper referred to above (p. 153), shows how a system of paths is defined by a projective connection. From (21) of his paper we see that these are the same as the paths defined by Π^i_{jk}, considered as components of an affine connection.

The equations (1.1) are completely integrable if and only if

$$(1.10)\qquad \begin{cases} \Pi^\alpha_{\beta\gamma} = \Pi^\alpha_{\gamma\beta}, \\ R^\alpha_{\beta\gamma\delta} = 0, \end{cases}$$

where $$R^\alpha_{\beta\gamma\delta} = \frac{\partial \Pi^\alpha_{\beta\gamma}}{\partial x^\delta} - \frac{\partial \Pi^\alpha_{\beta\delta}}{\partial x^\gamma} + \Pi^\sigma_{\beta\gamma}\, \Pi^\alpha_{\sigma\delta} - \Pi^\sigma_{\beta\delta}\, \Pi^\alpha_{\sigma\gamma}.$$

These equations may be written

$$(1.11)\qquad \begin{cases} (a) & \Pi^{\alpha}_{\beta\gamma} = \Pi^{\alpha}_{\gamma\beta}, \\ (b) & R^{i}_{jkl} = B^{i}_{jkl} + \delta^{i}_{l}\,\Pi^{0}_{jk} - \delta^{i}_{k}\,\Pi^{0}_{jl} = 0, \\ (c) & \Pi^{0}_{jk;\,l} = \Pi^{0}_{jl;\,k}, \end{cases}$$

where

$$(1.12)\qquad B^{i}_{jkl} = \frac{\partial \Pi^{i}_{jk}}{\partial x^{l}} - \frac{\partial \Pi^{i}_{jl}}{\partial x^{k}} + \Pi^{a}_{jk}\,\Pi^{i}_{al} - \Pi^{a}_{jl}\,\Pi^{i}_{ak},$$

and $\Pi^{0}_{jk;\,l}$ is the covariant derivative of Π^{0}_{jk} with respect to Π^{i}_{jk}.

$$R^{\alpha}_{0\gamma\delta} = R^{\alpha}_{\beta\gamma 0} = 0 \quad \text{by } (1.11\,a).$$

From $(1.11b)$ we have

$$(1.13)\qquad B_{jk} = B^{a}_{jak} = (n-1)\,\Pi^{0}_{jk}.$$

In view of $(1.11b)$, therefore, R^{i}_{jkl} bear the same relation to Π^{i}_{jk} as the Weyl tensor to the components of affine connection. It should, however, be observed that (1.13) need not hold unless $R^{i}_{jkl} = 0$.

If (1.1) yield a solution Z, we have, since $\Pi^{\alpha}_{0\beta} = \delta^{\alpha}_{\beta}$,

$$(1.14)\qquad \frac{\partial}{\partial x^{0}}\frac{\partial Z}{\partial x^{i}} = \frac{\partial Z}{\partial x^{i}};$$

hence it is easily seen that Z must be of the form

$$(1.15)\qquad Z = e^{x^{0}}\,\theta(x^{1}, \ldots, x^{n}) + K,$$

where K is a constant, which we shall take to be zero. Then

$$\frac{\partial Z}{\partial x^{0}} = Z.$$

Let $q^{1}, \ldots, q^{n}$ be the coordinates of a point for which $\Pi^{\alpha}_{\beta\gamma}$ are regular. Then, if (1.10) are satisfied, any $(n+1)^{2}$ constants, p^{α}_{β}, will determine a unique set of solutions, Z^{α}, such that, for $x^{0} = 0$, $x^{i} = q^{i}$,

$$(1.16)\qquad \frac{\partial Z^{\alpha}}{\partial x^{\beta}} = p^{\alpha}_{\beta}.$$

$\left(\text{Notice that } Z^{\alpha} = \dfrac{\partial Z^{\alpha}}{\partial x^{0}} = p^{\alpha}_{0}, \text{ initially.}\right)$ This may be written in the form

$$(1.17)\qquad Z^{\alpha} = e^{x^{0}}\,\theta^{\alpha}(x^{1}, \ldots, x^{n}).$$

To summarize these results, we start with a manifold of n dimensions, referred to a coordinate system x. In this coordinate system $\Pi^{\alpha}_{\beta\gamma}$ are a set of components determined by a projective connection. Any other set of components in the same coordinate system is given by equations (1 . 8).

Now (1 . 1) may be written, since we have taken $K = 0$,

$$\frac{\partial^2 Z}{\partial x^j \partial x^k} - \frac{\partial Z}{\partial x^i} \Pi^i_{jk} - Z\Pi^0_{jk} = 0. \tag{1 . 18}$$

We shall suppose (1 . 10) to be satisfied. Then (1 . 18) are completely integrable, and any solution is of the form

$$Z = e^{x_0} p_\alpha \theta^\alpha, \tag{1 . 19}$$

where θ^α are any $n+1$ linearly independent solutions, p_α are constants, and x^0 is any number. We may treat x^0 as an independent variable, which we shall call the *factor*. The manifold with which we started is a projective space of n dimensions*, referred to a coordinate system x and a factor x^0.

We shall call this a *representation* and denote it by $x+x^0$. Equations (1 . 5) and (1 . 7) will be said to define a change of coordinates and of factor respectively.

Then to each representation corresponds a unique set of components. In each coordinate system there are an infinite number of sets of components, related to each other by equations of the form (1 . 8); and to each factor corresponds an infinite number of sets of components, related to each other by equations of the form (1 . 6). From the form of equations (1 . 6) we see that each factor selects from the projective connection an affine connection and a tensor. The relation of the tensor to the affine connection is given by (1 . 13). We can think of any factor as showing, in cross-section, the affine properties of the projective connection. To study the purely projective relations we need apply transformations of factor only.

2. Return to equations (1 . 17).

$$Z^\alpha = e^{x^0} \theta^\alpha(x^1, \ldots, x^n), \tag{2 . 1}$$

where $|\partial Z^\alpha/\partial x^\beta| \neq 0$ in the domain $|x^i - q^i| < r$, for some $r > 0$. Then

* When we say that a manifold is such and such, we shall be referring to its local properties.

near $x = q$ we may solve for $x^0, \ldots, x^n$, and so obtain

$$(2.2) \qquad \begin{cases} (a) & e^{x^0} = \phi(Z), \\ (b) & x^i = \chi^i(Z). \end{cases}$$

Since

$$(2.3) \qquad e^\lambda Z^\alpha = e^{x^0+\lambda}\,\theta^\alpha(x),$$

we have, from (2.2) and (2.3),

$$(2.4) \qquad \begin{cases} \phi(e^\lambda Z) = e^{x^0+\lambda} = e^\lambda \phi(Z), \\ \chi^i(e^\lambda Z) = x^i = \chi^i(Z). \end{cases}$$

Hence $\phi(Z)$ and $\chi^i(Z)$ are homogeneous of degree unity and zero respectively. Any change of coordinates will leave the relation (2.2a) untouched.

The equations (2.2) are determined by a choice, in a particular coordinate system, of functions $\Pi^\alpha_{\beta\gamma}$ which satisfy (1.10). Then, if Γ^i_{jk} are the components, in this coordinate system, of any projectively flat affine connection, we can write

$$(2.5) \qquad \begin{cases} \Pi^\alpha_{\beta 0} = \delta^\alpha_\beta, \\ \Pi^i_{jk} = \Gamma^i_{jk}, \\ \Pi^0_{jk} = \dfrac{1}{n-1} B_{jk}, \end{cases}$$

and obtain equations of the form (2.1). Then the relation (2.2a) will be characteristic of the affine connection Γ^i_{jk}, as apart from any of the other affine connections which are associated with the projective connection $\Pi^\alpha_{\beta\gamma}$.

The functions $Z^0, \ldots, Z^n$ are coordinates defining a point in projective space of n dimensions, P_n say. The invariant theory of $\Pi^\alpha_{\beta\gamma}$ under all changes of representation is the geometry of P_n, and, as we have just seen, under transformations of coordinates only is the geometry of P_n subject to the condition (2.2a). So, in view of (1.6), (1.11b), and (2.2a), we have the theorem:

A. *Any projectively flat affine space determines a relation between the points of a projective space, which, in homogeneous projective coordinates, is given by an equation of the form* $\phi(Z) = e^{x^0}$, *where* $\phi(Z)$ *is homogeneous of degree unity.*

If in (2.1) and (2.2) we put $x^0 = 0$, the variables Z^α will no longer be homogeneous, but rather supernumerary coordinates. The relations (2.2*b*) will be unaltered, and (2.2*a*) will become

$$\phi(Z) = 1. \tag{2.6}$$

We shall call $\phi(Z)$ the characteristic function of the affine space in question. In case the locus $\phi(Z) = 0$ exists, it is clear that no point on it can be represented by any representation with the factor x^0. We shall therefore call it the "missing $(n-1)$-space"* corresponding to the factor x^0.

Conversely, let a projective space, P_n, be referred to homogeneous coordinates $Z^0, \ldots, Z^n$, and let $\phi(Z)$ be any function which is homogeneous of degree unity. Let $\chi^i(Z)$ be any functions of Z, homogeneous of degree zero, and regular in a certain domain D, which is such that $\phi(Z)$ is regular, non-zero, and one-valued in D. If necessary, we select a particular branch of $\phi(Z)$. Further, suppose that the Jacobian of $\phi(Z)$ and $\chi^i(Z)$ does not vanish in D.

Let

$$\begin{cases} e^{x^0} = \phi(Z), \\ x^i = \chi^i(Z). \end{cases} \tag{2.7}$$

Then we can solve (2.7) near any particular point of D, and we shall obtain

$$Z^\alpha = e^{x^0}\theta^\alpha(x^1, \ldots, x^n). \tag{2.8}$$

---

* If, for example, $\phi(z) = +(a_{\alpha\beta}z^\alpha z^\beta)^{\frac{1}{2}}$ we should have non-Euclidean geometry with $a_{\alpha\beta}z^\alpha z^\beta = 0$ for the equation of the absolute. If, on the other hand, $\phi(z) = z^0 e^{z^1/z^0}$ the locus $\phi(z) = 0$ does not exist.

In dealing with the singularities of $\phi(z)$ I shall not do more than give the simplest possible example of a locus each point of which is a pole. In considering this situation I shall assume a knowledge of § 4.

Let $n = 2$ and let the characteristic function be

$$\frac{x^2 + y^2 + z^2}{2z}.$$

The missing curve is represented in the 3-dimensional picture by the isotropic cone through the origin. The equation to V_2 is

$$x^2 + y^2 + z^2 = 2z,$$

so V_2 is a sphere with centre at the point (0, 0, 1), and radius 1. All of the points of the projective plane which lie on $z = 0$ will be represented by a single point of V_2, and this point will be the origin of A_3. By this method stereographic projection can be handled.

A flat projective connection will be defined by

$$\Pi^{\alpha}_{\beta\gamma} = \frac{\partial x^{\alpha}}{\partial Z^{\sigma}} \frac{\partial^2 Z^{\sigma}}{\partial x^{\beta} \partial x^{\gamma}}. \tag{2.9}$$

The relations (2.7) will define a representation $x+x^0$ for this connection, and Π^{i}_{jk} will be a projectively flat affine connection. We have, therefore, the theorem converse to A:

B. *A relation $\phi(Z) = e^{x^0}$ between points of a projective space, where $\phi(Z)$ is homogeneous of degree unity, determines a projectively flat affine space.*

Finally, let $\phi(Z)$ and $\psi(Z)$ be any two functions of $Z^0, \ldots, Z^n$, homogeneous of degree unity, and having a non-singular point in common. Then there will be a domain D in which both are regular, one-valued, and non-zero. As before, we may have to select particular branches of $\phi(Z)$ and $\psi(Z)$. Then, by introducing suitable functions $\chi^i(Z)$, we can define

$$\begin{cases} e^{x^0} = \phi(Z), \\ x^i = \chi^i(Z). \end{cases} \tag{2.10}$$

We may suppose Z expressed as functions of x by the relations (2.8). Let $\bar{x}^0$ be the factor corresponding to $\psi(Z)$. Then

$$\begin{aligned} e^{\bar{x}_0} &= \psi(Z) \\ &= \psi\{e^{x^0}\theta(x)\} \\ &= e^{x^0}\rho(x), \end{aligned}$$

where $$\rho(x) = \psi\{\theta(x)\},$$

and we have $$\bar{x}_0 = x^0 + \log \rho.$$

So we have the theorem:

C. *The affine connections corresponding to two relations of the form $e^{x^0} = \phi(Z)$, $e^{\bar{x}^0} = \psi(Z)$ define, in any coordinate system, components belonging to the same projective connection.*

3. Let Z be any homogeneous coordinate system, related to a representation $x+x^0$ by the equations

$$Z^{\alpha} = e^{x^0}\theta^{\alpha}(x). \tag{3.1}$$

Let

$$\begin{cases} y^0 = x^0 + \log \theta^0(x), \\ y^i = \dfrac{\theta^i(x)}{\theta^0(x)}. \end{cases} \tag{3.2}$$

Then we have

$$\begin{cases} Z^i = e^{y^0} y^i, \\ Z^0 = e^{y^0}. \end{cases} \tag{3.3}$$

I shall call $y + y^0$ the normal representation corresponding to the coordinate system Z. If $P^\alpha_{\beta\gamma}$ are the components of projective connection corresponding to this representation, we see that

$$P^\alpha_{jk} = 0. \tag{3.4}$$

The factor y^0 is one of a class of factors which are characterized by the vanishing of P^0_{jk}. The corresponding affine connection is flat and the missing $(n-1)$-space is a hyperplane.

In the representation given by (3.3) the equivalence of these three statements is obvious.

Let

$$\bar{y}^0 = y^0 + \log \rho(y). \tag{3.5}$$

Then, if $\bar{P}^\alpha_{\beta\gamma}$ are the components of connection corresponding to $y + \bar{y}^0$, we have, from (1.8),

$$\bar{P}^0_{jk} = -\frac{1}{\rho} \frac{\partial^2 \rho}{\partial y^j \partial y^k}. \tag{3.6}$$

So $\bar{P}^0_{jk} = 0$ if and only if $\rho = a_i y^i + a_0$, where a_α are constants. In this case the affine connection corresponding to $\bar{y}^0$ will be flat, and the equation to the missing $(n-1)$-space corresponding to $\bar{y}^0$ will be

$$a_\alpha Z^\alpha = 0, \tag{3.7}$$

or in non-homogeneous coordinates

$$a_i y^i + a_0 = 0. \tag{3.8}$$

The Cartesian coordinates for the affine connection corresponding to $\bar{y}^0$ are given by $\bar{y}^i = y^i/(a_j y^j + a_0)$, and, if we put $\bar{Z}^0 = a_\alpha Z^\alpha$, $\bar{Z}^i = Z^i$, we shall have

$$\begin{cases} \bar{Z}^i = e^{\bar{y}^0} \bar{y}^i, \\ \bar{Z}^0 = e^{\bar{y}^0}. \end{cases} \tag{3.9}$$

So any one of this class of factors is the normal factor for some homogeneous projective coordinate system. These representations may be called Cartesian, and the factors Cartesian factors.

It is easy to see how we may pass from a normal factor to the factor x^0, associated with any homogeneous function $\phi(Z)$, of degree unity. From (3.3) we have

$$\left\{\begin{aligned} y^i &= \frac{Z^i}{Z^0}, \\ e^{y^0} &= Z^0, \end{aligned}\right. \tag{3.10}$$

whence
$$\begin{aligned} \phi(Z) &= Z^0 \phi\left(\frac{Z^1}{Z^0}, \ \ldots, \ \frac{Z^n}{Z^0}\right) \\ &= e^{y^0}\rho(y), \quad \text{say,} \end{aligned}$$

so
$$e^{x^0} = e^{y^0}\rho(y),$$

or

$$x^0 = y^0 + \log \rho(y). \tag{3.11}$$

Conversely, if (3.11) is taken to define a change of factor, where $\rho(y)$ is any function of $y^1, \ldots, y^n$, we have

$$\begin{aligned} e^{x^0} &= e^{y^0}\rho(y) \\ &= Z^0 \rho\left(\frac{Z^1}{Z^0}, \ \ldots, \ \frac{Z^n}{Z^0}\right) \\ &= \phi(Z). \end{aligned} \tag{3.12}$$

If the locus $\rho(y) = 0$ exists it will be the missing $(n-1)$-space corresponding to the factor x^0.

4. A projective space of n dimensions may be treated as an affine space of $n+1$ dimensions in which there is a special point, the centre. The points in the projective space are represented by straight lines through the centre in the affine space.

Prof. L. P. Eisenhart* has shown that any projectively flat space can be regarded as a sub-space in an affine space of $n+1$ dimensions.

If we write Π^i_{jk} for Γ^i_{jk}, and Π^0_{jk} for a_{jk}, the equations

$$\theta, _{jk} = \theta a_{jk}, \tag{4.1}$$

which are (61.1) on the page referred to, will be the same as (1.18).

---

* *Non-Riemannian geometry* (1927), 166.

Eisenhart has shown that the solutions to these, which may, by a slight deviation from his notation, be written

$$Z^a = e^{x^0}\theta^a(x), \tag{4.2}$$

determine a transformation of coordinates in the $(n+1)$-space to a Cartesian coordinate system Z.

Expanding (4.2) in terms of x^0 we have

$$Z^a = \theta^a + x^0\theta^a + \ldots. \tag{4.3}$$

Hence Π^i_{jk} is the affine connection induced* on the sub-space $x^0 = 0$ by the contravariant vector Z^a, *i.e.* the radius vector from the centre.

The equation to this sub-space is

$$\phi(Z) = 1, \tag{4.4}$$

where $\phi(Z)$ is the function given by (2.2a).

In the normal representation,

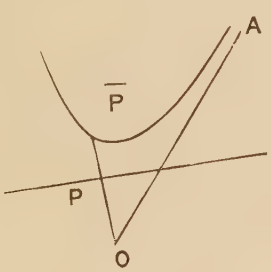
$$\begin{cases} Z^i = e^{y_0}y^i, \\ Z^0 = e^{y_0}. \end{cases} \tag{4.5}$$

In this case the sub-space is clearly the hyperplane defined by $Z^0 = 1$. Any other projectively flat space may be regarded as the projection of this hyperplane on to some hypersurface, the paths being the curves in which it is cut by planes through the centre.

We shall illustrate this situation on a cross-section by one of the paths, which are defined by equations of the form

$$Z^a = a^a u + b^a v, \tag{4.6}$$

where a and b are constants.



Let O be the centre and OP any ray in the affine $(n+1)$-space, representing a point in projective n-space. This meets the locus $Z^0 = 1$ in P

* Eisenhart, *loc. cit.*, Ch. 4 (50.11) and § 53.

and the locus $\phi(Z) = 1$ in $\bar{P}$. Let

$$O\bar{P} = \frac{1}{\rho} OP. \tag{4.7}$$

Then the coordinates of P and $\bar{P}$ in the affine $(n+1)$-space are

$$y^1, \ldots, y^n, 1 \quad \text{and} \quad \frac{y^1}{\rho}, \ldots, \frac{y^n}{\rho}, \frac{1}{\rho}, \quad \text{respectively,} \tag{4.8}$$

and

$$\phi\left(\frac{y^1}{\rho}, \ldots, \frac{y^n}{\rho}, \frac{1}{\rho}\right) = 1,$$

or

$$\rho = \phi(y^1, \ldots, y^n, 1). \tag{4.9}$$

The ray OA is parallel to, or in, the asymptotic cone, and satisfies $\phi(Z) = 0$.

5. In the last section we showed how our previous results are related to the theory of n-dimensional space, imbedded in an affine space of $n+1$ dimensions. We shall now exhibit this relation in greater detail by considering the change of parameter on the paths defined by the equations (1.9). The paths corresponding to the normal factor are represented by straight lines in the hyperplane A_n, defined by the equation $z^0 = 1$. On each of these straight lines a family of affine parameters $\alpha s + \beta$ will be defined, where α and β are arbitrary constants on each straight line, and s is any affine parameter. Let V_n be the hyper-surface defined by the equation (4.4). Then, by means of (1.9), an affine parameter $\bar{s}$ is defined on each path of V_n. Each of these paths is the intersection of V_n with a plane through the centre. These planes are defined by equations of the form (4.6), and each one is an affine space of two dimensions for which u and v are Cartesian coordinates.

With each pair of points on a plane curve, which is not a straight line, in affine space there is associated a relative scalar called the affine distance* between them.

If the parametric equations to the curve are

$$\begin{cases} u = u(\theta), \\ v = v(\theta), \end{cases} \tag{5.1}$$

* W. Blaschke, *Vorlesungen über Differentialgeometrie*, 2 (1923), 8.

the affine distance between two points whose parameters are θ_0 and θ_1 is given by

$$t = \int_{\theta_0}^{\theta_1} \left(\frac{du}{d\theta} \frac{d^2v}{d\theta^2} - \frac{d^2u}{d\theta^2} \frac{dv}{d\theta} \right)^{\frac{1}{3}} d\theta. \tag{5.2}$$

The affine distance from an arbitrary point is called the affine length. If we change to another Cartesian coordinate system $(\bar{u}, \bar{v})$, in which $\bar{t}$ is the affine length, we shall have

$$\bar{t} = \left| \frac{\partial(\bar{u}, \bar{v})}{\partial(u, v)} \right|^{\frac{1}{3}} t. \tag{5.3}$$

The Jacobian is, of course, constant.

Notice that there is no relation between two affine distances defined one on each of two different planes.

If we set up a Euclidean metric in an affine space, one of the affine parameters on each straight line will be the Euclidean length. The affine length on a curve which is not a straight line will be the Euclidean length, multiplied by a constant factor, on circles, and circles only, as is shown by Blaschke.

We shall in this section, answer these two questions:

Let t be the affine length on any path in V_n. What is the relation between $\bar{s}$ and t where $\bar{s}$ is defined by (1.9)?

If we set up a Euclidean metric in A_{n+1}, the $(n+1)$-dimensional affine space, what is the relation between $\bar{s}$ and σ where σ is the Euclidean length?

These questions are answered by equations (5.13) and (5.39) respectively.

Let P_2 be any plane, defined by equations of the form (4.6), which intersects A_n and V_n in curves P_1 and C_1 respectively. We may take $a^0 = 0$, $b^0 = 1$, and the equations to P_2 will be

$$\begin{cases} Z^i = a^i u + b^i v, \\ Z^0 = v. \end{cases} \tag{5.4}$$

The hyperplane A_n corresponds to the normal factor y^0, and is referred to a Cartesian coordinate system y. The hypersurface V_n corresponds to the factor $x^0 = y^0 + \log \rho$ and is referred, we shall suppose, to the same coordinate system y. Let $P^{\alpha}_{\beta\gamma}$ and $\Pi^{\alpha}_{\beta\gamma}$ be the components of the projective

connection in the representations $y+y^0$ and $y+x^0$ respectively. Then

$$(5.5)\qquad \begin{cases} (a) & P^a_{jk} = 0, \\ (b) & \Pi^i_{jk} = -\dfrac{1}{\rho}\left(\delta^i_j \dfrac{\partial \rho}{\partial y^k} + \delta^i_k \dfrac{\partial \rho}{\partial y^j}\right), \\ (c) & \Pi^0_{jk} = -\dfrac{1}{\rho}\dfrac{\partial^2 \rho}{\partial y^j \partial y^k}. \end{cases}$$

Let s and $\bar{s}$ be affine parameters on P_1 and C_1 respectively. Then, if V_n is flat, C_1 will be a straight line, and it follows from elementary principles of projective geometry, and the relation $d\bar{s} = ds/(\alpha s+\beta)^2$, that $\bar{s}$ is an affine parameter on C_1. When C_1 is not a straight line the element of affine length on C_1 is given by

$$(5.6)\qquad dt = (du\, d^2v - d^2u\, dv)^{\frac{1}{3}}.$$

We have, for the equation to C_1 as a locus in P_2,

$$(5.7)\qquad \rho(u, v) = 1,$$

where $\rho = \phi(au+bv)$, and is homogeneous of degree unity. The equations to P_1, as a locus in A_n, are

$$(5.8)\qquad y^i = b^i + a^i s,$$

where b^i and a^i are constants, $s = u/v$.

For values of u and v satisfying (5.7), we have, therefore,

$$(5.9)\qquad \begin{cases} v\rho(s, 1) = 1, \\ u = vs, \\ \quad = \dfrac{s}{\rho}, \end{cases}$$

and so

$$(5.10)\qquad \begin{cases} \dfrac{du}{dv} = \dfrac{s\, d\rho - \rho\, ds}{d\rho}, \\ \dfrac{dv\, d^2u - d^2v\, du}{(dv)^2} = d\dfrac{du}{dv} = \dfrac{\rho\, ds\, d^2\rho}{(d\rho)^3}, \end{cases}$$

taking $d^2s = 0$.

From (5 . 6) we have

$$dt = \left\{-(dv)^2\, d\,\frac{du}{dv}\right\}^{\frac{1}{3}} \tag{5 . 11}$$

$$= \left\{-\frac{(d\rho)^2}{\rho^4}\cdot\frac{\rho\, ds\, d^2\rho}{(d\rho)^2}\right\}^{\frac{1}{3}}$$

$$= \frac{ds}{\rho}\left(-\frac{d^2\rho}{ds^2}\right)^{\frac{1}{3}}.$$

Now

$$\frac{d^2\rho}{ds^2} = \frac{\partial^2\rho}{\partial y^j\,\partial y^k}\frac{dy^j}{ds}\frac{dy^k}{ds} + \frac{\partial\rho}{\partial y^i}\frac{d^2y^i}{ds^2} \tag{5 . 12}$$

$$= -\rho\Pi^0_{jk}\frac{dy^j}{ds}\frac{dy^k}{ds},$$

from (5 . 5c), and since $$\frac{d^2y^i}{ds^2} = 0.$$

So $$\rho^3\left(\frac{dt}{ds}\right)^3 = \rho\Pi^0_{jk}\frac{dy^j}{ds}\frac{dy^k}{ds}.$$

In view of (1 . 9), therefore, the relation between t and s is given by

$$\frac{dt}{ds} = \Pi^0_{jk}\frac{dy^j}{dt}\frac{dy^k}{dt}. \tag{5 . 13}$$

The ratio $dt/d\bar{s}$ has a constant value a if, and only if,

$$\Pi^0_{jk}\frac{dy^j}{d\bar{s}}\frac{dy^k}{d\bar{s}} = a^3. \tag{5 . 14}$$

If the condition (5 . 14) is satisfied, $\Pi^0_{jk}\frac{dy^j}{ds}\frac{dy^k}{ds}$ will be a first integral of the equations to the paths in V_n. The conditions that this should be the case are*

$$\Pi^0_{jk;\,l} + \Pi^0_{kl;\,j} + \Pi^0_{lj;\,k} = 0, \tag{5 . 15}$$

where the semicolon denotes covariant differentiation with respect to Π^i_{jk}, which will be regarded as components of an affine connection on V_n. From (1 . 11c) it is easily seen that (5 . 15) reduce to

$$\Pi^0_{jk;\,l} = 0. \tag{5 . 16}$$

---

* O. Veblen and T. Y. Thomas, "The geometry of paths", *Trans. American Math. Soc.*, 25 (1923), 599. Also L. P. Eisenhart, *Non-Riemannian geometry*, 84.

Let the rank of the matrix (Π^0_{jk}) be r. Then, if $r = n$ and (5 . 16) hold, Π^i_{jk} will be Christoffel symbols for Π^0_{jk}, which may be taken to define a metric. From (1 . 11*b*) it is seen that V_n will be a Riemannian space* of constant curvature.

If, therefore, the affine parameter on the paths of a projectively flat affine space is proportional to the affine length (with the same origin), and if the determinant $|\Pi^0_{jk}|$ does not vanish, then the space is Riemannian, has constant curvature, and we are led back to non-Euclidean geometry.

We shall show that, whatever the rank r may be, the missing $(n-1)$-space is a quadric if (5 . 16) are satisfied. Moreover, if the equation to this quadric is

$$a_{\alpha\beta} Z^\alpha Z^\beta = 0,$$

then the rank of the matrix $(a_{\alpha\beta})$ will be $r+1$. To this end we refer to (5 . 5), and see that the equations (5 . 16) amount to

$$-\frac{\partial}{\partial y^l}\left(\frac{1}{\rho}\frac{\partial^2\rho}{\partial y^j \partial y^k}\right) - \frac{1}{\rho^2}\left\{\frac{\partial^2\rho}{\partial y^a \partial y^k}\left(\delta^a_j \frac{\partial\rho}{\partial y^l} + \delta^a_l \frac{\partial\rho}{\partial y^j}\right)\right.$$

$$\left. + \frac{\partial^2\rho}{\partial y^a \partial y^j}\left(\delta^a_k \frac{\partial\rho}{\partial y^l} + \delta^a_l \frac{\partial\rho}{\partial y^k}\right)\right\} = 0,$$

or

$$\rho \frac{\partial^3\rho}{\partial y^j \partial y^k \partial y^l} + \frac{\partial\rho}{\partial y^j}\frac{\partial^2\rho}{\partial y^k \partial y^l} + \frac{\partial\rho}{\partial y^k}\frac{\partial^2\rho}{\partial y^l \partial y^j} + \frac{\partial\rho}{\partial y^l}\frac{\partial^2\rho}{\partial y^j \partial y^k} = 0, \tag{5 . 17}$$

which may be written

$$\frac{\partial}{\partial y^j}\left(\rho\frac{\partial^2\rho}{\partial y^k \partial y^l} + \frac{\partial\rho}{\partial y^k}\frac{\partial\rho}{\partial y^l}\right) = 0,$$

or

$$\tfrac{1}{2}\frac{\partial^3\rho^2}{\partial y^j \partial y^k \partial y^l} = 0. \tag{5 . 18}$$

So we have

$$\rho^2 = a_{jk} y^j y^k + 2a_{0i} y^i + a_{00}, \tag{5 . 19}$$

where $a_{\alpha\beta}$ are constants.

We shall show that the rank of the matrix $(a_{\alpha\beta})$ is $r+1$. Let $\theta = \rho^2$ and let ρ_i, θ_i, ρ_{ij}, and θ_{ij} stand for $\partial\rho/\partial y^i$, $\partial\theta/\partial y^i$, $\partial^2\rho/\partial y^i \partial y^j$, and

* L. P. Eisenhart, *Riemannian geometry*, 83.

$\partial^2\theta/\partial y^i\,\partial y^j$ respectively. Then

$$(5.20)\qquad \begin{cases} \theta_i = 2\rho\rho_i, \\ \theta_{jk} = 2(\rho\rho_{jk}+\rho_j\rho_k), \\ \dfrac{1}{\rho}\,\rho_{jk} = \dfrac{1}{4\theta^2}\,(2\theta\theta_{jk}-\theta_j\theta_k). \end{cases}$$

We shall now find the value of a p-th minor of (Π^0_{jk}). For convenience we may suppose this to be $|\,\Pi^0_{\lambda\mu}\,|$, $(\lambda,\ \mu = 1,\ \ldots,\ p)$. Then

$$(-)^p(2\theta)^{2p+1}\,|\,\Pi^0_{\lambda\mu}\,| = (-1)^p(2\theta)^{2p+1}\left|-\frac{1}{\rho}\,\rho_{\lambda\mu}\right|$$

$$= 2\theta\,|\,2\theta\,\theta_{\lambda\mu}-\theta_\lambda\,\theta_\mu\,|$$

$$= \begin{vmatrix} 2\theta & \theta_1 & \ldots & \theta_p \\ \theta_1 & \theta_{11} & \ldots & \theta_{1p} \\ \vdots & \vdots & & \vdots \\ \theta_p & \theta_{p1} & \ldots & \theta_{pp} \end{vmatrix}$$

$$= 2^{p+1}\begin{vmatrix} \theta & a_{\alpha 1}y^\alpha & \ldots & a_{\alpha p}y^\alpha \\ a_{1\alpha}y^\alpha & a_{11} & \ldots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{p\alpha}y^\alpha & a_{p1} & \ldots & a_{pp} \end{vmatrix},$$

where we have written y^0 for 1. Multiplying the λ-th row by y^λ and subtracting from the first, for $\lambda = 1,\ \ldots,\ p$, then doing the same thing for the columns, we have

$$(-)^p\,2^p\,\theta^{2p+1}\,|\,\Pi^0_{\lambda\mu}\,| = \begin{vmatrix} a_{\rho\sigma}y^\rho y^\sigma & a_{\rho 1}y^\rho & \ldots & a_{\rho p}y^\rho \\ a_{1\rho}y^\rho & a_{11} & \ldots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{p\rho}y^\rho & a_{p1} & \ldots & a_{pp} \end{vmatrix},$$

each element of the first row and column bring summed for

$$\rho,\ \sigma = 0,\ p+1,\ \ldots,\ n.$$

So we have

$$(5.21)\qquad (-)^p 2^p \theta^{2p+1} \left| \Pi^0_{\lambda\mu} \right| = \sum_{\rho,\sigma} \begin{vmatrix} a_{\rho\sigma} & a_{\rho 1} & \dots & a_{\rho p} \\ a_{1\sigma} & a_{11} & \dots & a_{1p} \\ \vdots & \vdots & & \vdots \\ a_{p\sigma} & a_{p1} & \dots & a_{pp} \end{vmatrix} y^\rho y^\sigma.$$

Since $|\Pi^0_{\lambda\mu}| = 0$ for $p > r$, all the coefficients on the right-hand side of (5.21) must in this case be zero. If $p \leqslant r$ at least one of these coefficients will not be zero, hence the rank of $(a_{\alpha\beta})$ is $r+1$, and we have the theorem :

D. *If* Π^i_{jk} *are the components of a projectively flat affine connection for which* $\Pi^0_{jk;\,e} = 0$, *then the characteristic function is the square root of a quadric,* $a_{\alpha\beta} Z^\alpha Z^\beta$ *say. If the rank of* $(a_{\alpha\beta})$ *is* $r+1$ *the rank of* (Π^0_{jk}) *will be* r.

In this case we can, by linear homogeneous transformations on $Z^0, \dots, Z^n$, reduce the equation of V_n to

$$(5.22)\qquad e_0 Z^{0^2} + \dots + e_r Z^{r^2} = 1.$$

If $a_{\alpha\beta}$ are real we can do this in the field of real numbers and $e_\lambda = \pm 1$.

Then some of the paths on V_n will be straight lines and others will be conic sections. The affine parameters on the former will be their affine parameters when they are regarded as straight lines in A_{n+1}. The affine parameters on the latter will be of the form $at+\beta$, where a and β are constant on each path and t is the affine length.

It is perhaps worth remarking that the components Π^0_{jk} are proportional to the coefficients of the affine quadratic form* for V_n. The equation

$$(5.23)\qquad \Pi^0_{jk}\, dy^j\, dy^k = 1$$

defines, at any point, a locus in the tangent space analogous to the Dupin indicatrix. We shall justify these statements, and then consider the situation when A_{n+1} is endowed with a Euclidean metric.

Blaschke, in the 2-dimensional case, defines a covariant tensor-density of the second order by means of the determinants

$$(5.24)\qquad \begin{cases} L = (Z_u Z_v Z_{uu}), \\ M = (Z_u Z_v Z_{uv}), \\ N = (Z_u Z_v Z_{vv}), \end{cases}$$

* W. Blaschke, *loc. cit.*, 105.

where Z_u, Z_{uu}, etc., denote $\partial Z/\partial u$, $\partial^2 Z/\partial u^2$, etc., and $Z = Z(u, v)$ is the equation, in vector notation, to a surface in 3-dimensional affine space. In case $LN - M^2 \neq 0$ he defines a tensor by means of the relations

$$E, F, G = \frac{L, M, N}{\sqrt{(LN - M^2)}}. \tag{5.25}$$

In $(n+1)$-dimensional space equations analogous to (5.25) are

$$\omega_{ij} = \frac{1}{\sqrt{l}}\, l_{ij}, \tag{5.26}$$

where the sub-space in question is defined by the equations

$$Z^\alpha = \theta^\alpha(x^1, \ldots, x^n), \tag{5.27}$$

and l_{ij} and l are given by

$$l_{ij} = \begin{vmatrix} \dfrac{\partial^2 \theta^0}{\partial x^i \partial x^j} & \cdots & \dfrac{\partial^2 \theta^n}{\partial x^i \partial x^j} \\ \dfrac{\partial \theta^0}{\partial x^1} & \cdots & \dfrac{\partial \theta^n}{\partial x^1} \\ \vdots & & \vdots \\ \dfrac{\partial \theta^0}{\partial x^n} & \cdots & \dfrac{\partial \theta^n}{\partial x^n} \end{vmatrix}, \qquad l = |\, l_{ij}\,|. \tag{5.28}$$

In order to obtain a tensor whose components shall be proportional to l_{ij} we need not assume that $l \neq 0$. If we write

$$Z^\alpha = e^{x^0} \theta^\alpha(x^1, \ldots, x^n), \tag{5.29}$$

we see at once that

$$l_{ij} = \left|\frac{\partial Z}{\partial x}\right| \frac{\partial x^0}{\partial Z^\alpha} \frac{\partial^2 Z^\alpha}{\partial x^i \partial x^j}, \tag{5.30}$$

and, since Z^α are independent solutions of (1.1), the components of the tensor in question will be

$$\Pi^0_{jk} = \left|\frac{\partial Z}{\partial x}\right|^{-1} l_{ij}. \tag{5.31}$$

We shall now suppose that A_{n+1} is the bearer of a metric defined by

$$d\sigma^2 = \delta_{\alpha\beta}\, dZ^\alpha\, dZ^\beta \quad (\delta_{\alpha\beta} = \delta^\alpha_\beta), \tag{5.32}$$

and find the relation between σ and $\bar{s}$. Let

$$Z^\alpha = a^\alpha u + b^\alpha v \tag{5.33}$$

be the equations to any plane P_2 through the centre of A_{n+1}. We require that a and b satisfy

$$(5.34) \qquad \begin{cases} \delta_{\alpha\beta} a^\alpha a^\beta = \delta_{\alpha\beta} b^\alpha b^\beta = 1, \\ \delta_{\alpha\beta} a^\alpha b^\beta = 0, \end{cases}$$

in which case u and v will be rectangular Cartesian coordinates in P_2. We further suppose that P_2 meets A_n, defined by $Z^0 = 1$, and V_n, defined by (4.4). The equations to P_1 and C_1, the curves in which P_2 intersects A_n and V_n, will be, respectively,

$$(5.35) \qquad \begin{cases} (a) \quad a^0 u + b^0 v = 1, \\ (b) \quad \phi(au + bv) = 1. \end{cases}$$

ϕ will be homogeneous of degree unity. Let u, v and $\bar{u}$, $\bar{v}$ be two sets of variables subject to the conditions $a^0 u + b^0 v = 1$, $\phi(a\bar{u} + b\bar{v}) = 1$, respectively. Let $\rho(\xi, \eta) \equiv \phi(a\xi + b\eta)$. Then ρ will be homogeneous of degree unity. Let ds and $d\sigma$ be, respectively, the element of arc on P_1 and C_1. Then, from (5.35),

$$(5.36) \qquad \begin{cases} \dfrac{du}{b^0} = \dfrac{-dv}{a^0} = \dfrac{ds}{A}, \\ \dfrac{d\bar{u}}{\rho_{\bar{v}}} = \dfrac{-d\bar{v}}{\rho_{\bar{u}}} = \dfrac{d\sigma}{\sqrt{(\Delta\rho)}}, \end{cases}$$

where $\rho_{\bar{u}} = \dfrac{\partial\rho(\bar{u}, \bar{v})}{\partial\bar{u}}$, $\rho_{\bar{v}} = \dfrac{\partial\rho(\bar{u}, \bar{v})}{\partial\bar{v}}$, $A = \sqrt{(a^{0^2} + b^{0^2})}$, and $\Delta\rho = \rho_{\bar{u}}^2 + \rho_{\bar{v}}^2$.

Since ρ is homogeneous of degree unity, we have

$$(5.37) \qquad \begin{cases} \dfrac{ds}{A} = v\,du - u\,dv, \quad \text{since} \quad a^0 u + b^0 v = 1, \\ \dfrac{d\sigma}{\sqrt{(\Delta\rho)}} = \bar{v}\,d\bar{u} - \bar{u}\,d\bar{v}, \quad \text{since} \quad \rho(\bar{u}, \bar{v}) = 1, \end{cases}$$

whence $$\frac{A}{\sqrt{(\Delta\rho)}} \frac{d\sigma}{ds} = \frac{\bar{v}\,d\bar{u} - \bar{u}\,d\bar{v}}{v\,du - u\,dv} = \left(\frac{\bar{v}}{v}\right)^2 \frac{d(\bar{u}/\bar{v})}{d(u/v)}.$$

But from (4.8) we have

$$\frac{\bar{u}}{u} = \frac{\bar{v}}{v} = \frac{1}{\rho},$$

and therefore

$$(5.38) \qquad \frac{A}{\sqrt{(\Delta\rho)}}\, d\sigma = \frac{1}{\rho^2}\, ds.$$

Since $\bar{s}$ was only defined as an affine parameter we may take $\bar{s}A$ for $\bar{s}$ and from (5.38) and (1.9) we see that the relation between $\bar{s}$ and σ is given by

$$d\bar{s} = \frac{d\sigma}{\sqrt{(\Delta\rho)}}. \tag{5.39}$$

The ratio $d\bar{s}/d\sigma$ will be constant if, and only if, $\Delta\rho$ is constant on C_1. This will be the case if, and only if*,

$$\Delta\rho = F(\rho). \tag{5.40}$$

Since ρ is homogeneous of degree unity, $\Delta\rho$ and so $F(\rho)$ are homogeneous of degree zero. From this it follows that $F(\rho)$ is a constant, which we may take to be unity since it can be absorbed into $\bar{s}$. Then we shall have

$$\rho_u^2+\rho_v^2 = 1. \tag{5.41}$$

We shall show that the only solutions to this differential equation which are homogeneous of degree unity are

$$\begin{cases} u\cos\theta+v\sin\theta, \\ \pm\sqrt{(u^2+v^2)}. \end{cases} \tag{5.42}$$

From (5.41), and since ρ_u and ρ_v are homogeneous of degree zero, we have

$$\begin{cases} (a) & \begin{cases} \rho_u\rho_{uu}+\rho_v\rho_{vu} = 0, \\ u\rho_{uu}+ v\rho_{uv} = 0, \end{cases} \\ (b) & \begin{cases} \rho_u\rho_{uv}+\rho_v\rho_{vv} = 0, \\ u\rho_{vu}+ v\rho_{rv} = 0. \end{cases} \end{cases} \tag{5.43}$$

Hence either

$$\rho_{uu} = \rho_{uv} = \rho_{vv} = 0, \tag{5.44}$$

* If $\psi(x, y)$ is constant on the curves defined by $\phi(x, y) = \text{const.}$, we can find dx and dy, not both zero, which satisfy

$$\phi_x dx+\phi_y dy = 0,$$

$$\psi_x dx+\psi_y dy = 0.$$

It follows that

$$\frac{\partial(\phi, \psi)}{\partial(x, y)} = 0,$$

and so

$$\psi = F(\phi).$$

$\Delta\rho$ is homogeneous and so, if it is constant on C_1, it will be constant on the curves defined by $\rho = \text{const.}$

in which case, since $\Delta\rho = 1$,

$$\rho = u\cos\theta + v\sin\theta, \tag{5.45}$$

or else

$$\begin{cases} \rho_u = \lambda u, \\ \rho_v = \lambda v. \end{cases} \tag{5.46}$$

Then $\lambda^2(u^2+v^2) = 1$, from (5.41),

and $\rho = \rho_u u + \rho_v v = \lambda(u^2+v^2)$.

So we have

$$\rho = \pm\sqrt{(u^2+v^2)}. \tag{5.47}$$

Hence the intersection of P_2 and V_n must be either a straight line or a circle with its centre at the origin. But P_2 is any plane which intersects A_n and V_n. Therefore V_n must be either a hyperplane or a hypersphere with its centre at the centre of A_{n+1}. In the first case the projectively flat space associated with it is Euclidean, and in the second case it is elliptic.

THE SOLUTION OF LINEAR CONGRUENCES IN QUATERNIONS

By D. E. LITTLEWOOD.

[Received 27 November, 1929.—Read 12 December, 1929.]

1. It is the purpose of this paper to establish methods of solution for linear congruence equations in quaternions. Both the simple form, $Ax = B$, and the general linear form, $Ax+Bxi+Cxj+Dxk = E$, will be considered. In general, real moduli, both prime and composite, over the real field, will be taken. The right-handed or left-handed congruences* with prime quaternion, or composite quaternion moduli, will be shown to be dependent upon these cases.

2. We first consider sets of residues (mod N), where N is a real and complex prime, *i.e.* a prime of the form $4n-1$. For real residues mod N, -1 is a non-quadratic residue, and thus the complex residues $a+ib$, mod N, form a Galois field. If α is a primitive root, then 0 and α^r, $0 < r \leqslant N^2 - 1$, form a complete set of residues mod N.

Every rational function of α is a complex residue, and is equivalent to a single power of α or to zero. We shall show how to obtain this power.

Let $\alpha = a+ib$.

1, α, and α^2 must be linearly dependent over the real field, and thus α must satisfy a quadratic equation. With a suitable multiplier to bring the constant term to unity, α satisfies an equation of the form

$$1+p\alpha+q\alpha^2 \equiv 0.$$

Now α^N may be found by means of the binomial theorem, and, since N is prime and divides all the middle terms, these vanish (mod N).

* A. R. Richardson, "Congruences in real quaternions", *Proc. London Math. Soc.*, 31 (1930), 117–122.

Thus $$\alpha^N = (a+ib)^N \equiv a^N + i^N b^N \equiv a - ib = \bar{\alpha}.$$

Therefore $\alpha^{N+1} \equiv \alpha.\bar{\alpha} = a^2+b^2$, which is real.

All powers of the form $\alpha^{r(N+1)}$ will have real values, and thus a table of the real powers of α may be constructed.

To obtain the equivalent power of α from a polynomial in α, first replace each $\alpha^{r(N+1)}$ by the equivalent real, and thus obtain a polynomial of degree less than or equal to N. Divide this polynomial, mod N, by $1+p\alpha+q\alpha^2$ until the remainder is a multiple of a single power of α. This will occur within N terms of the highest power of α, for, since the polynomial is equivalent to a single power of α, it is equivalent to a multiple of a power in this range.

Replacing the coefficient by the equivalent power of α, we obtain the result.

A rational function is the quotient of two polynomials, and may thus be expressed as the quotient of two powers of α.

Example. Mod 7, $\alpha = 1+2i$ is a primitive root.

$$1+\alpha+3\alpha^2 \equiv 0.$$

$$\alpha^8 \equiv 1^2+2^2 = 5, \qquad \alpha^{16} \equiv 4, \qquad \alpha^{24} \equiv 6,$$

$$\alpha^{32} \equiv 2, \qquad \alpha^{40} \equiv 3, \qquad \alpha^{48} \equiv 1.$$

$$P = \frac{\alpha+5\alpha^9+5\alpha^{16}}{4+3\alpha^2+\alpha^7}.$$

$$\alpha+5\alpha^9+5\alpha^{16} \equiv \alpha+4\alpha+6 = 6+5\alpha.$$

Using detached coefficients and addition to 7 or a multiple, we have

6	5					
6	6	4				
	6	6	4			
		4	4	5		
			6	6	4	
				3	3	2
6	$+5\alpha$					$+2\alpha^6 \equiv 0,$

and so $$6+5\alpha \equiv 5\alpha^6 \equiv \alpha^{14}.$$

$4+3a^2+a^7$.

4	0	3	0	0	0	0	1			
4	4	5								
	3	3	2							
		2	2	6						
			3	3	2					
				5	5	1				
					6	6	4			
						2	2	6		
							1	1	3	
4	$+3a^2$						$+a^7$		$+3a^{10}\equiv 0$.	

$$4+3a+a^7 \equiv 4a^{10} \equiv a^{26}.$$

$$P \equiv \frac{a^{14}}{a^{26}} \equiv a^{-12} \equiv a^{36}.$$

To avoid exceptional cases, we say, conventionally, that $0 = a^{-\infty}$.

A complete set of residues is given by a^r, $0 < r \leqslant N-1$, or $r = -\infty$. It will be seen that, if $-\infty$ is supposed to have the usual properties, the following equations still hold for this index.

Quaternion residues.

3. A complete set of quaternion residues is given by

$$a^r+a^sj, \quad 0 < r \leqslant N^2-1, \quad \text{or} \quad r = -\infty, \quad 0 < s \leqslant N^2-1, \quad \text{or} \quad s = -\infty.$$

$$a^rj = j\bar{a}^s = ja^{N_s}.$$

The product of two residues a^r+a^sj, a^t+a^uj is clearly

$$(a^{s+t}-a^{s+N_u})+(a^{s+N_t}+a^{r+u})j.$$

The sums or differences may be expressed as single powers by the method described above.

Factors of zero.

We are now in a position to obtain the condition that a given residue is a factor of zero (mod N), and to estimate the number of such factors of zero. We obtain the condition that a^r+a^sj should be a left-hand factor of zero, but it is readily seen that the same condition ensures that it shall be a right-hand factor of zero.

There must be indices t and u, not both equal to $-\infty$, such that the product of $\alpha^r+\alpha^s j$ and $\alpha^t+\alpha^u j$ vanishes. Hence

$$\begin{aligned}
\alpha^{r+t}-\alpha^{s+Nu}+(\alpha^{r+u}+\alpha^{s+Nt}) &\equiv 0 && \pmod N,\\
\alpha^{r+t}-\alpha^{s+Nu} &\equiv 0 && \pmod N,\\
\alpha^{r+u}+\alpha^{s+Nt} &\equiv 0 && \pmod N,\\
r+t-s-Nu &\equiv 0 && [\mathrm{mod}\ (N^2-1)],\\
r+u-s-Nt &\equiv \tfrac{1}{2}(N^2-1) && [\mathrm{mod}\ (N^2-1)].
\end{aligned}$$

For we have $\alpha^{\frac{1}{2}(N^2-1)} = -1$.

Eliminating t and u, we obtain

$$\begin{aligned}
(N+1)r-(N+1)s &\equiv \tfrac{1}{2}(N^2-1) && [\mathrm{mod}\ (N^2-1)],\\
r-s &\equiv \tfrac{1}{2}(N-1) && [\mathrm{mod}\ (N-1)]. \qquad (1)
\end{aligned}$$

This condition (1) ensures that $\alpha^r+\alpha^s j$ is a factor of zero.

The index r may take any finite value, mod $(N-1)$, but not $-\infty$. The corresponding values of s will be congruent to $r+\frac{1}{2}(N-1)$, mod $(N-1)$. There will be $N+1$ such, and therefore $(N+1)(N^2-1)$ distinct factors of zero.

Let m be any factor of zero, α^{N+1} is scalar and therefore commutative with m. Then $\alpha^r m\alpha^s$, $0 < r \leqslant N+1$, $0 < s \leqslant N^2-1$ represent the $(N+1)(N^2-1)$ distinct factors of zero.

These are factors of zero, for, if m is a factor of zero, then

$$m\bar{m} \equiv 0,$$

$$\alpha^r m\alpha^s . \alpha^{-s}\bar{m} \equiv 0.$$

Secondly, they are all distinct, for, if not, let $\alpha^{r_1} m\alpha^{s_1} = \alpha^{r_2} m\alpha^{s_2}$. Suppose that $m = \alpha^p+\alpha^q j$, and that p and q are both finite,

$$\alpha^{r_1}(\alpha^p+\alpha^q j)\alpha^{s_1} \equiv \alpha^{r_2}(\alpha^p+\alpha^q j)\alpha^{s_2},$$

therefore

$$\begin{aligned}
r_1+s_1 &\equiv r_2+s_2 && [\mathrm{mod}\ (N^2-1)],\\
r_1+Ns_1 &\equiv r_2+Ns_2 && [\mathrm{mod}\ (N^2-1)],\\
(N-1)r_1 &\equiv (N-1)r_2 && [\mathrm{mod}\ (N^2-1)],\\
r_1 &\equiv r_2 && [\mathrm{mod}\ (N+1)].
\end{aligned}$$

By hypothesis $s_1 = s_2$ and $r_1 = r_2$.

Thus all factors of zero may be expressed in the form $\alpha^r m\alpha^s$, where m is any factor of zero.

If A is not a factor of zero it has a unique inverse, and the equations $Ax = B$ and $xA = B$ have unique solutions.

4. We give three methods of solutions for the equations.

First method. Solution of $Ax \equiv B$.

Suppose that

$$A \equiv a^{a_1} + a^{a_2} j \quad \text{and} \quad B \equiv a^{b_1} + a^{b_2} j.$$

Let $$x \equiv a^r + a^s j.$$

From $Ax \equiv B$, equating the independent term and coefficient of j,

$$a^{a_1+r} - a^{a_2+Ns} \equiv a^{b_1},$$

$$a^{a_1+s} + a^{a_2+Nr} \equiv a^{b_2}.$$

Raising the second to the power N, we have

$$a^{Na_1+Ns} + a^{Na_2+r} \equiv a^{Nb_2}.$$

We have thus two linear equations in the unknowns a^r and a^{Ns}, and solving we have

$$a^r \equiv \frac{a^{Na_1+b_1} + a^{Nb_2+a_2}}{a^{(N+1)a_1} + a^{(N+1)a_2}},$$

$$a^s \equiv \frac{a^{b_2+Na_1} - a^{Nb_1+a_2}}{a^{(N+1)a_1} + a^{(N+1)a_2}}.$$

This gives a unique solution, provided that the denominator does not vanish. The vanishing of the denominator is, however, the condition that A is a factor of zero.

Second method. $Ax \equiv B$.

If A is not a factor of zero, then $A\bar{A}$, the product of A with its conjugate, is a non-vanishing scalar s, and we have

$$s^{N-1} \equiv 1,$$

$$(A\bar{A})^{N-2}\bar{A}Ax \equiv (A\bar{A})^{N-2}\bar{A}B,$$

$$x \equiv (A\bar{A})^{N-2}\bar{A}B.$$

This method again fails if A is a factor of zero, for then $A\bar{A} = 0$.

Third method. We have the following extension of Fermat's theorem for quaternion residues*. Every quaternion residue x satisfies the

* A. R. Richardson and D. E. Littlewood, "Fermat's theorem in quaternions". (Not yet published.)

equation $$x^2(x^{N(N^2-1)}-1) \equiv 0.$$

If x is not a factor of zero, then $x^{N(N^2-1)} \equiv 1$.

The solution, then, of $Ax \equiv B$ is

$$x \equiv A^{N^3-N-1}B.$$

By any of these methods the solution of $xA \equiv B$ may be found.

The solution of $Ax \equiv B$ when A is a factor of zero.

5. If $Ax \equiv B$, we have also

$$A(x+\bar{A}C) \equiv B,$$

C being any quaternion.

If, then, there is a solution x, there are N solutions, all congruent to x mod $\bar{A}$, considered as a right-handed congruence*. But 0 and a^s, $0 < s \leqslant N^2-1$, form a complete set of residues mod $\bar{A}$, right-hand congruence. In consequence there is a solution of the form a^s.

If $A \equiv a^{a_1}+a^{a_2}j$, $B \equiv a^{b_1}+a^{b_2}j$, then clearly

$$x \equiv a^{l_1-a_1}$$

is a solution if the equation is solvable, and the condition that it should be solvable is

$$a^{a_2}jx \equiv a^{b_2}j,$$

i.e., $$a_2+N(b_1-a_1) = b_2,$$

or $$a_2-Na_1 = b_2-Nb_1.$$

In this case the general solution is given by

$$x \equiv a^{b_1-a_1}+\bar{A}C,$$

or $$a^{b_1-a_1}+\bar{A}a^s$$

$$(0 < s \leqslant N^2-1 \text{ or } s = -\infty).$$

The solution of $xA \equiv B$ may be obtained in exactly the same way. The condition for a solution, if A is a factor of zero, is

$$a_2-a_1 = b_2-b_1.$$

Extension for prime moduli of the form $4n+1$.

6. Let c be the least real integer which is not a quadratic residue mod N, where $N = 4n+1$. Then also $-c$ is not a quadratic residue.

* A. R. Richardson, *loc. cit.*

If $\epsilon^2 = -c$, then the residues $a+b\epsilon$, where a and b are real residues mod N, form a Galois field, and there is a primitive root α giving $0, \alpha, \alpha^2, \ldots, \alpha^{N^2-1} \equiv 1$ as a complete set of residues.

$c-1$ is a quadratic residue congruent to d^2 mod N.

Consider quaternions of the form $a+b(di+k)$, where a and b are real residues mod N. These are simply isomorphic with the Galois field considered above, and we shall have a primitive root α.

We have, as before,
$$\alpha^N \equiv \bar{\alpha} = a-b\epsilon,$$
$$\alpha j \equiv j\bar{\alpha} \equiv j\alpha^N.$$

The quaternions
$$\alpha^r + \alpha^s j \quad (0 < r \leqslant N-1 \text{ or } r = -\infty)$$
$$(0 < s \leqslant N-1 \text{ or } s = -\infty)$$

form a complete set of quaternion residues mod N.

As before, α will satisfy an equation $1+p\alpha+q\alpha^2 \equiv 0$, and any rational function of α may be expressed as a power of α by the method given.

The whole of the previous work extends to prime moduli of this form without modification.

The solution of the general linear congruence.

7.
$$Ax+Bxi+Cxj+Dxk \equiv P \pmod{N}, \tag{1}$$

A, B, C, D, and P being quaternion residues mod N.

Multiply on the right by i, j, and k respectively,

$$-Bx+Axi+Dxj-Cxk \equiv Pi,$$
$$-Cx-Dxi+Axj+Bxk \equiv Pj,$$
$$-Dx+Cxi-Bxj+Axk \equiv Pk.$$

Solving for x, we have

$$\begin{vmatrix} A & B & C & D \\ -B & A & D & -C \\ -C & -D & A & B \\ -D & C & -B & A \end{vmatrix} x \equiv \begin{vmatrix} P & B & C & D \\ Pi & A & D & -C \\ Pj & -D & A & B \\ Pk & C & -B & A \end{vmatrix}. \tag{2}$$

If the determinant coefficient of x, which we shall call Δ, is neither zero nor a factor of zero, this equation may be solved uniquely by the methods given above, and we obtain the unique solution of our equation.

If Δ vanishes the solution is not unique, and indeed there may be no solution. The first condition that the equation is solvable is that the left

rank of the right-hand determinant in (2) does not exceed the rank of Δ. If this condition is satisfied, and if, further, the non-vanishing minors of highest degree in Δ are not all factors of zero, then the equation can be solved in terms of parameters*. There are N, N^2, or N^3 solutions according as the rank of Δ is 3, 2, or 1.

This method breaks down, however, if the determinant Δ is a factor of zero, or if the non-vanishing minors of highest degree are factors of zero. An example will show this.

Take $N = 5$. $$x+(1+i)xi \equiv a.$$

It is sufficient in this instance to multiply on the right by i only. We obtain

$$-(1+i)x+xi \equiv ai.$$

Eliminating xi, we have

$$(1+2i)x \equiv a-(1+i)ai.$$

Since $1+2i$ is a factor of zero, there are N^2 solutions, if the equation is solvable. But not all of these will satisfy the original equation. This has but N solutions. The N^2 solutions of the second equations include the N solutions of each of the equations

$$x+(1+i)xi \equiv y,$$

where y takes the $N-1$ values which are, apart from a, solutions of

$$y-(1+i)yi \equiv a-(1+i)ai.$$

We have a similar difficulty when the non-vanishing minors of highest degree are all factors of zero.

Linear congruences with N solutions, for which the determinant is a factor of zero, may be solved as follows.

Suppose that our equation is

$$ax+bxi+cxj+dxk \equiv e. \tag{1}$$

From this, as has been shown, we obtain

$$\Delta x \equiv D. \tag{2}$$

Since Δ is a factor of zero, this has N^2 solutions. But (1) may also be written in the form

$$xa'+ixb'+jxc'+kxd' \equiv e. \tag{3}$$

* A. R. Richardson, "Simultaneous linear equations over a division algebra", *Proc. London Math. Soc.* (2), 28 (1928), 395.

By multiplying this on the right by i, j, and k, and by right-hand determinants,

$$x\Delta' = D'. \tag{4}$$

From (2), $$x = \alpha^r + \bar{\Delta}\alpha^n,$$

or $$\alpha^r + \alpha^s m \alpha^n, \tag{5}$$

where m is any given factor of zero, r and s can be found, but n is arbitrary.

Substituting in (4), and taking the term in α^r to the right-hand side, and also substituting for the factor of zero Δ' in the form $\alpha^p m \alpha^q$, $\alpha^s m \alpha^n \alpha^p m \alpha^q$ is equal to a factor of zero. This may be written

$$\alpha^s m \alpha^n \alpha^p m \alpha^q = \alpha^t m \alpha^u.$$

This can be satisfied only if $s = t$ [mod $(N+1)$] and $q = u$ [mod $(N+1)$]. Our equation then becomes

$$m\alpha^{p_1+n} m \equiv m \pmod{N}.$$

Since m is any factor of zero, we may take it to be $1+\alpha^{\frac{1}{2}(N-1)}j$. We have

$$(1+\alpha^{\frac{1}{2}(N-1)}j)\,\alpha^{p_1+n}(1+\alpha^{\frac{1}{2}(N-1)}j) \equiv 1+\alpha^{\frac{1}{2}(N-1)}j,$$

whence $$\alpha^{n+p_1} + \alpha^{N(n+p_1)} \equiv 1,$$

$$\alpha^{n+p_1} + \bar{\alpha}^{n+p_1} \equiv 1.$$

The scalar part of α^{n+p_1} is $\frac{1}{2}(N+1)$.

Let the N powers of α with scalar part equal to $\frac{1}{2}(N+1)$ be α^{e_1}, α^{e_2}, α^{e_3}, ..., α^{e_n}.

Then $$\alpha^n = \alpha^{e_i - p_1} \quad (i = 1, 2, 3, 4, \ldots, N).$$

If these values be substituted in (5), the N solutions of (1) will be obtained.

The case when Δ vanishes, and the non-vanishing minors of highest degree are all factors of zero, remains outstanding. A similar treatment to that given above would present great difficulty, but we can always substitute $x = x_0 + x_1 i + x_2 j + x_3 k$, each x_i being real, and solve the four resulting simultaneous equations in real variables, obtained by equating to zero the real part and the coefficients of i, j, and k respectively.

Quaternion moduli.

8. Linear equations may be given as left- (or right-) handed congruences, with a prime quaternion modulus*. Suppose m to be the prime

* A. R. Richardson, "Congruences in real quaternions", *loc. cit.*

quaternion, and let the norm of m be N. Then N is a real prime. One solution is always given by the same equation taken with modulus N.

Solution of $Ax \equiv B \pmod{m}$.

This is supposed to be a left-handed congruence.

Let x_0 be the solution of $Ax \equiv B \pmod{N}$.

Let $x = x_0 + \alpha$. Then $A\alpha \equiv 0 \pmod{m}$.

If A is not a factor of zero, $\alpha \equiv 0 \pmod{m}$.

The unique solution is given by the solution of the equation taken with modulus N, provided that this solution be taken as a residue mod m.

In the solution of other linear congruences with a prime quaternion modulus, the solution will only be given as a residue mod N, for there will be quaternions congruent to the solution mod m which will not satisfy the equation.

For example, if $xA \equiv B \pmod{m}$, then

$$(x+\mu m)A \equiv B + \mu mA \not\equiv B \pmod{m}.$$

Since an equation, given mod m, is equivalent to N^2 alternative equations taken mod N, there will in general be N^2 solutions.

If $$Ax + Bxi + Cxj + Dxk \equiv E \pmod{m},$$

one solution is given by x_0, the solution of the same equation mod N.

Put $$x = x_0 + y.$$

Then $$Ay + Byi + Cyj + Dyk \equiv 0 \pmod{m},$$

$$\equiv \alpha^s m \pmod{N}.$$

Hence
$$y \equiv \begin{vmatrix} \alpha^s m & B & C & D \\ \alpha^s mi & A & D & -C \\ \alpha^s mj & -D & A & B \\ \alpha^s mk & C & -B & A \end{vmatrix}.$$

The other N^2-1 values of x are given by the N^2-1 different indices s.

Alternatively we may multiply our equation on the right by $\bar{m}$, and thus obtain a singular linear equation with modulus N, having N^2 solutions.

Simultaneous equations with prime quaternion moduli.

Since the linear equation with a prime quaternion modulus m has N^2 solutions mod N, we may expect to get a unique solution from two such equations. These may have different quaternion moduli, provided that these have the same norm N. Suppose that they are

$$Ax + Bxi + Cxj + Dxk \equiv E \pmod{m},$$

$$A'x + B'xi + C'xj + D'xk \equiv E' \pmod{m'}.$$

Multiply by $x\bar{m}$ and $\bar{m}'$ respectively, and thus obtain two singular equations mod N, say $\phi = 0$ and $\psi = 0$.

If the two original equations are independent, then a value for λ can be found, such that the linear form $\phi + \lambda\psi$ has a non-vanishing determinant.

The unique solution can be obtained from the equation

$$\phi + \lambda\psi = 0.$$

Composite moduli.

9. We shall assume henceforward, for simplicity, that the equations are non-singular, and thus have a unique solution. The singular cases present no new difficulties.

Let $\phi(x)$ represent any homogeneous linear form.

To solve $$\phi(x) \equiv \alpha \pmod{N^r}.$$

Let $$x = x_0 + x_1 N + x_2 N^2 + X_{r-1} N^{r-1}.$$

We shall show how to find values consecutively for x_0, x_1, etc.; x_0 is the solution of

$$\phi(x) \equiv \alpha \pmod{N}.$$

We have $$\phi(x - x_0) = N\phi(x_1) \pmod{N^2},$$

$\alpha - \phi(x_0)$ is divisible by N. So we have x_1 as the solution of

$$\phi(x_1) \equiv [\alpha - \phi(x_0)]/N \pmod{N}.$$

Similarly $$\phi(x_2) \equiv [\alpha - \phi(x_0) - N\phi(x_1)]/N^2.$$

Proceeding thus we obtain the complete solution.

The solution of $\phi(x) \equiv \alpha \pmod{N^r M^s}$ may be obtained as follows: Let the solution of $\phi(x) \equiv \alpha \pmod{N^r}$ be x_1, and the solution of $\phi(x) \equiv \alpha \pmod{M^s}$ be x_2.

Then $$x \equiv x_1 + AM^s \equiv x_2 + BN^r \pmod{N^r M^s}.$$

Since N^r and M^s are mutually prime, there are integers p and q such that

$$pN^r+qM^s \equiv 1 \pmod{N^rM^s},$$

$$pN^rx+qM^sx \equiv pN^rx_1+qM^sx_2 \pmod{N^rM^s},$$

$$x \equiv pN^rx_1+qM^sx_2.$$

A repetition of this procedure gives the result for any real composite modulus.

There remains the case of the left- or right-handed congruence with a composite quaternion modulus. Suppose the modulus to be m, a quaternion with norm $N = N_1N_2$, and suppose it to be a left-hand congruence,

$$\phi(x) \equiv a \pmod{m}.$$

The equation $\phi(x) \equiv a \pmod{m, N_1}$ is equivalent to the simple left-hand congruence we have already considered, m being considered as a factor of zero, mod N_1. There is a solution $x = x_1 \pmod{N_1}$. x_1 may take any of N_1^2 values.

Similarly $\phi(x) \equiv a \pmod{m, N_2}$ has the solution $x \equiv x_2 \pmod{N_2}$, where x_2 may take any of N_2^2 values.

As we have already shown, these lead to the solution

$$x = pN_1x_1+qN_2x_2 \pmod{N_1N_2},$$

where

$$pN_1+qN_2 = 1 \pmod{N_1N_2}.$$

There will thus be $N_1^2N_2^2 = N^2$ solutions. More composite moduli may be treated by the same method.

10. *An example.* To illustrate the methods, the following congruence is here solved.

$$x+(1+i)xi+(i+j+k)xj \equiv 1 \pmod{175}.$$

Multiplying on the right by i, j, k respectively, and solving the four linear equations, we obtain

$$\begin{vmatrix} 1 & 1+i & 1+j+k & \\ -1-i & 1 & & -1-j-k \\ -1-i-j & & 1 & 1+i \\ & 1+j+k & -1-i & 1 \end{vmatrix} x$$

$$\equiv \begin{vmatrix} 1 & 1+i & 1+j+k & \\ i & 1 & & -1-j-k \\ j & & 1 & 1+i \\ k & 1+j+k & -1-i & 1 \end{vmatrix}.$$

Whence $(10-2i+2j-6k)x \equiv 8-14i-6j-8k \pmod{175}$,

$$(-2i+2j-k)x \equiv 3+i-j+2k \pmod{5}.$$

Take $\alpha = 1+i+k$. This satisfies $1+\alpha+2\alpha^2 \equiv 0 \pmod{5}$,

$$[i+k+(2+3i+3k)j]x \equiv 3+4i+4k+(-1+3i+3k)j$$

$$(\alpha^3+\alpha^{23}j)x \equiv \alpha^{20}+\alpha^{10}j$$

$$(1+\alpha^{20}j)x \equiv \alpha^{17}+\alpha^7 j$$

$$(1-\alpha^{20}j)(1+\alpha^{20}j)x \equiv (1-\alpha^{20}j)(\alpha^{17}+\alpha^7 j)$$

$$2x \equiv \alpha^{17}+\alpha^7+(\alpha^7-\alpha^9)j$$

$$\equiv \alpha^{16}+\alpha^6 j$$

$$x \equiv 1+2i-j+2k \pmod{5}.$$

Take $x_0 = 1+2i-j+2k$,

$$(10-2i+2j-6k)(x-x_0) \equiv 8-14i-6j-8k \pmod{175}$$

$$-28-16i+16j-12k$$

$$\equiv -20-30i+10j-20k,$$

$$(10-2i+2j-6k)\tfrac{1}{5}(x-x_0) \equiv -4-6i+2j-4k \pmod{35}.$$

Put $x = 5x_1+x_0 \pmod{25}$.

Then $(\alpha^3+\alpha^{23}j)x_1 \equiv 1-i+2j+k \pmod{5}$

$$\equiv 1+\alpha^8 j$$

$$(1+\alpha^{20}j)x_1 \equiv \alpha^{21}+\alpha^5 j$$

$$(1-\alpha^{20}j)(1+\alpha^{20}j)x_1 \equiv (1-\alpha^{20}j)(\alpha^{21}+\alpha^5 j)$$

$$\equiv \alpha^{21}+\alpha^{21}+(\alpha^5-\alpha^5)j$$

$$2x_1 \equiv \alpha^{15}$$

$$x_1 \equiv 2i+2k \pmod{5}.$$

So that $x \equiv 1+12i-j+12k \pmod{25}$.

Also $(3-2i+2j+k)x \equiv 1+j-k \pmod{7}$.

Take $\alpha = 1+2i$. This satisfies $1+\alpha+3\alpha^2 \equiv 0 \pmod{7}$.

Then
$$\begin{aligned}
(\alpha^3+\alpha^{43}j)x &\equiv 1+\alpha^{46}j \\
(1+\alpha^{40}j)x &\equiv \alpha^{45}+\alpha^{43}j \\
(1-\alpha^{40}j)(1+\alpha^{40}j)x &\equiv (1-\alpha^{40}j)(\alpha^{45}+\alpha^{43}j) \\
&\equiv (\alpha^{45}+\alpha^5)+(\alpha^{43}+\alpha^{19})j \\
\alpha^{40}x &\equiv \alpha^{21}+\alpha^{27}j \\
x &\equiv \alpha^{29}+\alpha^{35}j \\
&\equiv 1-4i+6j-4k \qquad (\text{mod } 7), \\
x &\equiv 1-12i-j+12k \qquad (\text{mod } 25), \\
18\times 7+2\times 25 &\equiv -1 \qquad (\text{mod } 175),
\end{aligned}$$

so that
$$\begin{aligned}
x &\equiv 126(1+12i-j+12k)+50(1-4i+6j-4k) \\
&\equiv 1+87i+174j+87k \qquad (\text{mod } 175).
\end{aligned}$$

I wish to acknowledge my debt to Professor A. R. Richardson for introducing the subject to me, and for valuable advice.

A PROBLEM CONCERNING MEAN VALUES OF ANALYTIC FUNCTIONS

By L. S. BOSANQUET.

[Received and read 16 January, 1930.]

In a paper by Hardy, Ingham, and Pólya, published in 1927, the following theorem was stated and partly proved*.

If $f(z)$ is an analytic function of $z=x+iy$, regular in the strip $\alpha<x<\beta$, and if $p>0$ and $\frac{1}{2T}\int_{x-iT}^{x+iT}|f(z)|^p dz$ is bounded in the strip, and if $\lim_{T\to\infty}\frac{1}{2T}\int_{x-iT}^{x+iT}|f(z)|^p dz$ exists on an infinity of lines in the strip which have a limit line in the strip, then $\lim_{T\to\infty}\frac{1}{2T}\int_{x-iT}^{x+iT}|f(z)|^p dz$ exists on all interior lines of the strip, and uniformly in any strip $\alpha<\alpha'\leqslant x\leqslant\beta'<\beta$.

They proved the theorem for the case in which p is an even integer. Besicovitch has since shown† that the theorem is true for any $p>0$ if $n/T\to 0$, where n is the number of zeros in the rectangle $\alpha<x<\beta$, $-T<y<T$.

The object of this paper is to show that the theorem as stated is a best possible theorem, in the sense that there do exist functions of the type defined whose moduli possess mean values on any given number of lines in the strip and on no others, or on an infinity of lines in the strip and on no others, when they have no limit line in the strip. This problem has already been solved by other methods. M. Riesz has constructed a regular function, the modulus of which possesses a mean value on two lines and on no others, and Hardy, Ingham, and Pólya adapted his method to construct regular functions whose moduli possess mean values on a set of equally spaced lines. Independently, and by a

* "Mean values of analytic functions", *Proc. Royal Soc.* (A), 113 (1927), 542–569 (564).

† *Proc. London Math. Soc.* (2), 27 (1928), 373–388 (375).

different method, Besicovitch has constructed analytic functions whose moduli possess mean values on any finite number of lines, or on an infinity of lines. The method which I use is independent of both these, though it resembles Besicovitch's method and was suggested to me by him.

For simplicity I consider the case $p = 1$, but the same method applies without change when p has any positive value. The method is first to construct two auxiliary functions $\phi(z)$ and $\psi(z)$, which are regular and bounded in the strip and periodic with respect to y, the moduli of which therefore possess mean values $M_\phi(x)$ and $M_\psi(x)$ on every line of the strip. A third function $f(z)$ is then constructed, which is regular in the strip, and the modulus of which differs by a small amount from $|\phi(z)|$ and $|\psi(z)|$ in alternate stretches along the lines of the strip. And it is shown that these stretches may be chosen in such a way that

$$(1)\qquad \begin{cases} \overline{\lim}_{T\to\infty} \dfrac{1}{2T}\displaystyle\int_{-T}^{T} |f(x+iy)|\,dy = \operatorname{Max}\{M_\phi(x),\, M_\psi(x)\} \\ \text{and} \\ \underline{\lim}_{T\to\infty} \dfrac{1}{2T}\displaystyle\int_{-T}^{T} |f(x+iy)|\,dy = \operatorname{Min}\{M_\phi(x),\, M_\psi(x)\}. \end{cases}$$

It follows that $|f(z)|$ possesses a mean value only on those lines for which $M_\phi(x) = M_\psi(x)$. If, therefore, $\phi(z)$ and $\psi(z)$ be chosen in such a way that $M_\phi(x)$ is equal to $M_\psi(x)$ on an appropriate number of lines, $f(z)$ is a function of the type required.

I first consider the case of a finite number of lines. Let λ_m be the interval $(m-1)! \leqslant y < m!$ and λ_{-m} the interval $-m! \leqslant y < -(m-1)!$, where, for $m = 1$, $(m-1)!$ is replaced by 0. λ_m will also denote the *length* of the interval, and the intervals for which m is odd or even will be spoken of as the *odd* or *even intervals* respectively.

Let l be a positive integer, and let $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ and $\beta_1, \beta_2, \ldots, \beta_\kappa$ be two sets of real numbers.

Let

$$(2)\qquad \begin{cases} \phi(z) = \displaystyle\prod_{r=1}^{\kappa} \cosh(z-\alpha_r)\,l\pi, \\ \psi(z) = \displaystyle\prod_{r=1}^{\kappa} \cosh(z-\beta_r)\,l\pi, \\ f(z) = \displaystyle\lim_{m\to\infty} \prod_{r=1}^{\kappa} \prod_{n=-m}^{m-1} \left\{1 - \frac{2(z-\gamma_{r,n})\,l}{(2n+1)\,i}\right\}, \end{cases}$$

where $\gamma_{r,n} = \alpha_r$ if $(n+\frac{1}{2})/l$ belongs to an odd interval, and $\gamma_{r,n} = \beta_r$ if $(n+\frac{1}{2})/l$ belongs to an even interval. Thus $f(z)$ possesses those zeros of

$\phi(z)$ whose imaginary parts belong to odd intervals, and those zeros of $\psi(z)$ whose imaginary parts belong to even intervals.

Suppose first that y belongs to an odd interval. Let

$$\tau = \tau(y) = \text{Min}\,|\,(n+\tfrac{1}{2})/l - y\,|,$$

where n has any value such that $(n+\frac{1}{2})/l$ belongs to an *even* interval. Evidently $\tau \geqslant 1/(2l)$.

It follows from the definitions of $f(z)$ and $\phi(z)$ that

$$\frac{|f(z)|}{|\phi(z)|} = \prod_{r=1}^{\kappa} \prod_{n}' \left| \frac{(2n+1-2ly)i - 2(x-\beta_r)l}{(2n+1-2ly)i - 2(x-a_r)l} \right|, \tag{3}$$

where n runs through those values for which $(n+\frac{1}{2})/l$ belongs to an even interval. It will be seen that this is an absolutely convergent product, so that the order of the factors is immaterial. It may be written in the form

$$\prod_{r=1}^{\kappa} \prod_{n}' \left\{ \frac{(2n+1-2ly)^2 + 4(x-\beta_r)^2 l^2}{(2n+1-2ly)^2 + 4(x-a_r)^2 l^2} \right\}^{\frac{1}{2}}$$

$$= \prod_{r=1}^{\kappa} \prod_{n}' \left\{ \frac{1 + [(x-\beta_r)^2 / \{(n+\frac{1}{2})/l - y\}^2]}{1 + [(x-a_r)^2 / \{(n+\frac{1}{2})/l - y\}^2]} \right\}^{\frac{1}{2}}.$$

Now

$$\sum_{n}' \frac{1}{\{(n+\frac{1}{2})/l - y\}^2} < 2 \sum_{m=0}^{\infty} \frac{1}{\{\tau + (m/l)\}^2} < \frac{2}{\tau^2} + 2\int_0^{\infty} \frac{dt}{\{\tau + (t/l)\}^2}$$

$$= \frac{2}{\tau^2} + \frac{2l}{\tau} \leqslant \frac{6l}{\tau},$$

since $\tau \geqslant 1/(2l)$. It follows* that $A(x)$ can be chosen, independent of y, such that

$$\left| \frac{|f(x+iy)|}{|\phi(x+iy)|} - 1 \right| < \frac{A(x)}{\tau}. \tag{4}$$

provided only that y belongs to an odd interval.

Now, since $\phi(x+iy)$ is continuous and periodic with respect to y, it follows that $|\phi(x+iy)|$ is bounded along every line $x = \text{constant}$. Therefore $B(x)$ can be chosen, independent of y, such that

$$\Big| |f(x+iy)| - |\phi(x+iy)| \Big| < \frac{B(x)}{\tau}, \tag{5}$$

provided only that y belongs to an odd interval.

---

* If $x_n \geqslant 0$ and $\Sigma x_n \leqslant K$, then

$$0 \leqslant \Pi(1+x_n) - 1 \leqslant e^{\Sigma x_n} - 1 \leqslant M \Sigma x_n,$$

where $M = M(K)$.

Next suppose that y belongs to an even interval. Let

$$\sigma = \sigma(y) = \text{Min}\,|\,(n+\tfrac{1}{2})/l - y\,|,$$

where n has any value such that $(n+\frac{1}{2})/l$ belongs to an *odd* interval. Then $\sigma \geqslant 1/(2l)$, and $B(x)$ can also be chosen so that

$$\left|\,|\,f(x+iy)\,|-|\,\psi(x+iy)\,|\,\right| < \frac{B(x)}{\sigma}, \tag{6}$$

provided only that y belongs to an even interval.

It also follows that $C(x)$ can be chosen such that $|\,f(x+iy)\,| < C(x)$, for all values of y. Moreover, if $z = x+iy$ be restricted to a strip $a < x < b$, the functions $A(x)$, $B(x)$, and $C(x)$ are bounded, so that $|\,f(x+iy)\,|$ is bounded in the strip.

Since $|\,\phi(x+iy)\,|$ and $|\,\psi(x+iy)\,|$ are periodic with respect to y, with unit period, it follows that the mean value of each function along a line $x =$ constant is equal to its mean value along a unit length of the line. We have then

$$M_\phi(x) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |\,\phi(x+iy)\,|\,dy = \int_0^1 |\,\phi(x+iy)\,|\,dy, \tag{7}$$

and $M_\psi(x)$ is given by a similar formula. Evidently $M_\phi(x)$ and $M_\psi(x)$ are continuous functions of x.

I next prove the relations (1). Since $|\,f(x+iy)\,|$ is an even function of y, it is sufficient to consider the mean value over an interval $(0, T)$. Let U be the greatest integer not greater than T, and suppose that T and U lie in the interval λ_m, $m > 2$. Then $(m-1)! \leqslant U \leqslant T < m!$. *Suppose first that m is even*, and let $I = \frac{1}{T}\int_0^T |\,f(x+iy)\,|\,dy$. Then write

$$\begin{aligned} I = \left(\frac{1}{T}-\frac{1}{U}\right)\int_0^U |\,f\,|\,dy &+ \frac{1}{T}\int_U^T |\,f\,|\,dy + \frac{1}{U}\int_0^{(m-2)!} |\,f\,|\,dy \\ &+ \frac{1}{U}\int_{(m-2)!}^{(m-1)!} (|\,f\,|-|\,\phi\,|)\,dy + \frac{1}{U}\int_{(m-1)!}^{U} (|\,f\,|-|\,\psi\,|)\,dy \\ &+ \frac{1}{U}\int_{(m-2)!}^{U} |\,\phi\,|\,dy + \frac{1}{U}\int_{(m-1)!}^{U} (|\,\psi\,|-|\,\phi\,|)\,dy, \\ = I_1+I_2+I_3+I_4+I_5+I_6+I_7, & \end{aligned}$$

and let ϵ be any positive number.

Since $0 \leqslant T-U < 1$ and $|\,f\,| < C(x)$, we have

$$|\,I_1\,| < \frac{T-U}{TU}\int_0^U C(x)\,dy < \frac{C(x)}{T} \leqslant \frac{C(x)}{(m-1)!} < \epsilon,$$

provided that $m > m_1$, where $m_1 = m_1(x, \epsilon)$.

Since $0 \leqslant T-U < 1$, and $|f| < C(x)$, we have

$$|I_2| < \frac{1}{T}\int_U^T C(x)\,dx = \frac{T-U}{T}\,C(x) < \frac{C(x)}{T} \leqslant \frac{C(x)}{(m-1)!} < \epsilon,$$

provided that $m > m_2$, where $m_2 = m_2(x, \epsilon)$.

Since $|f| < C(x)$, we have

$$|I_3| < \frac{1}{U}\int_0^{(m-2)!} C(x)\,dy = \frac{(m-2)!}{U}\,C(x) \leqslant \frac{(m-2)!}{(m-1)!}\,C(x) = \frac{C(x)}{m-1} < \epsilon,$$

provided that $m > m_3$, where $m_3 = m_3(x, \epsilon)$.

Since $\big||f|-|\phi|\big| < B(x)/\tau$, when y lies in the odd interval λ_{m-1}, we have

$$|I_4| < \frac{1}{U}\int_{(m-2)!}^{(m-1)!} \frac{B(x)}{\tau}\,dy = \frac{2}{U}\int_{1/(2l)}^{1/(2l)+\frac{1}{2}\lambda_{m-1}} \frac{B(x)}{\tau}\,d\tau = \frac{2B(x)}{U}\log(1+l\lambda_{m-1})$$

$$< \frac{2B(x)}{\lambda_{m-1}}\log(1+l\lambda_{m-1}) < \epsilon,$$

provided that $m > m_4$, where $m_4 = m_4(x, \epsilon)$.

If $(m-1)!+\frac{1}{2}\lambda_m \leqslant U < m!$, we have, since $\big||f|-|\psi|\big| < B(x)/\sigma$, when y lies in the even interval λ_m,

$$|I_5| < \frac{1}{U}\int_{(m-1)!}^{U} \frac{B(x)}{\sigma}\,dy < \frac{1}{U}\int_{(m-1)!}^{m!} \frac{B(x)}{\sigma}\,dy = \frac{2B(x)}{U}\log(1+l\lambda_m)$$

$$< \frac{4B(x)}{\lambda_m}\log(1+l\lambda_m) < \epsilon,$$

provided that $m > m_5$, where $m_5 = m_5(x, \epsilon)$.

If $(m-1)! \leqslant U < (m-1)!+\frac{1}{2}\lambda_m$, then, since $\big||f|-|\psi|\big| < B(x)/\sigma$, $l \geqslant 1$, $U \geqslant (m-1)! > 1$, and $t^{-1}\log t$ is a decreasing function for $t > e$, we have

$$|I_5| < \frac{1}{U}\int_{(m-1)!}^{U} \frac{B(x)}{\sigma}\,dy = \frac{1}{U}\int_{1/(2l)}^{1/(2l)+U-(m-1)!} \frac{B(x)}{\sigma}\,d\sigma$$

$$= \frac{B(x)}{U}\log[1+2l\{U-(m-1)!\}]$$

$$< \frac{B(x)}{U}\log 3lU \leqslant \frac{B(x)}{(m-1)!}\log 3l(m-1)! < \epsilon,$$

provided that $m > m_6(x, \epsilon)$.

Since U is an integer, we have

$$I_6 = \frac{U-(m-2)!}{U}\,\frac{1}{U-(m-2)!}\int_{(m-2)!}^{U} |\phi|\,dy = \frac{U-(m-2)!}{U}\,M_\phi(x).$$

Therefore

$$|I_6 - M_\phi(x)| = \frac{(m-2)!}{U} M_\phi(x) \leqslant \frac{(m-2)!}{(m-1)!} M_\phi(x) < \epsilon,$$

provided that $m > m_7$, where $m_7 = m_7(x, \epsilon)$.

Finally, we have

$$I_7 = \frac{U-(m-1)!}{U} \{M_\psi(x) - M_\phi(x)\} = \theta \{M_\psi(x) - M_\phi(x)\},$$

where $0 \leqslant \theta < 1$. Thus, if T lies in an even interval λ_m, and η is any positive number, we have

$$|I - (1-\theta) M_\phi(x) - \theta M_\psi(x)| < \eta \quad \text{where} \quad 0 \leqslant \theta < 1, \tag{8}$$

provided that $m > m_8$, where $m_8 = m_8(x, \eta)$. [The case $\theta = 0$ occurs when $T = (m-1)!$.]

If T lies in an odd interval λ_m, it may be shown in a similar way that

$$|I - (1-\theta') M_\psi(x) - \theta' M_\phi(x)| < \eta, \quad \text{where} \quad 0 \leqslant \theta' < 1, \tag{9}$$

provided that $m > m_9$, where $m_9 = m_9(x, \eta)$. [The case $\theta' = 0$ occurs when $T = (m-1)!$.]

It follows that the upper and lower limits of I, when $T \to \infty$, are Max $\{M_\phi(x), M_\psi(x)\}$ and Min $\{M_\phi(x), M_\psi(x)\}$ respectively. Thus $|f(x+iy)|$ has a mean value on the lines $x =$ constant for which $M_\phi(x) = M_\psi(x)$, and on no others.

I next show that the functions $\phi(z)$ and $\psi(z)$ may be chosen in such a way that the function $M(x) = M_\phi(x) - M_\psi(x)$ has any given number of zeros in the interval $a < x < b$.

Since

$$|\cosh(z-\alpha) l\pi|^2 = \cosh^2(x-\alpha) l\pi \cos^2 yl\pi + \sinh^2(x-\alpha) l\pi \sin^2 yl\pi, \tag{10}$$

it follows that

$$\sinh(x-\alpha) l\pi \leqslant |\cosh(z-\alpha) l\pi| \leqslant \cosh(x-\alpha) l\pi,$$

and hence that

$$\prod_{r=1}^{\kappa} \sinh(x-\alpha_r) l\pi \leqslant |\phi(x+iy)| \leqslant \prod_{r=1}^{\kappa} \cosh(x-\alpha_r) l\pi$$

and

$$\prod_{r=1}^{\kappa} \sinh(x-\beta_r) l\pi \leqslant |\psi(x+iy)| \leqslant \prod_{r=1}^{\kappa} \cosh(x-\beta_r) l\pi.$$

Therefore, on every line for which

$$\prod_{r=1}^{\kappa} \cosh(x-\alpha_r)\,l\pi < \prod_{r=1}^{\kappa} \sinh(x-\beta_r)\,l\pi,$$

we have $|\phi(x+iy)| < |\psi(x+iy)|$, for all values of y, and so

$$M_\phi(x) < M_\psi(x).$$

Also on every line for which

$$\prod_{r=1}^{\kappa} \sinh(x-\alpha_r)\,l\pi > \prod_{r=1}^{\kappa} \cosh(x-\beta_r)\,l\pi,$$

we have $|\phi(x+iy)| > |\psi(x+iy)|$, for all values of y, and so

$$M_\phi(x) > M_\psi(x).$$

Now the ratios $2\cosh X/e^X$ and $2\sinh X/e^X$ may be made as near to unity as we please by choosing X sufficiently large. It follows that, if $\sum_{r=1}^{\kappa} |x-\alpha_r| < \sum_{r=1}^{\kappa} |x-\beta_r|$, where $x \neq \beta_r$ $(r = 1, 2, \ldots, \kappa)$, then $L = L(x)$ can be chosen so that

$$\prod_{r=1}^{\kappa} \cosh(x-\alpha_r)\,l\pi < \prod_{r=1}^{\kappa} \sinh(x-\beta_r)\,l\pi,$$

provided that $l > L$.

Similarly, if $\sum_{r=1}^{\kappa} |x-\alpha_r| > \sum_{r=1}^{\kappa} |x-\beta_r|$, where $x \neq \alpha_r$ $(r = 1, 2, \ldots, \kappa)$, then $L = L(x)$ can be chosen such that

$$\prod_{r=1}^{\kappa} \sinh(x-\alpha_r)\,l\pi > \prod_{r=1}^{\kappa} \cosh(x-\beta_r)\,l\pi,$$

provided that $l > L$.

Now suppose that $\alpha_1 < \beta_1 \leqslant \beta_2 < \alpha_2 \leqslant \alpha_3 < \beta_3 \leqslant \ldots$, and consider the function $F(x) = \sum_{r=1}^{\kappa} |x-\alpha_r| - \sum_{r=1}^{\kappa} |x-\beta_r|$. We have

$$F'(x) = \sum_{r=1}^{\kappa} \frac{x-\alpha_r}{|x-\alpha_r|} - \sum_{r=1}^{\kappa} \frac{x-\beta_r}{|x-\beta_r|}, \tag{11}$$

whenever this exists, and so $F(x)$ is constant for $x < \alpha_1$, in the intervals (β_1, β_2), (α_2, α_3), (β_3, β_4), ..., and for $x > \text{Max}(\alpha_\kappa, \beta_\kappa)$, while $F'(x)$ assumes the values $+2$ and -2 alternately in the intervals (α_1, β_1), (β_2, α_2), (α_3, β_3), ... Also $F(+\infty) = -F(-\infty)$. Therefore the numbers α_r, β_r may be chosen in such a way that $F(x)$ is zero and changes sign at any given finite set of values $x_1, x_2, \ldots, x_p$ in the interval $a < x < b$, and at no others. Now

let δ be chosen so that none of the intervals $(x_i-\delta,\ x_i+\delta)$ $(i=1, 2, \ldots, p)$ contains an α_r or β_r, then $L=L(\delta)$ can be chosen so that, if $l>L$, $M(x)$ has different signs at the ends of the intervals $(x_i-\delta,\ x_i+\delta)$, and a constant sign throughout each of the remaining intervals. Therefore the zeros of $M(x)$ are all in the intervals $(x_i-\delta,\ x_i+\delta)$ and, since $M(x)$ is continuous, each interval contains at least one. Moreover L may be chosen such that, for $l>L$, there is just one zero in each interval. For

$$M'(x)=\int_0^1 \frac{d}{dx}|\phi(x+iy)|\,dy-\int_0^1 \frac{d}{dx}|\psi(x+iy)|\,dy$$

and

$$\frac{1}{|\phi|}\frac{d}{dx}|\phi|=\tfrac{1}{2}\sum_{r=1}^{\kappa}\frac{d}{dx}\log\{\cosh^2(x-\alpha_r)\,l\pi\cos^2 yl\pi+\sinh^2(x-\alpha_r)\,l\pi\sin^2 yl\pi\}.$$

Now

$$\tfrac{1}{2}\frac{d}{dX}\log(\cosh^2 X\cos^2 Y+\sinh^2 X\sin^2 Y)=\frac{\sinh X\cosh X}{\cosh^2 X\cos^2 Y+\sinh^2 X\sin^2 Y},$$

which lies between $\tanh X$ and $\coth X$, and so it may be made as near to $X/|X|$ as we please by choosing X large enough. It follows that

$$\frac{d}{dx}|\phi|-\frac{d}{dx}|\psi|=|\phi|\,l\pi\left\{\sum_{r=1}^{\kappa}\frac{x-\alpha_r}{|x-\alpha_r|}+r_1(x,y)\right\}$$
$$-|\psi|\,l\pi\left\{\sum_{r=1}^{\kappa}\frac{x-\beta_r}{|x-\beta_r|}+r_2(x,y)\right\},$$

and $L=L(\delta)$ may be chosen so large that, for $l>L$, $|r_1|<\frac{1}{2}$ and $|r_2|<\frac{1}{2}$, uniformly in y and in every interval $(x_i-\delta,\ x_i+\delta)$. Therefore

$$M'(x)=M_\phi(x)\,l\pi\left\{\sum_{r=1}^{\kappa}\frac{x-\alpha_r}{|x-\alpha_r|}+r_3(x)\right\}-M_\psi(x)\,l\pi\left\{\sum_{r=1}^{\kappa}\frac{x-\beta_r}{|x-\beta_r|}+r_4(x)\right\},$$

where $|r_3|<\frac{1}{2}$ and $|r_4|<\frac{1}{2}$ throughout the intervals $(x_i-d,\ x_i+\delta)$. Now suppose that $M(x)=0$ for $x=x_i'$. Then $M'(x)$ has the sign of

$$\sum_{r=1}^{\kappa}\frac{x-\alpha_r}{|x-\alpha_r|}-\sum_{r=1}^{\kappa}\frac{x-\beta_r}{|x-\beta_r|}+r_3(x)-r_4(x)$$

for $x=x_i'$, and so has the sign of $F'(x_i')$. But $F'(x)$ is positive and negative alternately throughout the intervals $(x_i-\delta,\ x_i+\delta)$. It follows that $M(x)$ has one and only one zero $x=x_i'$ in each interval $(x_i-\delta,\ x_i+\delta)$. Thus, since $|f(x+iy)|$ is bounded for $a<x<b$, $f(z)$ is a function of the type required, and the first part of the problem is solved.

I next consider the case of an infinite number of lines. The above functions are inadequate to deal with this case, for κ is essentially finite.

The process must therefore be modified, in order to obtain functions which lend themselves to a limiting process.

Let λ_m denote the same interval as before. Let

$$(12)\qquad \begin{cases} \phi_\kappa(z) = \prod_{r=1}^{\kappa} \dfrac{\cosh(l_r z - \alpha_r)\pi}{\cosh(l_r z + \alpha'_r)\pi}, \\ \psi_\kappa(z) = \prod_{r=1}^{\kappa} \dfrac{\cosh(l_r z - \beta_r)\pi}{\cosh(l_r z + \beta'_r)\pi}, \\ f_\kappa(z) = \lim_{m\to\infty} \prod_{r=1}^{\kappa} \prod_{n=-m}^{m-1} \dfrac{(2n+1)i - 2(l_r z - \gamma_{r,n})\pi}{(2n+1)i - 2(l_r z + \gamma'_{r,n})\pi}, \end{cases}$$

where $\gamma_{r,n} = \alpha_r$, $\gamma'_{r,n} = \alpha'_r$, if $(n+\frac{1}{2})/l_r$ belongs to an odd interval,

$$\gamma_{r,n} = \beta_r,\ \gamma'_{r,n} = \beta'_r,\ \text{if } (n+\tfrac{1}{2})/l_r \text{ belongs to an even interval.}$$

Let a_r, a'_r, β_r, β'_r be positive and chosen (for $r = 1, 2, 3, \ldots$ to infinity) such that Σa_r, $\Sigma a'_r$, $\Sigma \beta_r$, $\Sigma \beta'_r$ are convergent, and suppose moreover that $a'_r \geqslant a_r$, $\beta'_r \geqslant \beta_r$, and that the numbers l_r are positive integers. Then, when $\kappa \to \infty$, the functions $\phi_\kappa(z)$ and $\psi_\kappa(z)$ both tend to limit functions $\phi(z)$ and $\psi(z)$, for $x > 0$. Moreover, the result is true uniformly along any line $x = \text{constant}$, $x > 0$. For

$$\begin{aligned} &\cosh(l_r z - \alpha_r)\pi \operatorname{sech} l_r z\pi \\ &\quad = (\cosh l_r z\pi \cosh \alpha_r \pi - \sinh l_r z\pi \sinh \alpha_r \pi) \operatorname{sech} l_r z\pi \\ &\quad = \{1 + \sinh^2 \alpha_r \pi\}^{\frac{1}{2}} \{1 - \tanh l_r z\pi \tanh \alpha_r \pi\}, \end{aligned}$$

and, since $$\frac{\tanh \alpha_r \pi}{\alpha_r \pi} \to 1 \quad \text{and} \quad \frac{\sinh \alpha_r \pi}{\alpha_r \pi} \to 1,$$

the convergence of $\Sigma \tanh \alpha_r \pi$ and $\Sigma \sinh^2 \alpha_r \pi$ follows from that of Σa_r and Σa_r^2. Now since $l_r \geqslant 1$, $\tanh l_r z\pi$ is uniformly bounded along any line $x = \text{constant}$, $x > 0$. Therefore $\Pi \cosh(l_r z - \alpha_r)\pi \operatorname{sech} l_r z\pi$ is uniformly convergent along any such line. The same is true for $\Pi \cosh(l_r z + \alpha'_r)\pi \operatorname{sech} l_r z\pi$ and the product has no zeros for $x > 0$. Hence $\phi_\kappa(z)$ tends uniformly to $\phi(z)$ along $x = \text{constant}$, $x > 0$. Similarly for $\psi_\kappa(z)$.

Next, since $a'_r \geqslant a_r$, we have, for $x \geqslant 0$,

$$(13)\qquad \left| \frac{\cosh(l_r z - \alpha_r)\pi}{\cosh(l_r z + \alpha'_r)\pi} \right| \leqslant 1;$$

therefore $|\phi_\kappa(z)| \leqslant 1$ for all κ and, for $x > 0$, $|\phi(z)| \leqslant 1$. Also these functions are all periodic with respect to y with unit period. Similarly for $\psi_\kappa(z)$, $\psi(z)$.

Again, since $\gamma'_{r,n} \geqslant \gamma_{r,n}$, we have, for $x \geqslant 0$,

$$\left| \frac{(2n+1)i - 2(l_r z - \gamma_{r,n})\pi}{(2n+1)i - 2(l_r z + \gamma'_{r,n})\pi} \right| \leqslant 1. \tag{14}$$

It follows that, for $x \geqslant 0$,

$$1 \geqslant \left| \frac{f_\kappa(z)}{f_{\kappa-1}(z)} \right| \geqslant \left| \frac{\phi_\kappa(z)\psi_\kappa(z)}{\phi_{\kappa-1}(z)\psi_{\kappa-1}(z)} \right|, \tag{15}$$

and hence that $|f_\kappa(z)|$ tends to a limit, when $\kappa \to \infty$, uniformly along any line $x = \text{constant}$, $x > 0$. Also $|f_\kappa(x)| = f_\kappa(x)$, since the factors may be taken in conjugate pairs, and so, for $x > 0$, $f_\kappa(x) \to f(x)$ when $\kappa \to \infty$. Therefore by Vitali's theorem*, since $|f_\kappa(z)| \leqslant 1$ for $x > 0$, $f_\kappa(z) \to f(z)$ when $\kappa \to \infty$, where $f(z)$ is regular and bounded for $x > 0$.

Let

$$M_{\phi_\kappa}(x) = \int_0^1 |\phi_\kappa(x+iy)|\,dy \quad \text{and} \quad M_\phi(x) = \int_0^1 |\phi(x+iy)|\,dy,$$

and let $M_{\psi_\kappa}(x)$ and $M_\psi(x)$ be defined in a similar way. Then these are the mean values of $|\phi_\kappa(z)|$, $|\phi(z)|$, $|\psi_\kappa(z)|$ and $|\psi(z)|$ respectively, along the line $x = \text{constant}$. Moreover, since $|\phi_\kappa(x+iy)| \to |\phi(x+iy)|$ uniformly in y, for a given $x > 0$, we have $M_\phi(x) = \lim_{\kappa\to\infty} M_{\phi_\kappa}(x)$; and similarly $M_\psi(x) = \lim_{\kappa\to\infty} M_{\psi_\kappa}(x)$. Also

$$\frac{1}{2T}\int_{-T}^{T} |f(x+iy)|\,dy = \lim_{\kappa\to\infty} \frac{1}{2T}\int_{-T}^{T} |f_\kappa(x+iy)|\,dy, \tag{16}$$

for a given $x > 0$.

Now, by an argument exactly similar to that used in the first part, it may be proved that, for $x > 0$,

$$\overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |f_\kappa(x+iy)|\,dy = \text{Max}\{M_{\phi_\kappa}(x), M_{\psi_\kappa}(x)\}, \tag{17}$$

and, since $|f_\kappa(x+iy)| \to |f(x+iy)|$ uniformly in y, it follows that, for $x > 0$,

$$\begin{aligned} \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |f(x+iy)|\,dy &= \lim_{\kappa\to\infty} \overline{\lim_{T\to\infty}} \frac{1}{2T}\int_{-T}^{T} |f_\kappa(x+iy)|\,dy \\ &= \text{Max}\{M_\phi(x), M_\psi(x)\}. \end{aligned} \tag{18}$$

* Or by an elementary argument.

Similarly it may be shown that, for $x > 0$,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x+iy)|\,dy = \text{Min}\{M_\phi(x), M_\psi(x)\}. \tag{19}$$

Thus $|f(x+iy)|$ has a mean value only on those lines $x =$ constant for which $M_\phi(x) = M_\psi(x)$.

I next show that $\phi(z)$ and $\psi(z)$ may be chosen so that the continuous function $M(x) = M_\phi(x) - M_\psi(x)$ has an infinite number of changes of sign in an interval $0 < x < A$, so that the equation $M_\phi(x) = M_\psi(x)$ has an infinite number of roots in that interval.

Let a sequence of positive numbers ϵ_r be given, such that $\Sigma\epsilon_r$ is convergent, and write $\sum_{r=1}^{\kappa} \epsilon_r = E_\kappa$, $\lim_{\kappa\to\infty} E_\kappa = E$. Let $\alpha_1+\alpha_1' > \beta_1+\beta_1' = \epsilon_1/\pi$, and $\alpha_r+\alpha_r' = \beta_r+\beta_r'$ for $r > 1$.

Let $l_1 = 1$, $\alpha_1 = \alpha_1'$, and $\beta_1' \geqslant \beta_1$. Then $|\phi_1(0+iy)| = 1$, and hence $M_{\phi_1}(0) = 1$, while $|\psi_1(0+iy)| < 1$, and hence $M_{\psi_1}(0) < 1$. Then, l_1, α_1, α_1' having been chosen, there is a lower bound to the possible values of $M_{\phi_2}(0)$. (This is not attainable since it is not permissible to take $\beta_2 = 0$.) Also $M_{\psi_1}(0)$ may be chosen as near to unity as we please, by choosing $\beta_1'-\beta_1$ sufficiently small. Let β_1, β_1' then be chosen such that $M_{\psi_1}(0)$ is greater than this lower bound. Then it is certainly possible to choose α_2, α_2' such that $M_{\phi_2}(0) < M_{\psi_1}(0)$.

Now $|\phi_1(x+iy)| \to e^{-(\alpha_1+\alpha_1')\pi}$ and $|\psi_1(x+iy)| \to e^{-(\beta_1+\beta_1')\pi}$ when $x\to\infty$, uniformly in y, and hence, since $\alpha_1+\alpha_1' > \beta_1+\beta_1'$, $M_1(x) = M_{\phi_1}(x) - M_{\psi_1}(x)$ is negative for sufficiently large values of x. But $M_1(0) > 0$, and therefore there is at least one positive number x for which $M_1(x) = 0$. Let x_1 be the largest. Then we can choose δ_1 and η_1 such that $x_1-\delta_1 > 0$, $M_1(x_1-\delta_1) > \eta_1$, and $M_1(x_1+\delta_1) < -\eta_1$.

Now, if x has any given positive value, l_2 may be chosen so large that

$$\left|\frac{\cosh(l_2 z-\alpha_2)\pi}{\cosh(l_2 z+\alpha_2')\pi}\right| \quad \text{and} \quad \left|\frac{\cosh(l_2 z-\beta_2)\pi}{\cosh(l_2 z+\beta_2')\pi}\right|$$

differ from $e^{-\epsilon_2}$ by a quantity as small as we please, uniformly in y and in the permissible values of α_2, α_2', β_2, β_2'. Let l_2 be chosen such that $M_2(x_1-\delta_1) > e^{-\epsilon_2}\eta_1$ and $M_2(x_1+\delta_1) < -e^{-\epsilon_2}\eta_1$ for all permissible values of α_2, α_2', β_2, β_2'.

Let $\beta_2 = \beta_2'$, so that $M_{\psi_2}(0) = M_{\psi_1}(0)$. Then, l_2, β_2, β_2' having been chosen, there is a lower bound to the possible values of $M_{\psi_2}(0)$. Let α_2, α_2' be chosen such that $M_{\phi_2}(0)$ is greater than this lower bound but less than $M_{\psi_1}(0) = M_{\psi_2}(0)$. Then $M_2(0) = M_{\phi_2}(0) - M_{\psi_2}(0) < 0$, and therefore there is at least one positive number less than $x_1-\delta_1$ for which $M_2(x) = 0$. Let

x_2 be the largest of these. Then we can choose δ_2 and η_2 such that $M_2(x_2-\delta_2) < -\eta_2$ and $M_2(x_2+\delta_2) > \eta_2$, where the interval $(x_2-\delta_2, x_2+\delta_2)$ is inside the interval $(0, x_1-\delta_1)$.

Next let l_3 be chosen such that $M_3(x_1-\delta_1) > e^{-\epsilon_2-\epsilon_3}\eta_1$, $M_3(x_1+\delta_1) < e^{-\epsilon_2-\epsilon_3}\eta_1$, $M_3(x_2-\delta_2) < -e^{-\epsilon_3}\eta_2$, and $M_3(x_2+\delta_2) > e^{-\epsilon_3}\eta_2$, for all permissible values of α_3, α_3', β_3, β_3'. Let $\alpha_3 = \alpha_3'$, so that $M_{\phi_3}(0) = M_{\phi_2}(0)$. Then let β_3 and β_3' be chosen such that $M_{\psi_3}(0)$ is greater than the lower bound of the permissible values of $M_{\phi_4}(0)$, but less than $M_{\phi_2}(0) = M_{\phi_3}(0)$. Then $M_3(0) = M_{\phi_3}(0) - M_{\psi_3}(0) > 0$, and therefore there is at least one positive number x less than $x_2-\delta_2$ for which $M_3(x) = 0$. Let x_3 be the greatest of these. Then we can choose δ_3 and η_3 such that $M_3(x_3-\delta_3) > \eta_3$ and $M_3(x_3+\delta_3) < -\eta_3$, where the interval $(x_3-\delta_3, x_3+\delta_3)$ is inside the interval $(0, x_2-\delta_2)$.

Proceeding in this way, we can define $\phi_\kappa(z)$ and $\psi_\kappa(z)$ such that $M_\kappa(x) = M_{\phi_\kappa}(x) - M_{\psi_\kappa}(x)$ has at least κ changes of sign, and moreover there is a set of non-overlapping intervals $(x_i-\delta_i, x_i+\delta_i)$ $(i = 1, 2, \ldots, \kappa)$, all included in the interval $0 < x < A$, and at the end-points of which $M_\kappa(x)$ has opposite signs, and $|M_\kappa(x_i \pm \delta_i)| > e^{E_i-E_\kappa}\eta_i$. It follows that there exist functions $\phi(z)$ and $\psi(z)$ such that $M(x)$ has opposite signs at the end-points of an infinite set of non-overlapping intervals $(x_i-\delta_i, x_i+\delta_i)$ $(i = 1, 2, 3, \ldots)$, all included in the interval $0 < x < A$, and $|M(x_i \pm \delta_i)| > e^{E_i-E}\eta_i$. Thus $M_\phi(x)$ is equal to $M_\psi(x)$ for an infinite number of lines of the strip $0 < x < A$, which have the line $x = 0$ as a limit line, and the corresponding function $|f(x+iy)|$ has a mean value on these lines and on no others. This completes the solution of the problem.

Further results. The proof of the theorem of Hardy, Ingham, and Pólya is by means of Vitali's theorem, which, stated for the unit circle, is:

If $f_n(z)$ is regular and uniformly bounded for $|z| < 1$, and if $f_n(x_i) \to f(x_i)$ $(i = 1, 2, 3, \ldots)$, where $x_i \to x$, and $|x| < 1$, then $f_n(z) \to f(z)$ uniformly throughout $|z| \leqslant 1-\delta$, for every $\delta > 0$, where $f(z)$ is regular for $|x| < 1$.

An extension of this theorem, due to Blaschke*, is as follows:

Vitali's theorem remains true when $|x| = 1$, provided that $\Sigma(1-|x_n|)$ is divergent.

A similar application of Blaschke's theorem yields the following theorem.

---

* Blaschke, *Leipziger Ber.*, 67 (1915), 194–200. Landau, *Leipziger Ber.*, 70 (1918), 156–159. *Encyk. Math. Wiss.*, II 3 (i), 495.

THEOREM A. *In those cases in which the theorem of Hardy, Ingham, and Pólya has been proved, the result still holds if* $\lim_{T\to\infty} \frac{1}{2T}\int_{x-iT}^{x+iT} |f(z)|^p\,dz$ *is only known to exist on an enumerable set of lines* $x = x_n$, *such that* $\lim_{n\to\infty} x_n = a$, *provided that* $\Sigma(x_n - a)$ *is divergent.*

Blaschke's theorem is applied to the function

$$F_T(\zeta) = \frac{1}{2T}\int_{-T}^{T} f^{\frac{1}{2}p}(\zeta+iv) f^{-\frac{1}{2}p}(\bar{\zeta}+iv)\,dv$$

in the circle $|\frac{1}{2}(\alpha+\beta)-\zeta| = \frac{1}{2}(\beta-\alpha)$, the extra condition taking the form that $\Sigma[\frac{1}{2}(\beta-\alpha)-\{\frac{1}{2}(\alpha+\beta)-x_n\}]$ is divergent, and the rest of the proof is the same.

The question whether Theorem A is a best possible theorem is one for further consideration. The function constructed above does not answer this question, but there seems no reason why a function of the same class should not be constructed to answer it in the affirmative.

Finally, I must thank Profs. G. H. Hardy and A. S. Besicovitch for their valuable advice during the early stages for the problem, and also Mr. W. L. Ferrar for drawing my attention to Blaschke's theorem.

ON THE DENSITY OF IRREGULAR LINEARLY MEASURABLE SETS OF POINTS

By A. S. Besicovitch *and* G. Walker.

[Received 15 November, 1929.—Read 12 December, 1929.]

1. Linearly measurable plane sets of points have been divided into two classes†, the first consisting of regular and the second of irregular sets. The classification has been made in terms of density. Regular sets are those which possess the property that, at almost all points, the density exists and is equal to unity; and irregular sets are those for which, either the density does not exist, or, if it does, then it differs from unity.

A general linearly measurable set can be divided into two subsets, one of which is regular and the other irregular.

Regular sets form a natural generalization of rectifiable curves, since the fundamental geometrical properties of regular sets and rectifiable curves are the same. But irregular sets are completely dissimilar to these, and *there is no gradual change of regular sets into irregular ones.*

It is very probable that *the set of points of a general linearly measurable set, at which the density exists and differs from one, is of measure zero.* If this be true, we could define regular sets as those for which, at almost all points, the density existed, and irregular sets as those for which the density did not exist. Apparently, to the discontinuity in geometrical properties there corresponds a discontinuity in the character of the density distribution at points of regular and irregular sets. In the paper cited above it has been shown that, *at almost all points of a linearly measurable set, the upper density is included in the interval* $(\frac{1}{2}, 1)$ *and the lower density in the interval* $(0, 1)$.

It seems probable that the discontinuity in the density distribution

† Besicovitch, *Math. Annalen*, 98 (1927), 422 464.

is due to the property of irregular sets that, *at points of irregular sets, the lower density cannot be arbitrarily near the upper density.*

A partial verification of these properties has been given in the above paper. In fact, it follows from the proof of the theorem on the existence of the tangent at points of a regular set that, at almost all points of an irregular set, the lower density is less than $1-\epsilon_0$, where ϵ_0 is a small fixed positive number†. Hence it follows that

(*a*) the value of the density cannot lie in the interval $(1-\epsilon_0, 1)$, and

(*b*) there do not exist irregular sets whose upper density is greater than $1-\epsilon_0$ and whose lower density is arbitrarily near the upper density.

But it is clear that the value of the difference which we can assert to exist between the upper and lower densities is very small (less than or equal to ϵ_0). It appears, then, that this does not correspond to the real nature of density distribution.

In this note we consider sets in which the upper density is near $\frac{1}{2}$, and show that for such sets the density does not exist, and that the value of the lower density is considerably less than the value of the upper density (in fact, less than $\frac{1}{3}$). Thus this note gives only a partial verification of the general result; but, taken in conjunction with the other results quoted above, we now have verification for values of the upper density at both ends of the interval $(\frac{1}{2}, 1)$ of all its possible values.

The complete density problem would be, *given the value of the upper density at points of an irregular set, to find the exact upper bound of possible values of the lower density.*

But this problem remains unsolved.

2. Let A be a plane set of points, $C(A, \delta)$ a finite or enumerable set of circles, each of diameter less than or equal to δ, and such that each point of A belongs to at least one circle of the set. We denote by

$$\sum_{C(A,\delta)} d = \sum_{C(A,\delta)} 2r$$

the sum of the diameters of all these circles. Given a fixed δ we consider the lower bound of the sum

$$\sum_{C(A,\delta)} d$$

† *Loc. cit.*, 454.

for all possible sets $C(A, \delta)$, and we write

$$\text{l. b.} \sum_{C(A,\delta)} d = L^*_{c,\delta} A$$

and

$$\lim_{\delta \to 0} L^*_{c\,\delta} A = L^*_c A,$$

l. b. denoting "lower bound of". We call $L^*_c A$ *the upper linear measure of the set* A.

$L^*_c A$ is different from the Carathéodory upper linear measure. Instead of sets of circles $C(A, \delta)$ Carathéodory considers sets $U(A, \delta)$ of arbitrary areas each of diameter less than or equal to δ, and correspondingly he considers

$$\text{l. b.} \sum_{U(A,\delta)} d = L^*_\delta A \quad \text{and} \quad \lim_{\delta \to 0} L^*_\delta A = L^* A.$$

We preserve Carathéodory's definition of measurable sets, and we write for a measurable set A

$$L^*_c A = L_c A,$$

and in the case of Carathéodory's measure

$$L^* A = LA,$$

and we call $L_c A$, LA the linear measures of A.

Carathéodory's definition of linear measure has the advantage from the axiomatic point of view that it involves the least number of conditions, but it is sometimes more convenient to use L_c measure for special problems where it may lead to a simplification of details. It is for this reason that we use L_c measure in this note, although the results remain equally valid for L measure also. We need the following lemma and theorem for our argument.

LEMMA. *Let* $C(\delta)$ *denote a finite or enumerable set of circles each of diameter less than or equal to* δ. *Then to any measurable set* A *and to any positive number* η *there corresponds a number* δ_1 *such that, for any set* $C(\delta)$, $\delta \leqslant \delta_1$, *the inequality*

$$L_c\{A \times C(\delta)\} < \sum_{C(\delta)} 2r + \eta$$

is satisfied.

THEOREM. *Suppose that the measurable set* A *is divided into two measurable sets* A_1 *and* A_2. *Then at almost all points of* A_1 *the density of* A_1 *coincides with the density of* A; *and a similar result is true at almost all points of* A_2.

The lemma and the theorem have been proved in §§ 7 and 14 of the paper quoted above for the case of Carathéodory measure; the proof applies without change for L_c measure.

3. We shall denote by the letters a, ϵ, γ, s, k, n the following numbers:

$$a = \epsilon = \gamma = 10^{-7}, \quad s = 10^6, \quad k = 10^3, \quad n = \tfrac{1}{2}.10^3. \tag{2}$$

Then the purpose of our note is to prove the following theorem.

THEOREM. *Let A be a linearly measurable plane set of points, B the set of those points of A for which the upper density lies in the closed interval $(\frac{1}{2}, \frac{1}{2}+a)$; then at almost all points of B the lower density is less than $\frac{1}{3}$.*

The theorem will be proved for the value of a given by (2), but it remains true for much larger values of a. We take a small for the purpose of simplifying the details.

4. We write $A = \mathfrak{A}+B$. Suppose that the theorem is not true, and let B' be a subset of B of positive measure, at all points of which the lower density of A is greater than or equal to $\frac{1}{3}$. On account of the theorem of § 2 we know that at almost all points of B' the densities of the set A are equal to the densities of the set B' itself. Consequently at almost all points of B' the upper density is greater than or equal to $\frac{1}{2}$ and less than or equal to $\frac{1}{2}+a$, and the lower density is greater than or equal to $\frac{1}{3}$. Let l be a positive number and β a point of B' such that the mean density of B' in the circle $c(\beta, r)$ (the circle with centre at β and of radius r) is less than $\frac{1}{2}+a+\epsilon$ for all $r<l$. We denote by B_1 the set of all such points β of B'. Obviously we have

$$L_c B_1 \to L_c B' > 0, \quad \text{as} \quad l \to 0. \tag{3}$$

Consequently we can assume l so small that $b_1 = L_c B_1 > 0$. Again, from the theorem of § 2 we conclude that, at almost all points of the set B_1, its upper density and its lower density coincide with those of B', and thus the lower density of B_1 is greater than or equal to $\frac{1}{3}$ at almost all points.

On the other hand, we shall define a subset H of B_1, of positive measure, at all points of which the lower density of B_1 is less than $\frac{1}{3}$. In this way we shall prove that the assumption that the theorem is not true, is false.

Let $\delta < l$ be a number less than or equal to the number δ_1 of the lemma of § 2, corresponding to the set B_1 and to $\eta = \frac{1}{6}\gamma b_1$, so that, for any $C(\delta)$, the inequality

$$\sum_{C(\delta)} 2r > L_e\{B_1 \times C(\delta)\} - \eta \quad (\eta = \tfrac{1}{6}\gamma b_1), \tag{4}$$

is satisfied.

Now it may be proved by Vitali's argument that there exists a set C of non-overlapping circles, each of diameter less than or equal to δ, such that

$$b_1 + \eta > \sum_{C} 2r > b_1 - \eta \tag{5}$$

and C covers almost all points of B_1. We divide the set of circles C into three sets, and write

$$C = C_1 + C_2 + C_3, \tag{6}$$

where C_1 is the set of circles in each of which the mean density of B_1 is less than $1-\gamma$, C_2 the set for which the mean density lies in the closed interval $(1-\gamma, 1+\gamma)$, and C_3 the set for which the mean density is greater than $1+\gamma$.

Writing $C_3 = C(\delta)$ in (4), we have

$$\begin{aligned} \sum_{C_3} 2r &> L_e(B_1 \times C_3) - \eta \\ &> (1+\gamma) \sum_{C_3} 2r - \eta, \end{aligned} \tag{7}$$

and thus

$$\sum_{C_3} 2r < \frac{\eta}{\gamma}; \tag{8}$$

and consequently, by (7),

$$L_e(B_1 \times C_3) < \frac{\eta}{\gamma} + \eta. \tag{9}$$

We now estimate the value of $L_e(B_1 \times C_1)$. We have

$$L_e(B_1 \times C_1) + L_e\{B_1 \times (C_2 + C_3)\} \geqslant L_e(B_1 \times C) = b_1, \tag{10}$$

$$L_e(B_1 \times C_1) < (1-\gamma) \sum_{C_1} 2r. \tag{11}$$

Hence

$$L_e\{B_1 \times (C_2 + C_3)\} > b_1 - (1-\gamma) \sum_{C_1} 2r. \tag{12}$$

On the other hand, putting $C(\delta) = C_2 + C_3$ in (4), we may write

$$\sum_{C_2+C_3} 2r > L_c\{B_1 \times (C_2+C_3)\} - \eta. \tag{13}$$

By (12), (13),

$$\sum_{C_2+C_3} 2r + \eta > b_1 - (1-\gamma) \sum_{C_1} 2r \tag{14}$$

or

$$\sum_{C_1+C_2+C_3} 2r = \sum_{C} 2r > b_1 + \gamma \sum_{C_1} 2r - \eta, \tag{15}$$

and, on account of (5),

$$b_1 + \eta > b_1 + \gamma \sum_{C_1} 2r - \eta, \tag{16}$$

and thus

$$\sum_{C_1} 2r < \frac{2\eta}{\gamma}, \tag{17}$$

whence

$$L_c(B_1 \times C_1) < (1-\gamma)\frac{2\eta}{\gamma}. \tag{18}$$

By (9) and (18),

$$\begin{aligned} L_c(B_1 \times C_2) &> b_1 - \frac{\eta}{\gamma} - \eta - \frac{2\eta}{\gamma}(1-\gamma) \\ &> b_1 + \eta - \frac{3\eta}{\gamma}, \end{aligned} \tag{19}$$

and, substituting for η the value given in (4),

$$L_c(B_1 \times C_2) > \frac{b_1}{2}. \tag{20}$$

Thus, *given a positive number* δ, *we can always find a set of circles* C_2, *each of diameter less than or equal to* δ, *covering a part of the set* B_1, *of measure greater than* $\frac{1}{2}b_1$, *and such that the mean density of the set* B_1 *in each of them, lies in the interval* $(1-\gamma, 1+\gamma)$.

5. Now let $c(0, r) = c$ be a circle of the set C_2, and β a point of B_1 belonging to c. We denote the distance 0β by r_1 and draw the circle $c(\beta, r+r_1)$. Observing that

$$c(\beta, r+r_1) \supset c(0, r),$$

we have

$$L_c\{B_1\times c(\beta,\, r+r_1)\} \geqslant L_c\{B_1\times c(0,\, r)\} \geqslant (1-\gamma)\,2r. \tag{22}$$

On the other hand, since $r+r_1 \leqslant 2r < \delta$, the mean density of the set B, and *a fortiori* of the set B_1 in the circle $c(\beta,\, r+r_1)$, is less than $\frac{1}{2}+a+\epsilon$. Thus

$$L_c\{B_1\times c(\beta,\, r+r_1)\} < (\tfrac{1}{2}+a+\epsilon)\,2(r+r_1). \tag{23}$$

From (22) and (23) it follows that

$$(\tfrac{1}{2}+a+\epsilon)\,2(r+r_1) > (1-\gamma)\,2r, \tag{24}$$

and hence

$$\begin{aligned}\frac{r_1}{r} &> \frac{\frac{1}{2}-a-\epsilon-\gamma}{\frac{1}{2}+a+\epsilon}\\ &> 1-4a-4\epsilon-2\gamma\\ &= 1-\sigma,\end{aligned} \tag{25}$$

where we have written

$$\sigma = 4a+2\epsilon+2\gamma.$$

Thus *in every circle* $c(0,\, r)$ *of* C_2 *all points of* B_1 *are included in the narrow ring between the circles* $c(0,\, r)$ *and* $c\{0,\, (1-\sigma)r\}$.

As before, let β be a point of B_1 belonging to $c(0,\, r)$. We draw a circle $c(\beta,\, \rho)$, where $\rho \leqslant 2r < l$, and define $L_c[\{B_1-c(0,\, r)\}\times c(\beta,\, \rho)]$ to mean the measure of the part of the set B_1 which is included in the circle $c(\beta,\, \rho)$ but is outside the circle $c(0,\, r)$. We have

$$\begin{aligned}L_c[\{B_1-c(0,\, r)\}\times c(\beta,\, \rho)] &\leqslant L_c[\{B_1-c(0,\, r)\}\times c(\beta,\, 2r)]\\ &= L_c\{B_1\times c(\beta,\, 2r)\}-L_c\{B_1\times c(0,\, r)\}\\ &< (\tfrac{1}{2}+a+\epsilon)\,4r-(1-\gamma)\,2r = r(4a+4\epsilon+2\gamma) = \sigma r.\end{aligned} \tag{26}$$

Thus *the set of points of* B_1 *contained in the circle* $c(\beta,\, \rho)$ *consists of two parts: one part is included in the ring* $c(0,\, r)-c\{0,\, (1-\sigma)r\}$, *and the measure of the other part is less than* σr.

6. We take again any circle $c=c(0,\, r)$ of C_3. We divide c into $s\,(=10^6)$ equal sectors, and we shall agree that each radius of division belongs only to one of two adjacent sectors. In this way the ring

$c(0, r)-c\{0, (1-\sigma)r\}$ will be also divided into s equal sectors, the length of the larger arc of each sector being $2\pi r/s = 2\pi r \,.\, 10^{-6}$, and the breadth $\sigma r = 10^{-6} r$. Thus "the length of each sector is 2π times its breadth".

We have

$$(1-\gamma)\,2r \leqslant L_c\{B_1 \times c(0, r)\} \leqslant (1+\gamma)\,2r. \tag{27}$$

Thus we may write

$$L_c\{B_1 \times c(0, r)\} = (1+\gamma_1)\,2r \quad (|\gamma_1| \leqslant \gamma). \tag{28}$$

We now divide all sectors of the ring $c(0, r)-c\{0, (1-\sigma)r\}$ into two classes. We call a sector *compact* if the measure of the part of B_1 belonging to this sector is greater than or equal to $(1+\gamma_1)2r/(sn)$ $(n = 500)$, otherwise we call it *non-compact*.

We shall next consider two different cases according as the ring $c(0, r)-c\{0, (1-\sigma)r\}$ contains, or does not contain, $k = 10^3$ consecutive non-compact sectors.

7. (i) There is an interval of not less than k consecutive non-compact sectors. Let de be such an interval of the ring (Fig. 1) (instead of a circular arc we draw a straight segment for the sake of simplicity),

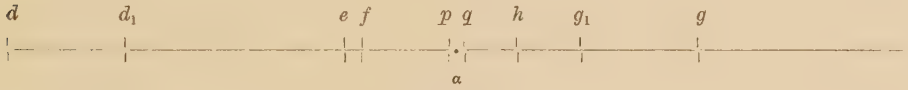


Fig. 1.

and ef an adjacent compact interval. We take the interval eg of k consecutive sectors to the right from the point c, and we divide it into two equal parts by the point h. Let pq be the sector nearest to h from the left-hand side, which contains points of B_1, and let a be such a point, and $k_2\,(2\pi/s)$ the angular distance from e to a $(k_2 \leqslant \frac{1}{2}k$; k_2 need not be an integer). Put $k_1 = \max\,(k_2, \frac{1}{4}k)$, and mark sectors ed_1, fg_1 of angle $(2k_1-1)(2\pi/s)$. Consider now $L_c(B_1 \times ed_1)$ and $L_c(B_1 \times eg_1)$. We have

$$L_c(B_1 \times ed_1) < [2k_1]\,\frac{2r(1+\gamma_1)}{sn} \leqslant 2k_1\,\frac{2r(1+\gamma_1)}{sn}, \tag{29}$$

where $[2k_1]$ denotes the integral part of $2k_1$.

Now all points of the set $B_1 \times eg_1$ are included in the circle $c\{a, (k_2+1)(2\pi r/s)\}$. But, from the definition of B_1, we know that the mean density of B_1 is less than $\frac{1}{2}+a+\epsilon$ in any circle with centre at a

point of B_1 and of radius less than l. Thus we have

$$(30)\quad L_c(B_1\times eg_1) \leqslant L_c\left[B_1\times c\left\{a,\ (k_2+1)\,\frac{2\pi r}{s}\right\}\right]$$

$$< (\tfrac{1}{2}+a+\epsilon)\,2(k_2+1)\,\frac{2\pi r}{s} \leqslant (\tfrac{1}{2}+a+\epsilon)\,2(k_1+1)\,\frac{2\pi r}{s}.$$

Let now β be any point of $B_1\times ef$. Consider the circle $c[\beta,\ 2r(1-\sigma)\sin\{(2k_1-1)\pi/s\}]$; it is clear that the part of the ring included in this circle belongs to the sector d_1g_1. We write

$$(31)\qquad B_1\times c\left\{\beta,\ 2r(1-\sigma)\sin\frac{(2k_1-1)\,\pi}{s}\right\} = E_1+E_2,$$

where E_1 is the part of the set B_1 belonging to the sector of the ring d_1g_1, and E_2 is the part of B_1 outside the circle $c(0,\ r)$. We have

$$(32)\qquad L_cE_1 \leqslant L_c\{B_1\times d_1g_1\} = L_c\{B_1\times d_1e\}+L_c\{B_1\times eg_1\}.$$

By (29), (30), (32),

$$(33)\qquad L_cE_1 < 2k_1\,\frac{2r(1+\gamma_1)}{sn}+(\tfrac{1}{2}+a+\epsilon)\,2(k_1+1)\,\frac{2\pi r}{s},$$

and, by (26),

$$(34)\qquad L_cE_2 < \sigma r.$$

By (31), (33), (34) we obtain the following inequality for D_1, the mean density of B_1 in the circle $c[\beta,\ 2r(1-\sigma)\sin\{(2k_1-1)\pi/s\}]$,

$$(35)\qquad D_1 < \frac{2k_1\,\dfrac{2r(1+\gamma_1)}{sn}+(\tfrac{1}{2}+a+\epsilon)\,2(k_1+1)\,\dfrac{2\pi r}{s}+\sigma r}{4r(1-\sigma)\,\sin\dfrac{(2k_1-1)\,\pi}{s}}$$

$$< \frac{2k_1\,\dfrac{2r(1+\gamma_1)}{sn}+(\tfrac{1}{2}+a+\epsilon)\,2(k_1+1)\,\dfrac{2\pi r}{s}+\sigma r}{4r(1-\sigma)\,\dfrac{(2k_1-1)\,\pi}{s}\left\{1-\dfrac{2k_1^2\pi^2}{3s^2}\right\}}.$$

Taking into account the values of a, ϵ, γ, s, n, k, σ, given by (2) and (25), and observing that $\frac{1}{4}k \leqslant k_1 \leqslant \frac{1}{2}k$, we can easily see that

$$(36)\qquad D_1 < 0{\cdot}26.$$

8. (ii) There is no interval of k consecutive non-compact sectors of the ring. In this case we proceed in the following way. We take a

compact sector $e_1 f_1$ (Fig. 2) of the ring, and then we define the interval $d_1 d_2$ of $k+1$ consecutive intervals of the ring with $e_1 f_1$ as the central sector, so that the angular distance $d_1 e_1$, $f_1 d_2$ is equal to $\frac{1}{2}k(2\pi/s)$. Then we define the points g_2, h_2 so that $d_2 g_2 = \frac{1}{2}k(2\pi/s)$, $g_2 h_2 = k(2\pi/s)$ (angular distances). In the interval $g_2 h_2$ of k consecutive sectors there is at least one compact sector. Let it be $e_2 f_2$; then we define the interval

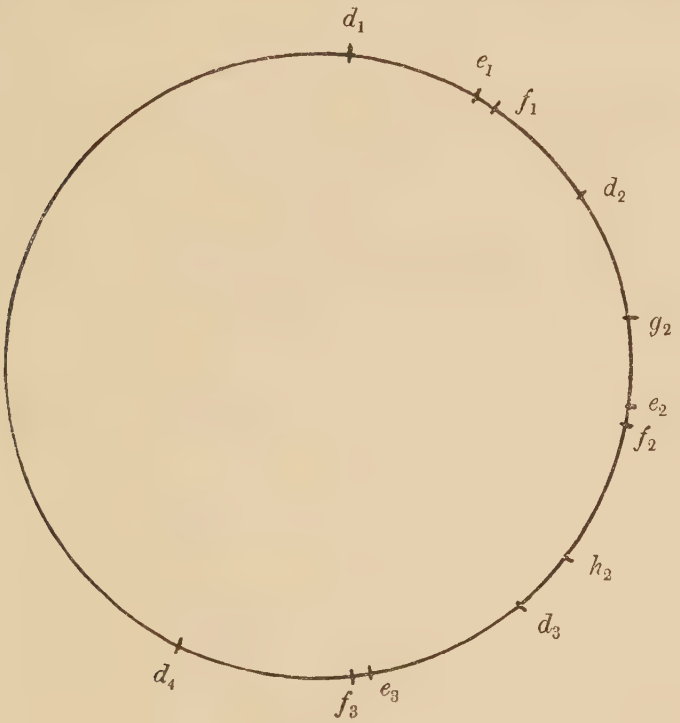


Fig. 2.

$d_2 d_3$ of consecutive sectors with $e_2 f_2$ as the central one. We have obviously,

$$\frac{k}{2}\,\frac{2\pi}{s} \leqslant d_2 e_2 = f_2 d_3 \leqslant (\tfrac{3}{2}k-1)\,\frac{2\pi}{s}. \tag{37}$$

In the same way we define an interval $d_3 d_4$ of consecutive sectors with a compact sector $e_3 f_3$ as the central one. Clearly we can continue in this way until we come to a point d_v such that the angular distance $d_v d_1$ is less than $(\frac{3}{2}k-1)(2\pi/s)$. Thus we finally have the intervals $d_1 d_2$, $d_2 d_3$, ..., $d_{v-1} d_v$; for each of them we have inequalities similar to (37).

$$\frac{k}{2}\,\frac{2\pi}{s} \leqslant d_i e_i = f_i d_{i+1} \leqslant (\tfrac{3}{2}k-1)\,\frac{2\pi}{s}. \tag{38}$$

We now define the mean density of B_1 in an interval $d_i d_{i+1}$ as $L_e(B_1 \times d_i d_{i+1})$, divided by the number of sectors belonging to $d_i d_{i+1}$. From the fact that $L_e\{B_1 \times c(0, r)\} = (1+\gamma_1)2r$, and that the number of sectors belonging to all intervals $d_1 d_2$, $d_2 d_3$, ..., $d_{v-1} d_v$ is greater than $s-3k$, we conclude that there is at least one interval $d_i d_{i+1}$ with the mean density of B_1 less than $\{2r(1+\gamma_1)/(s-3k)\}$. Denoting by k_1 the number of sectors in the interval $d_i e_i$, we have

$$\tfrac{1}{2}k \leqslant k_1 \leqslant \tfrac{3}{2}k-1. \tag{39}$$

Let β be any point of $B_1 \times e_i f_i$. Consider the circle

$$c\left\{\beta,\ 2r(1-\sigma)\sin\frac{k_1}{2}\frac{2\pi}{s}\right\}.$$

The intersection of this circle with the ring $c(0, r)-c\{0, (1-\sigma)r\}$ is obviously included in the interval $d_i d_{i+1}$. We write

$$B_1 \times c\left\{\beta,\ 2r(1-\sigma)\sin\frac{k_1}{2}\frac{2\pi}{s}\right\} = E_1+E_2, \tag{40}$$

where E_1 is the part included in the interval $d_i d_{i+1}$, and E_2 the part outside the circle $c(0, r)$.

By the definition of $d_i d_{i+1}$,

$$L_e(B_1 \times d_i d_{i+1}) < (2k_1+1)\frac{2r(1+\gamma_1)}{s-3k}, \tag{41}$$

and thus

$$L_e E_1 < (2k_1+1)\frac{2r(1+\gamma_1)}{s-3k}. \tag{42}$$

On the other hand, from (26),

$$L_e E_2 < \sigma r. \tag{43}$$

From (40), (42), (43) we obtain the following inequality for D_2, the mean density of B_1 in the circle $c\{\beta, 2r(1-\sigma)\sin\frac{1}{2}k_1(2\pi/s)\}$,

$$D_2 < \frac{\{2r(1+\gamma_1)(2k_1+1)/(s-3k)\}+\sigma r}{4r(1-\sigma)\sin(k_1\pi/s)}, \tag{44}$$

from which it is easy to see that

$$D_2 < \frac{1}{\pi}+0{\cdot}005 = t < \tfrac{1}{3}. \tag{45}$$

9. Thus in both cases (i) and (ii) we can find a compact sector of the ring $c(0, r)-c\{0, (1-\sigma)r\}$ such that about each point of B_1 belonging

to this sector we can draw a circle of radius less than $\frac{1}{100}r < \frac{1}{200}\delta$ in which the mean density of B_1 is less than $t < \frac{1}{3}$. The measure of the part of B_1 included in this compact interval is by definition

$$\geqslant \frac{2r(1+\gamma_1)}{sn} \quad i.e. \quad \geqslant \frac{L_c\{B_1 \times c(0, r)\}}{sn}.$$

The argument is applicable to each circle of C_2, and consequently we come to the following result :

Given δ there exists a subset G of B_1 such that, with each point of G as centre, there can be drawn a circle of radius less than $\frac{1}{200}\delta$, in which the mean density of B_1 is less than $t < \frac{1}{3}$. For the measure of G we have the inequality

$$L_c G \geqslant \frac{1}{sn} L_c(B_1 \times C_2) > \frac{1}{2sn} b_1. \tag{46}$$

We give now to δ an infinite succession of values tending steadily to zero,

$$\delta_1 > \delta_2 > \ldots, \ \delta_n \to 0, \ l > \delta_1, \tag{47}$$

and we call G_1, G_2,... the sets G corresponding to these values of δ. We have

$$L_c G_i > \frac{1}{2sn} b_1 \quad \text{for all } i. \tag{48}$$

Let H be the upper limit of the sequence of sets G_1, G_2, ... Then

$$L_c H \geqslant \frac{1}{2sn} b_1. \tag{49}$$

Let β be any point of H. It belongs to an infinite number of members of the sequence G_1, G_2, ..., say to sets

$$G_{i_1}, \ G_{i_2}, \ G_{i_3}, \ \ldots. \tag{50}$$

Consequently, about the point β as centre there can be drawn an infinite set of circles of radii respectively less than $\frac{1}{200}\delta_{i_1}$, $\frac{1}{200}\delta_{i_2}$, ..., *i.e.* of radii tending to zero, such that the mean density of B_1 in each of them is less than $t < \frac{1}{3}$.

Thus *the lower density of the set B_1 at the point β is less than* $t < \frac{1}{3}$. Similarly the lower density at each point of the set H (of positive measure) is less than $t < \frac{1}{3}$.

In this way our theorem is proved.

MULTIPLE TANGENTS TO SURFACES IN HIGHER SPACE

By L. Roth.

[Received 6 March, 1930.—Read 13 March, 1930.]

The present note is intended to supplement the work of a previous paper*, in which approximately the same ground was covered by use of the functional method. It is now shown that the formulae for *multiple tangents* to a surface in $[n]$ $(4 \leqslant n \leqslant 6)$ may be deduced from the formulae for multiple secants by means of elementary correspondence processes. At the same time, the opportunity has been taken to add some new results and to correct a few errors in the previous work.

The symbols and notation for surfaces are the same as before; any result which is quoted here has already been established in the former paper. It may be recalled that a general surface of order n depends upon four characters† n, a, j, m which, for surfaces to be mentioned later, have the following values :

n	a	j	m
F_4	8	4	12
F_4''	6	6	3
F_5'	12	8	20
ϕ_6	12	12	12
ϕ_7	14	16	12
J_9	34	38	87
J_{10}	40	50	105

A description of surfaces in four dimensions, with an enumeration of various types, will be found in a paper by Severi‡.

* "Some formulae for surfaces in higher space", *Proc. Camb. Phil. Soc.*, 25 (1929), 390. This will be referred to as F.

† For these reference may be made to *Proc. London Math. Soc.* (2), 32 (1931), 74. The significance of the four characters was originally pointed out by Severi.

‡ Severi, *Rend. di Palermo*, 15 (1901), 33.

The general correspondence theorem. This theorem*, which was first used extensively by Schubert, forms the basis of the following investigation, and may be enunciated thus :

If the points of two curves of orders α and β respectively are in (x, x') correspondence, then the number of united points is

$$x\beta+x'\alpha-\gamma,$$

where γ is the order of the scroll of lines joining corresponding points, counted twice in involutory cases. In particular, the two curves may be supposed to be one and the same curve, and the theorem holds without modification.

Space of four dimensions.

1. *The improper nodes.* The cone of chords to F from an arbitrary point P is of order

$$\mu_2 = \binom{n}{2} - \tfrac{1}{2}a \tag{1}$$

and the curve of intersection of the cone with F is of order $2\mu_2$. Consider the correspondence set up on this curve by the generators of the cone. The united points arise from the j tangents to F, which pass through P, and the d improper nodes, counted twice. Hence, by the theorem quoted above,

$$j+2d = 2\,.\,2\mu_2-2\mu_2 = n(n-1)-a, \tag{2}$$

a result first given by Severi†.

If the surface is projected from an ordinary point P of itself, the projection has order $n-1$ and rank $a-2$; if P is at an improper node, the order is $n-2$ and the rank $a-4$. It follows from (1) that the cone of trisecants through a point of F is of order

$$t_3 = \binom{n-1}{2} - \tfrac{1}{2}(a-2), \tag{3}$$

and that the cone of trisecants through an improper node is of order

$$\tau_3 = \binom{n-2}{2} - \tfrac{1}{2}(a-4). \tag{4}$$

From the formula

$$\mu_3 = \binom{n}{3} - \tfrac{1}{2}an + \tfrac{1}{3}(2a+2j+m), \tag{5}$$

---

* See Salmon-Rogers, *Analytic geometry of three dimensions*, 2 (1915), 289; also Segre, *Encyk. der Math. Wiss.* III, C 7, 911. The name given above is due to James.

† Severi, *op. cit.*

giving the number of trisecants to F from an arbitrary point*, may be deduced the number of quadrisecants through an arbitrary or a double point of F. For the projection has class m and $j-4$ pinch-points in the first case, and class m and $j-8$ pinch-points in the second. Thus the required numbers are

$$t_4 = \binom{n-1}{3} - \tfrac{1}{2}(a-2)(n-1) + \tfrac{1}{3}(2a+2j+m-12)$$

$$= \mu_3 - \mu_2 + 2n - 6, \tag{6}$$

and

$$\tau_4 = \binom{n-2}{3} - \tfrac{1}{2}(a-4)(n-2) + \tfrac{1}{3}(2a+2j+m-24)$$

$$= \mu_3 - 2\mu_2 + 5n - 16. \tag{7}$$

2. *The apparent double curve.* The cone M_2 of chords from P intersects F in the "apparent double curve", of order $2\mu_2$. We may find the class of immersion of this curve (*i.e.* the order of the form of tangent planes to F at the points of the curve) by the following method†. Let F be given a small displacement in the direction of some fixed point O in the space Ω at infinity. The displaced surface will meet M_2 in $n\mu_2$ points, consisting of

(i) the $3\mu_3$ points of intersection of F with the trisecants from P;

(ii) the $2\mu_2$ points in which the apparent double curve meets Ω; since these are unchanged by the displacement;

(iii) the ρ points at which the tangent plane to F meets OP.

Thus we obtain $$n\mu_2 = 3\mu_3 + 2\mu_2 + \rho,$$

giving $$\rho = an - a - 2j - m. \tag{8}$$

The unsimplified form is precisely that in which the result is obtained from the theory of reciprocal surfaces in ordinary space‡.

3. *Tangents elsewhere secant.* The trisecants of F form a complex of order μ_3 and class

$$\nu_3 = 2\binom{n}{3} - \tfrac{1}{2}a(n-2). \tag{9}$$

---

* F, § 5, equation (4).

† *Cf.* Zeuthen, *Lehrbuch der abzählenden Methoden* (Leipzig, 1914), 101.

‡ Salmon-Rogers, *op. cit.*

Now the trisecants which meet an arbitrary line l generate a scroll of order $\mu_3+\nu_3$ which meets F in a curve of order $3\nu_3$; on this curve the generators of the scroll define a correspondence $(2, 2)$, whose united points arise from

(i) the $2:1$-tangents which meet l, and

(ii) the lines through the improper nodes (counted twice) which meet l and F elsewhere. Hence the order $\mu_{2:1}$ of the form of these tangents is given by

$$\mu_{2:1}+2(n-2)\,d = 2\,.\,2\,.\,3\nu_3-3\,.\,2(\mu_3+\nu_3),$$

or
$$\mu_{2:1} = (a+j)\,n-2m-6j. \tag{10}$$

The number of $2:1$-tangents in an arbitrary space is

$$\nu_{2:1} = an-6a+4n. \tag{11}$$

4. *Inflexional tangents.* Consider the scroll of $2:1$-tangents which meet an arbitrary plane π; its order is obviously $\mu_{2:1}+\nu_{2:1}$. At each ordinary point of F there are $j-4$ secants elsewhere tangent and $n-4$ tangents elsewhere secant (lying in the tangent plane at the point in question). At a node of F there are $j-8$ secants elsewhere tangent and two sets of $n-5$ tangents elsewhere secant, one set lying in each tangent plane. It follows that the number of $2:1$-tangents meeting one such tangent (found by taking a plane section through the tangent) is

$$m_{2:1} = \mu_{2:1}-2j-2n+16. \tag{12}$$

As a verification we note that this vanishes for F_4, which does not possess $2:1$-tangents. Also the scroll touches F along a curve of order

$$\nu_{2:1}+n(n-4) = an+n^2-6a, \tag{13}$$

and intersects F in a curve of order

$$\nu_{2:1}+n(j-4) = an+jn-6a. \tag{14}$$

The correspondence between these curves gives (as in § 3) the number $\mu_{3:}$ of inflexional tangents meeting π:

$$\mu_{3:}+2d = (an+n^2-6a)+(an+jn-6a)-(\mu_{2:1}+\nu_{2:1}),$$

whence
$$\mu_{3:} = 7j-5a+2m-3n. \tag{15}$$

The scroll of inflexional tangents touches F along a curve whose order $m_{3:}$ is determined as follows. Consider the tangents to F at points of

an arbitrary space section σ; these generate a form f of order a which meets F again in a curve of order $a_{2:1}$, say, which is the curve of intersection of 2 : 1-tangents whose contacts are on σ. Now a second space section σ' of F meets f in an points, n of which (suitably counted) lie in the plane common to σ and σ'. Hence we have

$$a_{2:1} = an - \lambda n,$$

where λ is a constant to be determined.

Since $\nu_{2:1}$ vanishes for F_4, $a_{2:1}$ must in this case be equal to a certain multiple* of 16 (the number of lines on F_4); hence $\lambda = 4$, or else the result would be negative for F'_4, which possesses such tangents and which contains no lines. Thus

$$a_{2:1} = an - 4n.$$

But this curve meets σ in the simple intersections of the 2 : 1-tangents lying in σ and the $m_{3:}$ points of contact of the scroll of inflexional tangents. It follows that $a_{2:1} = \nu_{2:1} + m_{3:}$, giving

$$m_{3:} = 6a - 8n. \tag{16}$$

5. *The* 2 : 2*-tangents.* We proceed now to obtain formulae from the results for quadrisecants†. These generate a form of order

$$\mu_4 = 3\binom{n}{4} - a\binom{n}{2} + \tfrac{1}{3}n(5a + 2j + m) + \tfrac{1}{8}a^2 - \tfrac{1}{4}(7a + 11j + 6m) \tag{17}$$

and class

$$\nu_4 = 2\binom{n}{4} - \tfrac{1}{2}a\binom{n}{2} + an + \tfrac{1}{8}a^2 - \tfrac{11}{4}a + n. \tag{18}$$

The multiplicity of an ordinary and a double point of F on this form have already been given in (6) and (7).

Taking an arbitrary plane section of the form through a given quadrisecant, we have, for the number of quadrisecants meeting it at points external to F,

$$m_4 = \mu_4 - 4t_4 + 1. \tag{19}$$

This vanishes for ϕ_6, ϕ_7, J_9, J_{10}.

Consider the quadrisecants which meet an arbitrary plane π; they generate a scroll of order $\mu_4 + \nu_4$, meeting F in a curve of order $nt_4 + 4\nu_4$, on which the generators set up a (3, 3) correspondence. The united

* By the "principle of redundancy": F, § 4.

† See F, § 6.

points arise from the $\mu_{2:2}$ tangents elsewhere bisecant, which meet π, and τ_3 trisecants through each improper node which meet π (counted twice). Thus

$$\mu_{2:2}+2d\,.\,\tau_3 = 2\,.\,3(nt_4+4\nu_4)-4\,.\,3(\mu_4+\nu_4),$$

or
$$\mu_{2:2} = (2a+j)\binom{n}{2}+4n^2-2n(3a+3j+m)-\tfrac{1}{2}aj-\tfrac{1}{2}a^2$$
$$-7a+38j+18m-16n. \qquad (20)$$

6. The scroll of 2 : 2-tangents touches F along a curve of order $c_{2:2}$ and intersects it in a curve of order $m_{2:2}$. The only simple way of obtaining the former number appears to be by the functional method, which gives*

$$c_{2:2} = 3\binom{n}{3}+2\binom{n}{2}+a\binom{n}{2}-6an-\tfrac{1}{2}a^2+33a-24n. \qquad (21)$$

Now consider the quadrisecants of F which meet a given space section σ; these generate a scroll containing σ as a multiple curve of order t_4, and meeting the given space in ν_4 quadruple generators; hence it is of order $nt_4+4\nu_4$. The residual intersection with F is of order

$$a_4 = c_{2:2}+12\nu_4$$
$$= 24\binom{n}{4}+3\binom{n}{3}+2\binom{n}{2}-5a\binom{n}{2}+6an+a^2-12n. \qquad (22)$$

The (2, 2) correspondence set up on a_4 by the generators of the scroll has the following united points :

(i) The $m_{2:2}$ points of contact of the tangents to F which meet σ and F elsewhere;

(ii) the number of lines (counted twice) through each improper node which meet σ and F elsewhere.

To find this number we note that the lines through a node which are elsewhere bisecant meet F in a curve of order

$$2\tau_3+2(n-5).$$

This follows by taking an arbitrary space section through the node in question; for each tangent plane at the node meets F again in $n-5$

* F, § 11, equation (23).

points*. Thus we have

$$m_{2:2}+2d\{2\tau_3+2(n-5)\} = 2\,.\,2a_4-3\,.\,2(nt_4+4\nu_4),$$

giving†

$$m_{2:2} = 2\binom{n}{2}\{a+j\}-2n(3a+3j+m)-aj+66a-48n. \qquad (23)$$

7. *The bitangents.* The correspondence on the curve $m_{2:2}$ has united points which arise from

(i) the μ_{2-2} bitangents of F, counted twice;

(ii) the $j-8$ tangents from each improper node, each counted twice. Thus

$$2\mu_{2-2}+2(j-8)\,d = 2m_{2:2}-2\mu_{2:2},$$

whence
$$\mu_{2-2} = \tfrac{1}{2}(a^2+j^2)+69a-42j-18m-36n. \qquad (24)$$

8. *The* 3 : 1*-tangents.* The correspondence between $m_{2:2}$ and $c_{2:2}$ has united points due to

(i) the inflexional tangents which are elsewhere secant; and

(ii) the $2(n-5)$ tangents at each improper node, which are elsewhere secant. Hence

$$\mu_{3:1}+2(n-5)\,d = 2c_{2:2}+m_{2:2}-2\mu_{2:2},$$

and
$$\mu_{3:1} = n\mu_{3:}+141a-81j-36m-69n. \qquad (25)$$

9. *The* 2 : 3*-tangents.* The scroll of quintisecants is of order‡

$$\mu_5 = 5\binom{n}{5}-\tfrac{3}{2}a\binom{n}{3}+\tfrac{1}{3}\binom{n}{2}\{8a+2j+m+6\}-\tfrac{3}{2}mn-\tfrac{11}{4}jn-\tfrac{9}{2}an$$
$$-\tfrac{1}{3}aj-\tfrac{1}{6}am+\tfrac{1}{4}a^2n-\tfrac{5}{6}a^2+3a+12j+7m-3n, \qquad (26)$$

and its curve of intersection with F is of order

$$m_5 = 15\binom{n}{5}+3\binom{n}{3}-4a\binom{n}{3}+\tfrac{1}{3}\binom{n}{2}\{19a+4j+2m\}+\tfrac{5}{4}n\binom{a}{2}$$
$$-\tfrac{13}{6}a^2-n\{\tfrac{11}{4}j+\tfrac{97}{8}a+\tfrac{3}{2}m\}-\tfrac{2}{3}aj-\tfrac{1}{3}am+50a-24n. \qquad (27)$$

---

* Severi, *op. cit.*

† In order to obtain the correct result in F, § 11, it is necessary to deduct $4d$ from the formula there given.

‡ F, § 7.

The correspondence on this curve has united points arising from

(i) the tangents which are elsewhere trisecant;

(ii) the τ_4 quadrisecants (counted twice) which may be drawn through each improper node. Thus we obtain

$$\mu_{2:3}+2\tau_4 . d = 2 . 4m_5 - 5 . 4\mu_5,$$

which reduces* to

$$\mu_{2:3} = \binom{n}{3}\{2a+j+12\} - 2\binom{n}{2}\{3a+3j+m+12\}$$
$$+n(38j-7a+18m-\tfrac{1}{2}a^2-\tfrac{1}{2}aj)+\tfrac{1}{3}j(11a+2j+m)$$
$$+am+a^2+324a-256j-140m-132n. \qquad (28)$$

Space of five dimensions.

10. The trisecants† of F generate a V_3, whose order

$$\mu_3 = 3\binom{n}{3} - \tfrac{1}{2}jn - an + \tfrac{1}{3}(4a+7j+2m). \qquad (29)$$

The number of trisecants in an arbitrary prime is

$$\nu_3 = \binom{n}{3} - \tfrac{1}{2}an + 2a - n. \qquad (30)$$

The number t_3 of trisecants which may be drawn through a point of F is obviously the number of improper nodes of the projection of F from that point, whence

$$t_3 = \binom{n-1}{2} - \tfrac{1}{2}(a-2) - \tfrac{1}{2}(j-4)$$
$$= \binom{n}{2} - n - \tfrac{1}{2}a - \tfrac{1}{2}j + 4. \qquad (31)$$

11. *The tangents.* The tangents meeting a given S_3 generate a scroll of order $\mu_3+\nu_3$, and meet F in a curve of order $3\nu_3$ which has n t_3-ple points in S_3.

The correspondence on this curve gives the order $\mu_{2:1}$ of the scroll of tangents which are elsewhere secant; thus

$$\mu_{2:1} = 2 . 2(3\nu_3+nt_3) - 3 . 2(\mu_3+\nu_3).$$

Hence
$$\mu_{2:1} = (a+j)n + 4a - 14j - 4m + 6n. \qquad (32)$$

* The linear terms of the corresponding formula in F, § 12, are incorrect.

† See F, § 14. The formulae for t_3 and ν_3 are misprinted.

The order $c_{2:1}$ of the curve of contact is found by the functional method to be

$$c_{2:1} = an+\tfrac{4}{3}(5a-7j-2m). \tag{33}$$

The order $m_{2:1}$ of the curve of intersection is then determined as follows. Consider the trisecants which meet an arbitrary prime section σ of F; these generate a scroll of order $nt_3+3\nu_3$, which meets F again in a curve of order

$$a_3 = c_{2:1}+6\nu_3 = 6\binom{n}{3}-2an+19a-9j-2m-6n-\tfrac{1}{3}(a+j+2m). \tag{34}$$

The correspondence on this curve gives

$$m_{2:1} = 2a_3-2(nt_3-3\nu_3) = jn-12n+\tfrac{4}{3}(19a-14j-4m). \tag{35}$$

12. *The stationary tangents.* The correspondence between $c_{2:1}$ and $m_{2:1}$ gives for the number* of stationary tangents

$$\begin{aligned}\mu_{3:} &= c_{2:1}+m_{2:1}-\mu_{2:1}\\ &= 28a-14j-4m-18n.\end{aligned} \tag{36}$$

13. *Space of six dimensions.* The trisecants of F generate a scroll of order

$$\mu_3 = 2\binom{n}{3}-an-\tfrac{1}{2}jn+\tfrac{1}{3}(2a+14j+4m)-2n, \tag{37}$$

which meets F in a curve of order†

$$m_3 = 3\binom{n}{3}-\tfrac{3}{2}an-\tfrac{1}{2}jn+12a-9n. \tag{38}$$

From the correspondence on this curve we obtain the number of tangents which are elsewhere secant,

$$\mu_{2:1} = 2\,.\,2m_3-3\,.\,2\,.\,\mu_3,$$

whence

$$\mu_{2:1} = jn+44a-28j-8m-24n. \tag{39}$$

There is no similar tangent problem for surfaces in higher space.

---

* This is printed with the wrong sign in F, § 16.

† The formula previously given is erroneous in the linear terms.

ON THE LIMIT OF A FUNCTION AT A POINT

By S. VERBLUNSKY.

[Received 31 December, 1929.—Read 16 January, 1930.]

1. Suppose that we are given a function $f(t)$ defined in the interval $(x-\eta,\ x+\eta)$ and absolutely integrable in this interval. If $f(t)$ is not continuous at x, it is often possible and convenient to alter the value of $f(t)$ at $t=x$ in the following way. If

$$\frac{k}{\xi^k}\int_0^\xi \{f(x+t)+f(x-t)\}(\xi-t)^{k-1}\,dt \quad (k>0) \tag{1}$$

tends to a limit $2S$ as $\xi\to 0$, we write

$$f(x)=S.$$

In these circumstances

$$\frac{1}{\xi^k}\int_0^\xi \phi(t)(\xi-t)^{k-1}\,dt = o(1), \tag{2}$$

as $\xi\to 0$, where

$$\phi(t)=f(x+t)+f(x-t)-2S. \tag{3}$$

In this paper we consider the properties of integrals of the form (2). We suppose that $\phi(t)$ is absolutely integrable in a neighbourhood, $(0,\ \eta)$ say, of the origin, unless otherwise stated. The properties of these "fractional integrals" are naturally independent of equation (3), with which we shall have no further concern. We replace ξ by x, and write

$$\sigma^{(k)}(x)=\frac{1}{\Gamma(k)}\int_0^x \phi(u)(x-u)^{k-1}\,du \quad (k>0), \tag{4}$$

$$C^{(k)}(x)=\frac{k}{x^k}\int_0^x \phi(u)(x-u)^{k-1}\,du. \tag{5}$$

If $C^{(k)}(x)\to 0$ as $x\to 0$, we write

$$\phi(x)=o(1) \quad (C,\ k). \tag{6}$$

We shall further use the following notation:

$$\phi_0(t) = \phi(t),$$

$$\phi_1(t) = \frac{1}{t}\int_0^t \phi_0(u)\,du,$$

$$\dots \qquad \dots \qquad \dots$$

$$\phi_n(t) = \frac{1}{t}\int_0^t \phi_{n-1}(u)\,du,$$

where n is a positive integer;

$$\phi_{\alpha+r}(t) = \frac{1}{t^\alpha}\int_0^t \phi_r(u)(t-u)^{\alpha-1}\,du,$$

where $0<\alpha<1$, and r is zero or a positive integer. If $\phi_p(t)\to 0$ as $t\to 0$, we write

$$\phi(t) = o(1) \quad (H, p).$$

It will appear in a subsequent paper that it is more convenient to use the function $\phi_p(t)$ in certain questions concerning Fourier series than the function $C^{(p)}(t)$.

The functions $C^{(k)}(x)$ and $\phi_k(x)$ possess properties which are in many cases analogous to the properties of the Cesàro means of index k of a numerical series. It is well known that, if a series is summable (C, r) for some positive integer r, and if the means of order k are bounded, then the series is summable $(C, k+\delta)$ for all $\delta>0$. The following analogous property is one of the principal results of the paper.

If $$\frac{1}{t}\int_0^t |\phi_p(u)|\,du = O(1)$$

and $$\phi(t) = o(1) \quad (H, r)$$

for some positive r, then $$\phi_p(t) = o(1) \quad (H, 1+\delta)$$

for all $\delta>0$.

The other main theorem which we prove is the following:

If $$\phi(u) = o(1) \quad (C, k),$$

then $$\phi(u) = o(1) \quad (H, k),$$

and conversely.

The special case when k is an integer is due to Hardy and Littlewood*.

* Hardy and Littlewood, *Proc. London Math. Soc.* (2), 22 (1924), xl–xliii.

The methods of this paper are very elementary. Many of the proofs are rather long, but there is no occasion for any difficult or delicate arguments. The reason is that, though the results here obtained are sufficient for the applications to the theory of Fourier series which I have in view, they express the more immediate properties of the functions $C^{(k)}(x)$ and $\phi_k(x)$. Doubtless, in order to investigate the deeper properties, more advanced methods will be necessary.

2. It is well known that, if $\phi(u)$ is absolutely integrable in $(0, v)$, then, for almost all x in $(0, v)$,

$$\int_0^x (x-u)^{\delta-1}\,du\int_0^u \phi(y)(u-y)^{\alpha-1}\,dy = \frac{\Gamma(\alpha)\,\Gamma(\delta)}{\Gamma(\alpha+\delta)}\int_0^x \phi(y)(x-y)^{\alpha+\delta-1}\,dy \qquad (\delta>0,\ \alpha>0).$$

Our first problem is to investigate how far this result remains true when it is no longer assumed that $\phi(u)$ is absolutely integrable in $(0, v)$. A case of common occurrence, for example, is where

$$\phi(u) = \frac{1}{u}\int_0^u f(w)\,dw,$$

and where we know that $\lim_{\epsilon\to 0}\int_\epsilon^v \phi(u)\,du$ exists, but not that $\phi(u)$ is absolutely integrable in $(0, v)$. To fix the ideas, we shall suppose that $\phi(u)$ is integrable D in $(0, v)$, *i.e.* integrable in the sense of Denjoy-Perron. After obtaining some general lemmas (Lemmas 1, 2), we consider three important special cases of $\phi(u)$, and deduce Theorems I, II, III respectively.

LEMMA 1. *If $\phi(y)$ is integrable D in $(0, \eta)$, then*

$$\lim_{\epsilon\to 0}\int_\epsilon^v (v-u)^{\delta-1}\,du\int_0^{u-\epsilon}\phi(y)(u-y)^{\alpha-1}\,dy = \frac{\Gamma(\delta)\,\Gamma(\alpha)}{\Gamma(\alpha+\delta)}\int_0^v \phi(y)(v-y)^{\alpha+\delta-1}\,dy \qquad (\alpha>0,\ \delta>0,\ 0<v<\eta),$$

whenever the second member exists.

We first prove that

$$\int_\epsilon^v (v-u)^{\delta-1}\,du\int_0^{u-\epsilon}\phi(y)(u-y)^{\alpha-1}\,dy = \int_0^{v-\epsilon}\phi(y)\,dy\int_{y+\epsilon}^v (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du. \quad (7)$$

Writing $$\Phi(y) = \int_0^y \phi(u)\,du,$$

we have

$$\int_0^{u-\epsilon} \phi(y)(u-y)^{\alpha-1}\,dy = \Phi(u-\epsilon)\,\epsilon^{\alpha-1} + (\alpha-1)\int_0^{u-\epsilon} \Phi(y)(u-y)^{\alpha-2}\,dy.$$

Hence

$$\int_\epsilon^v (v-u)^{\delta-1}\,du \int_0^{u-\epsilon} \phi(y)(u-y)^{\alpha-1}\,dy$$

$$= \int_\epsilon^v (v-u)^{\delta-1}\,\Phi(u-\epsilon)\,\epsilon^{\alpha-1}\,du + (\alpha-1)\int_\epsilon^v (v-u)^{\delta-1}\,du \int_0^{u-\epsilon} \Phi(y)(u-y)^{\alpha-2}\,dy$$

$$= \int_\epsilon^v (v-u)^{\delta-1}\,\Phi(u-\epsilon)\,\epsilon^{\alpha-1}\,du + (\alpha-1)\int_0^{v-\epsilon} \Phi(y)\,dy \int_{y+\epsilon}^v (v-u)^{\delta-1}(u-y)^{\alpha-2}\,dy. \quad (8)$$

Further $$\int_0^{v-\epsilon} \phi(y)\,dy \int_{y+\epsilon}^v (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du$$

$$= -\int_0^{v-\epsilon} \Phi(y)\,\frac{d}{dy}\left\{\int_{y+\epsilon}^v (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du\right\} dy.$$

But $$\frac{d}{dy}\int_{y+\epsilon}^v (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du$$

$$= -\int_{y+\epsilon}^v (v-u)^{\delta-1}(u-y)^{\alpha-2}(\alpha-1)\,du - (v-y-\epsilon)^{\delta-1}\,\epsilon^{\alpha-1}.$$

When this is substituted in the second member of the preceding equation, the resulting expression is seen to be equal to (8). This proves (7).

3. We next show that

$$\lim_{\epsilon\to 0} \int_0^{v-\epsilon} \phi(y)\,dy \int_y^{y+\epsilon} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du = 0. \quad (9)$$

We have

$$\int_y^{y+\epsilon} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du = (v-y)^{\alpha+\delta-1} \int_0^{\epsilon/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt.$$

For small ξ, $$\int_0^\zeta (1-t)^{\delta-1}\,t^{\alpha-1}\,dt = O(\xi^\alpha).$$

The expression $$\int_0^{\epsilon/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt \quad (10)$$

is a monotone increasing function of y, and is $O(\epsilon^{\frac{1}{2}\alpha})$ for

$$0 \leqslant y \leqslant v-\epsilon^{\frac{1}{2}}.$$

Hence $$\lim_{\epsilon\to 0}\int_0^{v-\sqrt{\epsilon}} \phi(y)(v-y)^{\alpha+\delta-1}\,dy \int_0^{\epsilon/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt = 0, \tag{11}$$

since we have supposed that

$$\int_0^v \phi(y)(v-y)^{\alpha+\delta-1}\,dy$$

exists. Finally, (10) is $O(1)$ for

$$v-\sqrt{\epsilon} < y \leqslant v-\epsilon,$$

so that $$\lim_{\epsilon\to 0}\int_{v-\sqrt{\epsilon}}^{v-\epsilon} \phi(y)(v-y)^{\alpha+\delta-1}\,dy \int_0^{\epsilon/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt = 0. \tag{12}$$

From (11) and (12), equation (9) follows.

From (7) and (9) we have

$$\begin{aligned}\lim_{\epsilon\to 0}\int_\epsilon^v (v-u)^{\delta-1}\,du \int_0^{u-\epsilon} \phi(y)(u-y)^{\alpha-1}\,dy \\ &= \lim_{\epsilon\to 0}\int_0^{v-\epsilon} \phi(y)\,dy \int_y^v (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du \\ &= \frac{\Gamma(\alpha)\,\Gamma(\delta)}{\Gamma(\alpha+\delta)}\int_0^v \phi(y)(v-y)^{\alpha+\delta-1}\,dy.\end{aligned}$$

This proves Lemma 1.

4. LEMMA 2. *If $\phi(y)$ is integrable D in $(0, \eta)$, then*

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0} I(\epsilon, \epsilon') = \lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\int_\epsilon^v (v-u)^{\delta-1}\,du \int_{u-\epsilon}^{u-\epsilon'} \phi(y)(u-y)^{\alpha-1}\,dy = 0,$$

whenever $$\int_0^v \phi(y)(v-y)^{\alpha+\delta-1}\,dy \tag{13}$$

exists.

We first show that

$$\begin{aligned}I(\epsilon, \epsilon') = \Big(\int_{\epsilon-\epsilon'}^{v-\epsilon} dy \int_{y+\epsilon'}^{y+\epsilon} du + \int_{v-\epsilon}^{v-\epsilon} dy \int_{y+\epsilon'}^{v} du \\ + \int_0^{\epsilon-\epsilon'} dy \int_\epsilon^{y+\epsilon} du\Big)\,\phi(y)(v-u)^{\delta-1}(u-y)^{\alpha-1}.\end{aligned} \tag{14}$$

We have

$$\int_{u-\epsilon}^{u-\epsilon'} \phi(y)(u-y)^{\alpha-1}\,dy = \Phi(u-\epsilon')\,\epsilon'^{\alpha-1} - \Phi(u-\epsilon)\,\epsilon^{\alpha-1} + \int_{u-\epsilon}^{u-\epsilon'} \Phi(y)(\alpha-1)(u-y)^{\alpha-2}\,dy,$$

so that

$$I(\epsilon, \epsilon') = \int_{\epsilon}^{v} (v-u)^{\delta-1}\,du\,[\Phi(u-\epsilon')\,\epsilon'^{\alpha-1} - \Phi(u-\epsilon)\,\epsilon^{\alpha-1}] + \int_{\epsilon}^{v} (v-u)^{\delta-1}\,du \int_{u-\epsilon}^{u-\epsilon'} \Phi(y)(\alpha-1)(u-y)^{\alpha-2}\,dy. \quad (15)$$

The second term, as an absolutely convergent double integral, is equal to

$$\left(\int_{\epsilon-\epsilon'}^{v-\epsilon} dy \int_{y+\epsilon'}^{y+\epsilon} du + \int_{v-\epsilon}^{v-\epsilon'} dy \int_{y+\epsilon'}^{v} du + \int_{0}^{\epsilon-\epsilon'} dy \int_{\epsilon}^{y+\epsilon} du\right) \Phi(y)(v-u)^{\delta-1}(u-y)^{\alpha-2}(\alpha-1) = I_1 + I_2 + I_3.$$

Now

$$I_1 = \int_{\epsilon-\epsilon'}^{v-\epsilon} \Phi(y)\,dy \left[(v-y-\epsilon)^{\delta-1}\,\epsilon^{\alpha-1} - (v-y-\epsilon')^{\delta-1}\,\epsilon'^{\alpha-1} - \frac{d}{dy}\int_{y+\epsilon'}^{y+\epsilon} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du\right]$$

$$= \int_{\epsilon-\epsilon'}^{v-\epsilon} \Phi(y)\,dy\,[(v-y-\epsilon)^{\delta-1}\,\epsilon^{\alpha-1} - (v-y-\epsilon')^{\delta-1}\,\epsilon'^{\alpha-1}] - \Phi(v-\epsilon)\int_{v-\epsilon+\epsilon'}^{v} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du + \Phi(\epsilon-\epsilon')\int_{\epsilon}^{2\epsilon-\epsilon'} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du + \int_{\epsilon-\epsilon'}^{v-\epsilon} \phi(y)\,dy \int_{y+\epsilon'}^{y+\epsilon} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du.$$

In a similar manner,

$$I_2 = -\int_{v-\epsilon}^{v-\epsilon'} \Phi(y)(v-y-\epsilon')^{\delta-1}\,\epsilon'^{\alpha-1}\,dy + \Phi(v-\epsilon)\int_{v-\epsilon+\epsilon}^{v} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du + \int_{v-\epsilon}^{v-\epsilon} \phi(y)\,dy \int_{y+\epsilon'}^{v} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du,$$

and

$$I_3 = \int_0^{\epsilon-\epsilon'} \Phi(y)(v-y+\epsilon)^{\delta-1}\,\epsilon^{\alpha-1}\,dy - \Phi(\epsilon-\epsilon')\int_\epsilon^{2\epsilon-\epsilon'} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du$$
$$+\int_0^{\epsilon-\epsilon'} \phi(y)\,dy \int_\epsilon^{y+\epsilon} (v-u)^{\delta-1}(u-y)^{\alpha-1}\,du.$$

Substituting these expressions in (15), we obtain (14).

5. From (14) we deduce that

$$I(\epsilon,\epsilon') = \int_{\epsilon-\epsilon'}^{v-\epsilon} \phi(y)(v-y)^{\alpha+\delta-1}\,dy \int_{\epsilon'/(v-y)}^{\epsilon/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt$$
$$+\int_{v-\epsilon}^{v-\epsilon'} \phi(y)(v-y)^{\alpha+\delta-1}\,dy \int_{\epsilon'/(v-y)}^{1} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt$$
$$+\int_0^{\epsilon-\epsilon'} \phi(y)(v-y)^{\alpha+\delta-1}\,dy \int_{(\epsilon-y)/(v-y)}^{\epsilon/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt$$
$$= J_1+J_2+J_3.$$

We have

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0} J_1 = \lim_{\epsilon\to 0}\int_\epsilon^{v-\epsilon} \phi(y)(v-y)^{\alpha+\delta-1}\,dy \int_0^{\epsilon/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt$$
$$-\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\int_{\epsilon-\epsilon'}^{v-\epsilon} \phi(y)(v-y)^{\alpha+\delta-1}\,dy \int_0^{\epsilon'/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt.$$

The first term vanishes, as in the proof of equation (9). As regards the second term, since, for fixed ϵ,

$$\int_0^{\epsilon'/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt = O(\epsilon'^{\alpha}) \quad (0 < y \leqslant v-\epsilon)$$

and is a monotone increasing function of y, we have, by the second mean value theorem,

$$\lim_{\epsilon'\to 0}\int_{\epsilon-\epsilon'}^{v-\epsilon} \phi(y)(v-y)^{\alpha+\delta-1}\,dy \int_0^{\epsilon'/(v-y)} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt = 0.$$

Thus
$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0} J_1 = 0.$$

Next, since
$$\int_{\epsilon'/(v-y)}^{1} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt$$

is monotone in y and uniformly bounded,

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to} J_2 = 0.$$

Finally,

$$J_3 = \int_0^{\epsilon-\epsilon'} \phi(y)(v-y)^{\alpha+\delta-1}\,dy\left[\int_0^{\epsilon/(v-y)} - \int_0^{1-[(v-\epsilon)/(v-y)]} (1-t)^{\delta-1}\,t^{\alpha-1}\,dt\right],$$

and, since both integrals in the square brackets are monotone in y and uniformly bounded,

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0} J_3 = 0.$$

This proves Lemma 2.

6. It remains to investigate under what conditions

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\int_\epsilon^v (v-u)^{\delta-1}\,du\int_{u-\epsilon'}^u \phi(y)(u-y)^{\alpha-1}\,dy = 0. \tag{16}$$

For this purpose we introduce the following definition.

DEFINITION. If $\phi(y)$ is integrable in $(0, \xi)$, and if, for $0<\epsilon\leqslant u\leqslant\xi$ and $0<\epsilon'\leqslant\epsilon'(\epsilon)$, where $\epsilon'(\epsilon)$ denotes a number depending on ϵ,

$$\left|\int_{u-\epsilon'}^u \phi(y)(u-y)^{\alpha-1}\,dy\right| \leqslant \kappa_\epsilon(u),$$

where $\kappa_\epsilon(u)$ is a non-negative function absolutely integrable in $(0, \xi)$, and which may depend on ϵ, then we shall say that $\phi(y)$ satisfies the condition (κ). The following cases in which condition (κ) is satisfied are of special importance.

7. *The function $\phi(y)$ is absolutely integrable in (η, ξ) for all $0<\eta<\xi$, but not necessarily so in $(0, \xi)$.*

Having fixed a value of ϵ, take $\epsilon'<\frac{1}{2}\epsilon$. The function $\phi_\epsilon(y)$, defined as

$$\begin{aligned}\phi_\epsilon(y) &= \phi(y) \quad (\tfrac{1}{2}\epsilon\leqslant y\leqslant\xi)\\ &= 0 \quad (0<y<\tfrac{1}{2}\epsilon)\end{aligned}$$

is absolutely integrable in $(0, \xi)$. We may therefore write

$$\phi_\epsilon(y) = \phi_{1\epsilon}(y) - \phi_{2\epsilon}(y),$$

where

$$2\phi_{1\epsilon}(y) = \phi_\epsilon(y) + |\phi_\epsilon(y)|,$$

and where $\phi_{1\epsilon}(y)\geqslant 0$, $\phi_{2\epsilon}(y)\geqslant 0$ are integrable in $(0, \xi)$.

We have

$$\left|\int_{u-\epsilon'}^u \phi(y)(u-y)^{\alpha-1}\,dy\right| \leqslant \int_{u-\epsilon}^u \phi_{1\epsilon}(y)(u-y)^{\alpha-1}\,dy + \int_{u-\epsilon'}^u \phi_{2\epsilon}(y)(u-y)^{\alpha-1}\,dy$$

$$\leqslant \int_0^u \{\phi_{1\epsilon}(y)+\phi_{2\epsilon}(y)\}(u-y)^{\alpha-1}\,dy.$$

The preceding expression is of the form $\kappa_\epsilon(u)$, as may be seen from the equation

$$\int_0^\xi (\xi-u)^{-\alpha}\,du\int_0^u \{\phi_{1\epsilon}(y)+\phi_{2\epsilon}(y)\}(u-y)^{\alpha-1}\,dy$$

$$=\Gamma(\alpha)\,\Gamma(1-\alpha)\int_0^\xi \{\phi_{1\epsilon}(y)+\phi_{2\epsilon}(y)\}\,dy,$$

the existence of either member implying that of the other.

Since $$\int_0^v (v-u)^{\delta-1}\kappa_\epsilon(u)\,du$$

exists for almost all v in $(0,\ \xi)$, it follows, by a well known theorem of Lebesgue, that, for each such v,

$$\lim_{\epsilon'\to 0}\int_\epsilon^v (v-u)^{\delta-1}\,du\int_{u-\epsilon'}^u \phi(y)(u-y)^{\alpha-1}\,dy=0, \tag{17}$$

and hence that (16) is satisfied for almost all v in $(0,\ \xi)$.

Taking account of Lemmas 1 and 2, we deduce

THEOREM I. *If $\phi(y)$ is integrable D in $(0,\ \xi)$ and integrable L in $(\eta,\ \xi)$ for every $0<\eta<\xi$, then, for almost all v in $(0,\ \xi)$,*

$$\int_0^v (v-u)^{\delta-1}\,du\int_0^u \phi(y)(u-y)^{\alpha-1}\,dy=\frac{\Gamma(\alpha)\,\Gamma(\delta)}{\Gamma(\alpha+\delta)}\int_0^v \phi(y)(v-y)^{\alpha+\delta-1}\,dy$$

$$(\alpha>0,\ \delta>0).$$

8. *The function $\phi(y)$ is integrable D in $(0,\ \xi)$ and $\alpha\geqslant 1$.*

Writing $$\int_0^u \phi(y)\,dy=\Phi(u),$$

we have $$\int_{u-\epsilon'}^u \phi(y)(u-y)^{\alpha-1}\,dy=\epsilon'^{\alpha-1}[\Phi(u-\epsilon'')-\Phi(u-\epsilon')],$$

where $0<\epsilon''<\epsilon'$. Since $\Phi(u)$ is uniformly continuous, it follows in this case that, for fixed ϵ, $\kappa_\epsilon(u)$ is independent of u. Hence (17) is satisfied for *all* v in $(0,\ \xi)$, and so is (16). Since, further, (13) exists for all v in $(0,\ \xi)$, we have

THEOREM II. *If $\phi(y)$ is integrable D in $(0,\ \xi)$, then*

$$\int_0^v (v-u)^{\delta-1}\,du\int_0^u \phi(y)(u-y)^{\alpha-1}\,dy=\frac{\Gamma(\alpha)\,\Gamma(\delta)}{\Gamma(\alpha+\delta)}\int_0^v \phi(y)(v-y)^{\alpha+\delta-1}\,dy$$

$$(\alpha\geqslant 1,\ \delta>0)$$

for all v in $(0,\ \xi)$.

9. *The function $\phi(y)$ is integrable D in $(0, \xi)$, and is of the form*

$$\phi(y) = \frac{1}{y}\int_0^y f(w)\,dw.$$

Since $\phi(y)$ is continuous in (ϵ, ξ),

$$\left|\int_{u-\epsilon'}^{u} \phi(y)(u-y)^{\alpha-1}\,dy\right| < M_\epsilon \frac{\epsilon'^\alpha}{\alpha}$$
$$(0 < \epsilon < u \leqslant \xi;\ 0 < \epsilon' < \epsilon_{12}),$$

where $$M_\epsilon = \operatorname*{Max}_{\frac{1}{2}\epsilon \leqslant y \leqslant \xi} |\phi(y)|.$$

Hence (16) holds for all v in $(0, \xi)$. Further, it is easily seen that (13) exists for all v in $(0, \xi)$. Hence we have

THEOREM III. *If $\phi(y)$ is integrable D in $(0, \xi)$, and is of the form*

$$\phi(y) = \frac{1}{y}\int_0^y f(w)\,dw,$$

then

$$\int_0^v (v-u)^{\delta-1}\,du \int_0^u \phi(y)(u-y)^{\alpha-1}\,dy = \frac{\Gamma(\alpha)\,\Gamma(\delta)}{\Gamma(\alpha+\delta)}\int_0^v \phi(y)(v-y)^{\alpha+\delta-1}\,dv$$
$$(\alpha > 0,\ \delta > 0)$$

for all v in $(0, \xi)$.

10. In what follows, we shall occasionally be concerned with equations which hold for almost all values of the variables which occur. In such cases we shall replace the sign $=$ by $\simeq$. In a similar manner, if a function $\phi(t)$ tends to zero as $t \to 0$, provided that a set of values of t of zero measure is neglected, we shall write

$$\phi(t) \simeq o(1).$$

The following three lemmas are fundamental.

LEMMA 3. *If* $$\phi(x) - g \simeq o(1),$$

then $$C^{(k)}(x) - g = o(1).$$

We may, without loss of generality, suppose that $g = 0$. The proof is then immediate.

LEMMA 4. *If $\phi(y)$ is integrable D in $(0, \xi)$ and integrable L in (η, ξ) for all $0 < \eta < \xi$, and if*

$$C^{(k)}(x) - g \simeq o(1) \quad (k > 0),$$

then $$C^{(k+\delta)}(x) - g \simeq o(1) \quad (\delta > 0).$$

We may, without loss of generality, suppose that $g=0$. By Theorem I,

$$\sigma^{(k+\delta)}(x) \simeq \sigma^{(\delta)}\{\sigma^{(k)}(x)\}.$$

We now have

$$C^{(k+\delta)}(x) \simeq \frac{\Gamma(k+\delta+1)}{\Gamma(\delta)\,x^{k+\delta}} \int_0^x (x-u)^{\delta-1}\,\frac{C^{(k)}(u)\,u^k}{\Gamma(k+1)}\,du$$

$$\simeq o\left[\frac{1}{x^\delta}\int_0^x (x-u)^{\delta-1}\,du\right]$$

$$\simeq o(1).$$

COROLLARY. *When the condition*

$$\sigma^{(k+\delta)}(x) = \sigma^{(\delta)}\{\sigma^{(k)}(x)\} \tag{18}$$

is satisfied, then $C^{(k)}(x)-g \simeq o(1)$

implies $C^{(k+\delta)}(x)-g = o(1).$

By Theorem III, equation (18) is satisfied when $\phi(y)$ is of the form

$$\phi(y) = \frac{1}{y}\int_0^y f(w)\,dw.$$

11. LEMMA 5*. *If* $0<k<1$ *and*

$$g(x,\xi) = \frac{1}{\Gamma(k)}\int_0^\xi \phi(u)\,(x-u)^{k-1}\,du \quad (0<\xi<x),$$

then

$$|\,g(x,\xi)\,| \leqslant \operatorname*{Max}_{0\leqslant u\leqslant \xi} |\,\sigma^{(k)}(u)\,|,$$

where, in evaluating the maximum in the second member, we can neglect a set of measure zero of values of u; *provided that*

(i) $\phi(u)$ *is absolutely integrable in* $(0, x)$, *or*

(ii) $\phi(u)$ *is integrable D in* $(0, x)$ *and is of the form*

$$\phi(u) = \frac{1}{u}\int_0^u f(w)\,dw.$$

We write

$$M(\nu) = \frac{1}{\Gamma(k)\,\Gamma(1-k)}\int_\nu^\xi (x-t)^{k-1}(t-\nu)^{-k}\,dt$$

$$= 1-\frac{1}{\Gamma(k)\,\Gamma(1-k)}\int_\xi^x (x-t)^{k-1}(t-\nu)^{-k}\,dt. \tag{19}$$

* *Cf.* M. Riesz, *Acta Szeged*, 1 (1923), 114–126.

The function $M(\nu)$ is monotone, diminishing as ν increases from 0 to ξ, and in this interval

$$0 \leqslant M(\nu) < 1.$$

Proof of (i). Since

$$\int_t^\xi \frac{d}{d\nu}\{M(\nu)\}(\nu-t)^{k-1}\,d\nu = -(x-t)^{k-1}$$

and the integrand does not change sign,

$$\int_0^\xi |\phi(t)|\,dt \int_t^\xi \left|\frac{d}{d\nu}\{M(\nu)\}\right|(\nu-t)^{k-1}\,d\nu = \int_0^\xi |\phi(t)|(x-t)^{k-1}\,dt.$$

Hence

$$\int_0^\xi \frac{d}{d\nu}\{M(\nu)\}\,d\nu \int_0^\nu \phi(t)(\nu-t)^{k-1}\,dt = \int_0^\xi \phi(t)\,dt \int_t^\xi \frac{d}{d\nu}\{M(\nu)\}(\nu-t)^{k-1}\,d\nu, \quad (20)$$

or

$$-\int_0^\xi \sigma^{(k)}(\nu)\frac{d}{d\nu}\{M(\nu)\}\,d\nu = \frac{1}{\Gamma(k)}\int_0^\xi \phi(t)(x-t)^{k-1}\,dt.$$

The result follows by an application of the first mean value theorem, if we remember that we can neglect the values of $\sigma^k(\nu)$ for values of ν which form a set of measure zero.

Proof of (ii). We must show that (20) holds in this case. We have, by the same argument as for equation (7),

$$\int_\epsilon^\xi \frac{d}{d\nu}\{M(\nu)\}\,d\nu \int_0^{\nu-\epsilon} \phi(t)(\nu-t)^{k-1}\,dt = \int_0^{\xi-\epsilon} \phi(t)\,dt \int_{t+\epsilon}^\xi \frac{d}{d\nu}\{M(\nu)\}(\nu-t)^{k-1}\,d\nu.$$

Also, since $\phi(t)$ is continuous in $(\frac{1}{2}\epsilon, \xi)$ say,

$$\lim_{\epsilon'\to 0}\int_0^\xi \frac{d}{d\nu}\{M(\nu)\}\,d\nu \int_{\nu-\epsilon'}^\nu \phi(t)(\nu-t)^{k-1}\,dt = 0.$$

We next show that

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\int_\epsilon^\xi \frac{d}{d\nu}\{M(\nu)\}\,d\nu \int_{\nu-\epsilon}^{\nu-\epsilon'} \phi(t)(\nu-t)^{k-1}\,dt = 0.$$

The expression under the limiting signs is

$$\left(\int_{\epsilon-\epsilon'}^{\xi-\epsilon} dt\int_{t+\epsilon'}^{t+\epsilon} d\nu + \int_{\xi-\epsilon}^{\xi-\epsilon'} dt\int_{t+\epsilon'}^{\xi} d\nu + \int_0^{\epsilon-\epsilon'} dt\int_\epsilon^{t+\epsilon} d\nu\right)\left(\phi(t)(\nu-t)^{k-1}\frac{d}{d\nu}\{M(\nu)\}\right)$$

$$= H_1 + H_2 + H_3.$$

We have

$$\frac{d}{dv}\{M(v)\} = \frac{1}{\Gamma(k)\,\Gamma(1-k)}\Big[-(x-\xi)^{k-1}(\xi-v)^{-k} + (1-k)\int_v^{\xi}(x-y)^{k-2}(y-v)^{-k}\,dy\Big].$$

Considering H_1, the term

$$-\frac{(x-\xi)^{k-1}}{\Gamma(k)\,\Gamma(1-k)}\int_{\epsilon-\epsilon'}^{\xi-\epsilon}\phi(t)\,dt\int_{t+\epsilon'}^{t+\epsilon}(\xi-v)^{-k}(v-t)^{k-1}\,dv$$

vanishes after $\lim\limits_{\epsilon\to 0}\lim\limits_{\epsilon'\to 0}$ precisely as does J_1 in § 5. Neglecting a constant factor, we have to consider

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\int_{\epsilon-\epsilon'}^{\xi-\epsilon}\phi(t)\,dt\int_{t+\epsilon'}^{t+\epsilon}(v-t)^{k-1}\,dv\int_v^{\xi}(x-y)^{k-2}(y-v)^{-k}\,dy$$

$$=\lim_{\epsilon\to 0}\int_{\epsilon}^{\xi-\epsilon}\phi(t)\,dt\int_t^{t+\epsilon}(v-t)^{k-1}\,dv\int_v^{\xi}(x-y)^{k-2}(y-v)^{-k}\,dy.$$

This will certainly vanish if the total variation of

$$\int_t^{t+\epsilon}(v-t)^{k-1}\,dv\int_v^{\xi}(x-y)^{k-2}(y-v)^{-k}\,dy$$

is uniformly $o(1)$ in $(\epsilon,\ \xi-\epsilon)$ as $\epsilon\to 0$. We have

$$\int_v^{\xi}(x-y)^{k-2}(y-v)^{-k}\,dy = (x-v)^{-1}\int_0^{(\xi-v)/(x-v)}\tau^{-k}(1-\tau)^{k-2}\,d\tau$$

$$=(x-v)^{-1}F(v),$$

where $F(v)$ is bounded and monotone diminishing in $(0,\ \xi)$. We have

$$\frac{d}{dt}\int_t^{t+\epsilon}(v-t)^{k-1}(x-v)^{-1}F(v)\,dv=\int_t^{t+\epsilon}(v-t)^{k-1}\frac{d}{dv}\{(x-v)^{-1}F(v)\}\,dv,$$

$$\frac{d}{dv}F(v)=-\frac{1}{(\xi-x)^{1-k}(\xi-v)^{k}},$$

from which we easily infer that the total variation in question tends to zero with the required uniformity. Thus

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}H_1=0.$$

Clearly

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}H_2=0.$$

Now consider H_3; the term

$$-\frac{(x-\xi)^{k-1}}{\Gamma(k)\,\Gamma(1-k)}\int_0^{\epsilon-\epsilon'}\phi(t)\,dt\int_\epsilon^{t+\epsilon}(\xi-\nu)^{-k}(\nu-t)^{k-1}\,d\nu$$

vanishes after $\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}$ precisely as J_3 in § 5. Neglecting a constant factor, we are left with

$$\lim_{\epsilon\to 0}\int_0^{\epsilon}\phi(t)\,dt\int_\epsilon^{t+\epsilon}(\nu-t)^{k-1}\,d\nu\int_\nu^{\xi}(x-y)^{k-2}(y-\nu)^{-k}\,dy.$$

It is sufficient to show that the total variation of

$$\int_\epsilon^{t+\epsilon}(\nu-t)^{k-1}\,d\nu\int_\nu^{\xi}(x-y)^{k-2}(y-\nu)^{-k}\,dy=\int_\epsilon^{t+\epsilon}(\nu-t)^{k-1}(x-\nu)^{-1}F(\nu)\,d\nu$$

is uniformly bounded in $(0,\epsilon)$ as $\epsilon\to 0$. We find that

$$\frac{d}{dt}\int_\epsilon^{t+\epsilon}(\nu-t)^{k-1}(x-\nu)^{-1}F(\nu)\,d\nu$$
$$=(\epsilon-t)^{k-1}(x-\epsilon)^{-1}F(\epsilon)+\int_\epsilon^{t+\epsilon}(\nu-t)^{k-1}\frac{d}{d\nu}\{(x-\nu)^{-1}F(\nu)\}\,d\nu,$$

and the required result easily follows. We have now proved that the first member of (20) is equal to

$$\lim_{\epsilon\to 0}\int_0^{\xi-\epsilon}\phi(t)\,dt\int_{t+\epsilon}^{\xi}\frac{d}{d\nu}\{M(\nu)\}(\nu-t)^{k-1}\,d\nu.$$

It remains then to show that

$$\lim_{\epsilon\to 0}\int_0^{\xi-\epsilon}\phi(t)\,dt\int_t^{t+\epsilon}\frac{d}{d\nu}\{M(\nu)\}(\nu-t)^{k-1}\,d\nu=0.$$

Using the expression given above for $(d/d\nu)[M(\nu)]$, we have

$$\lim_{\epsilon\to 0}-\frac{(x-\xi)^{k-1}}{\Gamma(k)\,\Gamma(1-k)}\int_0^{\xi-\epsilon}\phi(t)\,dt\int_t^{t+\epsilon}(\xi-\nu)^{-k}(\nu-t)^{k-1}\,d\nu=0$$

by an argument precisely similar to that used for proving (9). It remains then to prove that

$$\lim_{\epsilon\to 0}\int_0^{\xi-\epsilon}\phi(t)\,dt\int_t^{t+\epsilon}(\nu-t)^{k-1}\,d\nu\int_\nu^{\xi}(x-y)^{k-2}(y-\nu)^{-k}\,dy=0.$$

In virtue of what was proved in discussing H_1, we need only show that

$$\lim_{\epsilon\to 0}\int_0^{\epsilon}\phi(t)\,dt\int_t^{t+\epsilon}(\nu-t)^{k-1}(x-\nu)^{-1}F(\nu)\,d\nu=0.$$

For this, it is sufficient that

$$\int_0^\epsilon \left| \frac{d}{dt} \int_t^{t+\epsilon} (\nu - t)^{k-1} (x - \nu)^{-1} F(\nu)\, d\nu \right| dt = O(1)$$

as $\epsilon \to 0$; and this follows simply from the expression given above for the integrand. This completes the proof.

12. THEOREM IV. *If*

$$\frac{1}{t} \int_0^t |\phi(u)|\, du = O(1)$$

and

$$\phi(t) = o(1) \quad (H, r),$$

where r is a positive integer, then

$$\phi(t) = o(1) \quad (H, 1+\delta)$$

for all $\delta > 0$.

We require the following lemma.

LEMMA. *If*

$$|f(t)| = O(1)$$

and

$$f(t) = o(1) \quad (H, r),$$

then

$$f(t) = o(1) \quad (H, 1).$$

This is a special case of a result due to Hardy and Littlewood*. Turning to the proof of the theorem, since, by hypothesis,

$$\frac{1}{t} \int_0^t |\phi(u)|\, du = O(1),$$

we have, *a fortiori*,

$$\frac{1}{t} \int_0^t \phi(u)\, du = O(1),$$

so that, by the lemma,

$$\phi(u) = o(1) \quad (H, 2).$$

We have to show that

$$\frac{1}{t^\delta} \int_0^t \phi_1(u)(t-u)^{\delta-1}\, du = o(1) \quad (\delta > 0).$$

* Hardy and Littlewood, *Journal London Math. Soc.*, 1 (1926), 134–138 (135).

Now,

$$\frac{1}{t^\delta}\int_0^t \phi_1(u)(t-u)^{\delta-1}\,du = \frac{1}{t^\delta}\left[\int_0^{\theta t} + \int_{\theta t}^t \phi_1(u)(t-u)^{\delta-1}\right] \quad (0<\theta<1)$$

and

$$\left|\frac{1}{t^\delta}\int_{\theta t}^t \phi_1(u)(t-u)^{\delta-1}\,du\right| = O\left[\frac{1}{t^\delta}\int_{\theta t}^t (t-u)^{\delta-1}\,du\right]$$

$$= O\left[\frac{(1-\theta)^\delta}{\delta}\right],$$

which vanishes after the operation $\lim_{\theta\to 1}\lim_{t\to 0}$. Finally

$$\left|\frac{1}{t^\delta}\int_0^{\theta t}\phi_1(u)(t-u)^{\delta-1}\,du\right|$$

$$= \left|\frac{1}{t^\delta}\left[\theta t\,\phi_2(\theta t)\,t^{\delta-1}(1-\theta)^{\delta-1} + \int_0^{\theta t} u\phi_2(u)\,\frac{\delta-1}{(t-u)^{2-\delta}}\,du\right]\right|$$

$$\to 0$$

as $t\to 0$. This proves the theorem.

13. THEOREM* V. *If*

$$\phi(u) = o(1) \quad (H, p),$$

where $p>0$, *then*

$$\phi(u) = o(1) \quad (C, p).$$

It is only necessary to consider the case $p = r+k$, where r is a positive integer and $0<k\leqslant 1$. We must show that

$$\frac{1}{t^{r+k}}\int_0^t \phi(u)(t-u)^{r+k-1}\,du = o(1).$$

We have

$$\int_0^t \phi(u)(t-u)^{r+k-1}\,du = \int_0^t \frac{d}{du}\{u\phi_1(u)\}(t-u)^{r+k-1}\,du$$

$$= (r+k-1)\int_0^t u\phi_1(u)(t-u)^{r+k-2}\,du$$

$$= \sum_{s=1}^{r} A_s \int_0^t \phi_r(u)\,u^s(t-u)^{k+r-s-1}\,du,$$

* We suppose that $\phi(u)$ is integrable D.

where the A_s are constants. Further,

$$\lim_{t\to 0}\frac{1}{t^{r+k}}\int_0^t \phi_r(u)\,u^s(t-u)^{k+r-s-1}\,du$$
$$=\lim_{t\to 0}\frac{1}{t^{r+k}}\left(\int_0^{\theta t}+\int_{\theta t}^t\right)\phi_r(u)\,u^s(t-u)^{k+r-s-1}\,du,$$

where $0<\theta<1$.

Now the hypothesis

$$\phi(u)=o(1)\quad (H,\,p)$$

is by definition the same as

$$\phi_r(u)=o(1)\quad (H,\,k),$$

and hence, by the corollary to Lemma 4,

$$\phi_{r+1}(u)=o(1).$$

Consequently

$$\int_0^{\theta t}\phi_r(u)\,u^s(t-u)^{k+r-s-1}\,du=\int_0^{\theta t}\frac{d}{du}\left\{u\phi_{r+1}(u)\right\}u^s(t-u)^{k+r-s-1}\,du$$
$$=o\left[t^{k+r}(1-\theta)^{k+r-s-2}\right],$$

and hence

$$\lim_{t\to 0}\frac{1}{t^{k+r}}\int_0^{\theta t}\phi_r(u)\,u^s(t-u)^{k+r-s-1}\,du=0.$$

Again,
$$\phi_{r+k}(t)=\frac{1}{t^k}\int_0^t\phi_r(u)(t-u)^{k-1}\,du,$$

and so, by Theorem III,

$$\int_0^t\phi_r(u)\,du=A\int_0^t\frac{\phi_{r+k}(u)\,u^k}{(t-u)^k}\,du,$$

where A is a constant. Hence

$$\psi(t)=\int_0^t\phi_r(u)\,du$$
$$=o(t).$$

For $s<r$,

$$\int_{\theta t}^t\phi_r(u)\,u^s(t-u)^{k+r-s-1}\,du$$
$$=\left[\psi(u)\,u^s(t-u)^{k+r-s-1}\right]_{\theta t}^t-\int_{\theta t}^t\psi(u)\,\frac{d}{du}\left\{u^s(t-u)^{k+r-s-1}\right\}du$$
$$=o(t^{k+r}).$$

For $s=r$,

$$\left|\frac{1}{t^{k+r}}\int_{\theta t}^{t}\phi_r(u)\,u^r(t-u)^{k-1}\,du\right|$$

$$=\left|\frac{(\theta t)^r}{t^{k+r}}\int_{\theta t}^{\theta' t}\phi_r(u)\,(t-u)^{k-1}\,du+\frac{1}{t^k}\int_{\theta' t}^{t}\phi_r(u)\,(t-u)^{k-1}\,du\right| \quad (\theta<\theta'<1)$$

$$\leqslant 4\ \operatorname*{Max}_{0\leqslant\xi\leqslant t}\left|\frac{1}{\xi^k}\int_0^{\xi}\phi_r(u)\,(\xi-u)^{k-1}\,du\right| \quad [\text{Lemma 5 (ii)}].$$

Hence, in any case,

$$\lim_{t\to 0}\frac{1}{t^{k+r}}\int_{\theta t}^{t}\phi_r(u)\,u^s(t-u)^{k+r-s-1}\,du=0,$$

and this proves the theorem.

14. Before proving the converse of Theorem V, we shall first consider the special case $p=1+k$ with $0<k\leqslant 1$. We have

THEOREM* VI. *If*

$$\phi(u)=o(1)\quad (C,\,1+k)$$

with $0<k\leqslant 1$, *then*

$$\phi(u)=o(1)\quad (H,\,1+k).$$

We are given that

$$\frac{1}{x^{1+k}}\int_0^x\phi(u)\,(x-u)^k\,du=o(1), \tag{22}$$

and, on integrating by parts, we obtain

$$\frac{1}{x^{1+k}}\int_0^x u\phi_1(u)\,(x-u)^{k-1}\,du=o(1). \tag{23}$$

Write

$$\Phi_1(u)=\int_0^u\phi(v)\,dv,$$

$$\Phi_2(u)=\int_0^u\Phi_1(v)\,dv.$$

By Theorem II, $\sigma^{(2)}(x)=\sigma^{(1-k)}\{\sigma^{(1+k)}(x)\}$,

and hence $C^{(2)}(x)=o(x^2)$,

or, what is equivalent, $\Phi_2(x)=o(x^2)$.

Now $\lim_{x\to 0}\frac{1}{x^k}\int_0^x\phi_1(u)\,(x-u)^{k-1}\,du=\lim_{x\to 0}\frac{1}{x^k}\int_0^{\theta x}+\int_{\theta x}^{x}\phi_1(u)\,(x-u)^{k-1}\,du.$

---

* We suppose that $\phi(u)$ is integrable D.

We have

$$\frac{1}{x^k}\int_0^{\theta x}\phi_1(u)\,(x-u)^{k-1}\,du$$

$$=\frac{1}{x^k}\int_0^{\theta x}(x-u)^{k-1}\frac{\Phi_1(u)}{u}\,du$$

$$=\frac{1}{x}\,k\left[(x-u)^{k-1}\frac{\Phi_2(u)}{u}\right]_0^{\theta x}-\frac{1}{x^k}\int_0^{\theta x}\Phi_2(u)\,\frac{d}{du}\left\{\frac{(x-u)^{k-1}}{u}\right\}du$$

$$=o\left[\theta(1-\theta)^{k-1}\right]+o\left[\frac{1}{x^k}\int_0^{\theta x}u(x-u)^{k-2}\,du\right]+o\left[\frac{1}{x^k}\int_0^{\theta x}(x-u)^{k-1}\,du\right]$$

$$=o\left[\theta(1-\theta)^{k-1}+\theta\left\{1-(1-\theta)^{k-1}\right\}+1-(1-\theta)^k\right].$$

Hence
$$\lim_{x\to 0}\frac{1}{x^k}\int_0^{\theta x}\phi_1(u)\,(x-u)^{k-1}\,du=0. \tag{24}$$

Again,

$$\left|\frac{1}{x^k}\int_{\theta x}^{x}(x-u)^{k-1}\,\phi_1(u)\,du\right|$$

$$=\left|\frac{1}{x^k}\int_{\theta x}^{x}(x-u)^{k-1}\,u\phi_1(u)\,\frac{du}{u}\right|$$

$$=\left|\frac{1}{\theta x^{k+1}}\int_{\theta x}^{\xi}(x-u)^{k-1}\,u\phi_1(u)\,du+\frac{1}{x^{k+1}}\int_{\xi}^{x}(x-u)^{k-1}\,u\phi_1(u)\,du\right|$$

$$\leqslant\frac{4}{\theta x^{k+1}}\operatorname*{Max}_{0\leqslant v\leqslant x}\left|\int_0^v(v-u)^{k-1}\,u\phi_1(u)\,du\right|$$

by Lemma 5 (i). Hence, by (23),

$$\lim_{x\to 0}\frac{1}{x^k}\int_{\theta x}^{x}(x-u)^{k-1}\,\phi_1(u)\,du=0. \tag{25}$$

Equations (24) and (25) together prove the theorem.

15. We now consider the converse of Theorem V in the general case.

THEOREM* VII. *If* $\phi(u)=o(1)\quad(C,p)$,

where $p>0$, *then* $\phi(u)=o(1)\quad(H,p)$.

We have to consider only the case $p=r+k$, r being a positive integer

* We suppose that $\phi(u)$ is integrable D.

and $0 < k \leqslant 1$. The theorem being true for $p = 1+k$, we assume it for

$$p = 1+k,\ 2+k,\ \ldots,\ r+k,$$

and prove it for $p = r+1+k$.

We are given that

$$\frac{1}{x^{r+1+k}} \int_0^x \phi(u)(x-u)^{k+r}\, du = o(1), \tag{26}$$

and we must show that $\phi(u) = o(1)$ $(H,\ r+1+k)$,

or, what is equivalent, that $\phi_1(u) = o(1)$ $(H,\ r+k)$.

By the theorem for $p = r+k$, it is sufficient to show that

$$\phi_1(u) = o(1) \quad (C,\ r+k),$$

i.e. that

$$\frac{1}{x^{r+k}} \int_0^x \phi_1(u)(x-u)^{k+r-1}\, du = o(1), \tag{27}$$

With the same meaning for $\Phi_1(u)$ as in § 14, we write generally

$$\Phi_n(u) = \int_0^u \Phi_{n-1}(v)\, dv\,;$$

then (26) gives, on successive integrations by parts,

$$\frac{1}{x^{r+1+k}} \int_0^x \Phi_{r+1}(u)(x-u)^{k-1}\, du = o(1). \tag{28}$$

Further,

$$\Phi_s(u) = o(u^{s-1}) \quad (s = 1,\ 2,\ \ldots,\ r+1), \tag{29}$$

and (26) implies

$$\frac{1}{x^{r+2}} \int_0^x \phi(u)(x-u)^{r+1}\, du = o(1),$$

or, what is equivalent,

$$\Phi_{r+2}(u) = o(u^{r+2}). \tag{30}$$

We have

$$\begin{aligned}\int_0^x \phi_1(u)(x-u)^{k+r-1}\, du &= \int_0^x \frac{d}{du}\{\Phi_2(u)\} \frac{(x-u)^{k+r-1}}{u}\, du \\ &= -\int_0^x \Phi_2(u) \frac{d}{du}\left\{\frac{(x-u)^{k+r-1}}{u}\right\} du \\ &= \sum_{s=1}^{r+1} A_s \int_0^x \Phi_{r+1}(u)\, u^{-s-1}(x-u)^{k+s-1}\, du,\end{aligned}$$

where the A_s are constants. Now

$$\begin{aligned}&\frac{1}{x^{r+k}} \int_0^x \Phi_{r+1}(u)\, u^{-s-1}(x-u)^{k+s-1}\, du \\ &\qquad = \frac{1}{x^{r+k}} \left(\int_0^{\theta x} + \int_{\theta x}^x\right) \Phi_{r+1}(u)\, u^{-s-1}(x-u)^{k+s-1}\, du\end{aligned}$$

and

$$\frac{1}{x^{r+k}}\int_0^{\theta x}\Phi_{r+1}(u)\,u^{-s-1}(x-u)^{k+s-1}\,du$$

$$=\frac{1}{x^{r+k}}\left[\Phi_{r+2}(u)\,u^{-s-1}(x-u)^{k+s-1}\right]_0^{\theta x}$$

$$-\frac{1}{x^{r+k}}\int_0^{\theta x}\Phi_{r+2}(u)\left[(k+s-2)(x-u)^{k+s-2}\,u^{-s-1}\right.$$

$$\left.-(s+1)\,u^{-s-2}(x-u)^{k+s-1}\right]du\to 0,$$

as $x\to 0$ by (30). Again,

$$\left|\frac{1}{x^{r+k}}\int_{\theta x}^{x}\Phi_{r+1}(u)\,u^{-s-1}(x-u)^{k+s-1}\,du\right|$$

$$=\left|\frac{1}{\theta^{s+1}x^{r+k+s+1}}\int_{\theta x}^{\xi}\Phi_{r+1}(u)(x-u)^{k+s-1}\,du\right.$$

$$\left.+\frac{1}{x^{r+k+s+1}}\int_{\xi}^{x}\Phi_{r+1}(u)(x-u)^{k+s-1}\,du\right|$$

$$\leqslant\left|\frac{(1-\theta)^s}{\theta^{s+1}x^{r+k+1}}\int_{\theta x}^{\xi'}\Phi_{r+1}(u)(x-u)^{k-1}\,du\right|$$

$$+\left|\frac{1}{\theta^{s+1}x^{r+k+1}}\int_{\xi'}^{\xi}\Phi_{r+1}(u)(x-u)^{k-1}\,du\right|$$

$$+\left|\frac{1}{x^{r+k+1}}\int_{\xi}^{\xi''}\Phi_{r+1}(u)(x-u)^{k-1}\,du\right|+\left|\frac{1}{x^{r+k+1}}\int_{\xi''}^{x}\Phi_{r+1}(u)(x-u)^{k-1}\,du\right|$$

$$\leqslant\frac{8}{\theta^{s+1}x^{r+k+1}}\operatorname*{Max}_{0\leqslant v\leqslant x}\left|\int_0^{v}\Phi_{r+1}(u)(v-u)^{k-1}\,du\right|\to 0\quad[\text{Lemma 5 (i)}],$$

as $z\to 0$ by (28). This proves the theorem.

16. THEOREM VIII. *If $\phi(u)$ is absolutely integrable, and*

$$\frac{1}{t}\int_0^t|\phi_r(u)|\,du=O(1)\quad(p\geqslant 0)$$

and

$$\phi(t)=o(1)\quad(H,\,r),$$

where r is a positive integer, then*

$$\phi_p(t)=o(1)\quad(H,\,1+\delta)$$

for all $\delta>0$.

The proof is rather long (§§ 16–21). In virtue of the definition of $\phi_p(t)$, there are only two distinct cases to consider, namely $p=0$ and

* This implies that $\phi_n(t)$ is absolutely integrable for every positive integer n. *Cf.* Hardy and Littlewood, *Math. Zeitschrift*, 19 (1923), 67–96 (89).

$p = a$ $(0 < a < 1)$. The result for $p = 0$ is Theorem IV. We consider then $p = a$.

We have

$$\lim_{t\to 0} \frac{1}{t^{1+\delta}} \int_0^t \phi_a(u)\,(t-u)^\delta\,du = \lim_{\theta\to 1} \lim_{t\to 0} \frac{1}{t^{1+\delta}} \int_0^{\theta t} + \int_{\theta t}^t \phi_a(u)\,(t-u)^\delta\,du$$

and

$$\left| \lim_{\theta\to 1} \lim_{t\to 0} \frac{1}{t^{1+\delta}} \int_{\theta t}^t \phi_a(u)\,(t-u)^\delta\,du \right| \leqslant \lim_{\theta\to 1} \lim_{t\to 0} \frac{(1-\theta)^\delta}{t} \int_0^t |\phi_a(u)|\,du = 0.$$

Now $$\frac{1}{t^{1+\delta}} \int_0^{\theta t} \phi_a(u)\,(t-u)^\delta\,du = \frac{1}{t^{1+\delta}} \int_0^{\theta t} (t-u)^\delta \frac{du}{u^a} \int_0^u \phi(v)\,(u-v)^{a-1}\,dv, \qquad (31)$$

and we must show that the second member of (31) vanishes after the operation $\lim_{\theta\to 1} \lim_{t\to 0}$. It will be necessary to express the second member of (31) as a repeated integral in which the order of integration is inverted. This requires examination, since we cannot assert that

$$\int_0^{\theta t} (t-u)^\delta \frac{du}{u^a} \int_0^u \phi(v)\,(u-v)^{a-1}\,dv = \int_0^{\theta t} \phi(v)\,dv \int_v^{\theta t} (t-u)^\delta (u-v)^{a-1} \frac{du}{u^a}.$$

It is sufficient for our purpose to show that

$$\int_0^{\theta t} (t-u)^\delta \frac{du}{u^a} \int_0^u \phi(v)\,(u-v)^{a-1}\,dv = \lim_{\epsilon\to 0} \int_0^{\theta t-\epsilon} \phi(v)\,dv \int_{v+\epsilon}^{\theta t} (t-u)^\delta (u-v)^{a-1} \frac{du}{u^a}. \qquad (32)$$

17. We have

$$\int_\epsilon^{\theta t} (t-u)^\delta \frac{du}{u^a} \int_0^{u-\epsilon} \phi(v)\,(u-v)^{a-1}\,dv = \int_0^{\theta t-\epsilon} \phi(v)\,dv \int_{v+\epsilon}^{\theta t} (t-u)^\delta (u-v)^{a-1} \frac{du}{u^a},$$

since each member is an absolutely convergent double integral. Again, since

$$\left| \int_{u-\epsilon'}^u \phi(v)\,(u-v)^{a-1}\,dv \right|$$

is, for $0 < \epsilon' < \frac{1}{2}\epsilon$ and fixed ϵ, less than an absolutely integrable function of u in $(\epsilon, \theta t)$ (*cf.* § 7), it follows that

$$\lim_{\epsilon'\to 0} \int_\epsilon^{\theta t} (t-u)^\delta \frac{du}{u^a} \int_{u-\epsilon'}^u \phi(v)\,(u-v)^{a-1}\,dv = 0.$$

To prove (32) it remains then to show that

$$\lim_{\epsilon\to 0} \lim_{\epsilon'\to 0} \int_\epsilon^{\theta t} (t-u)^\delta \frac{du}{u^a} \int_{u-\epsilon}^{u-\epsilon'} \phi(v)\,(u-v)^{a-1}\,dv = 0. \qquad (33)$$

The repeated integral in (33) is for fixed ϵ, ϵ' an absolutely convergent

double integral. The equation (33) is therefore equivalent to

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\left(\int_{\epsilon-\epsilon'}^{\theta t-\epsilon} dv\int_{v+\epsilon'}^{v+\epsilon} du+\int_{\theta t-\epsilon}^{\theta t-\epsilon'} dv\int_{v+\epsilon'}^{\theta t} du\right.$$
$$\left.+\int_{0}^{\epsilon-\epsilon'} dv\int_{\epsilon}^{v+\epsilon} du\right)\frac{\phi(v)(t-u)^{\delta}(u-v)^{\alpha-1}}{u^{\alpha}}=0, \quad (34)$$

which we write as

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}(J_1+J_2+J_3)=0. \quad (35)$$

Consider J_1. Writing

$$F(v,\epsilon')=\int_{v}^{v+\epsilon'}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}},$$

we see that $F(v,\epsilon')$ is $O(\epsilon'/v)^{\alpha}$. Hence

$$\lim_{\epsilon'\to 0}\int_{\epsilon-\epsilon'}^{\theta t-\epsilon}\phi(v)\,dv\int_{v}^{v+\epsilon'}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}=0. \quad (36)$$

Similarly $F(v,\epsilon)$ is $O(\epsilon/v)^{\alpha}$, and

$$\lim_{\epsilon'\to 0}\int_{\epsilon-\epsilon'}^{\theta t-\epsilon}\phi(v)\,dv\int_{v}^{v+\epsilon}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}$$
$$=\lim_{\epsilon'\to 0}\left(\int_{\epsilon}^{\lambda\epsilon}+\int_{\lambda\epsilon}^{\theta t-\epsilon}\right)\phi(v)\,dv\int_{v}^{v+\epsilon}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}.$$

Thus

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0}\int_{\epsilon-\epsilon'}^{\theta t-\epsilon}\phi(v)\,dv\int_{v}^{v+\epsilon}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}$$
$$=\lim_{\lambda\to\infty}\lim_{\epsilon\to 0}\int_{\epsilon}^{\lambda\epsilon}\phi(v)\,dv\int_{v}^{v+\epsilon}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}$$
$$+\lim_{\lambda\to\infty}\lim_{\epsilon\to 0}\int_{\lambda\epsilon}^{\theta t-\epsilon}\phi(v)\,dv\int_{v}^{v+\epsilon}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}$$
$$=\lim_{\lambda\to\infty}\lim_{\epsilon\to 0}(I_1+I_2).$$

For fixed λ we have

$$\lim_{\epsilon\to 0} I_1=\lim_{\epsilon\to 0} O\left[\int_{\epsilon}^{\lambda\epsilon}|\phi(v)|\,dv\right]=0$$

so that

$$\lim_{\lambda\to\infty}\lim_{\epsilon\to 0} I_1=0;$$

and

$$\lim_{\lambda\to\infty}\lim_{\epsilon\to 0} I_2=\lim_{\lambda\to\infty} O\left[\frac{1}{\lambda^{\alpha}}\right]=0.$$

This, with (36), proves that

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0} J_1 = 0.$$

Consider now J_2; since

$$\int_{v+\epsilon'}^{\theta t}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}$$

is bounded uniformly with respect to ϵ, ϵ' in $(\theta t-\epsilon,\ \theta t-\epsilon')$, we have

$$\lim_{\epsilon\to 0}\lim_{\epsilon'\to 0} J_2 = 0.$$

Consider finally J_3; we must show that

$$\lim_{\epsilon\to 0}\int_0^{\epsilon}\phi(v)\,dv\int_{\epsilon}^{v+\epsilon}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}} = 0.$$

This follows from the fact that

$$\int_{\epsilon}^{v+\epsilon}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}$$

is uniformly bounded in $0\leqslant v\leqslant\epsilon$. This completes the proof of (32)

18. Write

$$\Phi_1(v) = \int_0^v \phi(u)\,du,$$

$$\Phi_n(v) = \int_0^v \Phi_{n-1}(u)\,du,$$

where $n>1$ is a positive integer. We have, by definition,

$$\phi_{\alpha}(t) = \frac{1}{t^{\alpha}}\int_0^t \phi(u)(t-u)^{\alpha-1}\,du,$$

whence

$$\int_0^t \phi(u)\,du = A\int_0^t \frac{\phi_{\alpha}(u)\,u^{\alpha}}{(t-u)^{\alpha}}\,du$$

where A is a constant. We have, by (31) and (32), to consider the expression

$$\lim_{\theta\to 1}\lim_{t\to 0}\lim_{\epsilon\to 0}\frac{1}{t^{1+\delta}}\int_0^{\theta t-\epsilon}\phi(v)\,dv\int_{v+\epsilon}^{\theta t}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}. \qquad (37)$$

We find

$$\frac{d}{dv}\int_{v+\epsilon}^{\theta t}(t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}$$

$$= -\frac{(1-\theta)^{\delta}\,t^{\delta}}{(\theta t)^{\alpha}(\theta t-v)^{1-\alpha}} + \int_{v+\epsilon}^{\theta t}(u-v)^{\alpha-1}\frac{d}{du}\left\{\frac{(t-u)^{\delta}}{u^{\alpha}}\right\}du,$$

so that integration by parts gives

$$\int_0^{\theta t-\epsilon} \phi(v)\,dv \int_{v+\epsilon}^{\theta t} (t-u)^{\delta}(u-v)^{\alpha-1}\frac{du}{u^{\alpha}}$$

$$= -\int_0^{\theta t-\epsilon} \Phi_1(v)\,dv \left[-\frac{(1-\theta)^{\delta}\,t^{\delta}}{(\theta t)^{\alpha}(\theta t-v)^{1-\alpha}} + \int_{v+\epsilon}^{\theta t} (u-v)^{\alpha-1}\frac{d}{du}\left\{\frac{(t-u)^{\delta}}{u^{\alpha}}\right\} du\right].$$

Now

$$\lim_{\theta\to 1}\lim_{t\to 0}\lim_{\epsilon\to 0}\left|\frac{1}{t^{1+\delta}}\int_0^{\theta t-\epsilon}\Phi_1(v)\,\frac{(-\theta)^{\delta}\,t^{\delta}}{(\theta t)^{\alpha}(\theta t-v)^{1-\alpha}}\,dv\right|$$

$$\leqslant A\lim_{\theta\to 1}\lim_{t\to 0}\frac{(1-\theta)^{\delta}}{t(\theta t)^{\alpha}}\int_0^{\theta t}(\theta t-v)^{\alpha-1}\,dv\int_0^{v}\frac{|\phi_a(u)|}{(v-u)^{\alpha}}\,u^{\alpha}\,du$$

$$= A\lim_{\theta\to 1}\lim_{t\to 0}\frac{(1-\theta)^{\delta}}{t(\theta t)^{\alpha}}\int_0^{\theta t}|\phi_a(u)|\,u^{\alpha}\int_u^{\theta t}(\theta t-v)^{\alpha-1}(v-u)^{-\alpha}\,du$$

$$= A\,\Gamma(a)\,\Gamma(1-a)\lim_{\theta\to 1}\lim_{t\to 0}\frac{(1-\theta)^{\delta}}{t(\theta t)^{\alpha}}\int_0^{\theta t}|\phi_a(u)|\,u^{\alpha}\,du$$

$$< A\,\Gamma(a)\,\Gamma(1-a)\lim_{\theta\to 1}\lim_{t\to 0}\frac{(1-\theta)^{\delta}}{t}\int_0^{\theta t}|\phi_a(u)|\,du$$

$$= 0.$$

19. We must finally show that

$$\lim_{\theta\to 1}\lim_{t\to 0}\lim_{\epsilon\to 0}\frac{1}{t^{1+\delta}}\int_0^{\theta t-\epsilon}\Phi_1(v)\,dv\int_{v+\epsilon}^{\theta t}(u-v)^{\alpha-1}\frac{d}{du}\left\{\frac{(t-u)^{\delta}}{u^{\alpha}}\right\}du = 0. \quad (38)$$

We may suppose that, in the enunciation of Theorem VIII, $r \geqslant 3$. We first show that

$$\int_0^{\theta t}\Phi_s(v)(\theta t-v)^{\alpha-1}\,dv = o(t^{s+\alpha}) \quad (s = 2, 3, \ldots, r-1) \quad (39)$$

for fixed $\theta < 1$, or, what is equivalent,

$$\int_0^{t}\Phi_s(v)(t-v)^{\alpha-1}\,dv = o(t^{s+\alpha}). \quad (40)$$

We are given that

$$\frac{1}{t}\int_0^{t}|\phi_a(u)|\,du = O(1); \quad (41)$$

it follows from Theorem IXB (proved below, and independent of the pre-

sent theorem) that this implies

$$\frac{1}{t}\int_0^t |\phi_1(u)|\,du = O(1). \tag{42}$$

We have

$$\begin{aligned}\int_0^t \Phi_s(v)(t-v)^{\alpha-1}\,dv &= \int_0^t (t-v)^{\alpha-1}\,dv \int_0^v \Phi_{s-1}(u)\,du \\ &= \int_0^t \Phi_{s-1}(u)\,du \int_u^t (t-v)^{\alpha-1}\,dv \\ &= \int_0^t \Phi_{s-1}(u)\,\frac{(t-u)^\alpha}{\alpha}\,du.\end{aligned}$$

By repeating the process we find that

$$\begin{aligned}\int_0^t \Phi_s(v)(t-v)^{\alpha-1}\,dv &= C\int_0^t \phi(u)(t-u)^{\alpha+s-1}\,du \\ &= \sum_{j=1}^{s} A_j \int_0^t \phi_s(u)\,u^j(t-u)^{\alpha+s-j-1}\,du\end{aligned}$$

where C and the A_j are constants. Equation (42) implies

$$\frac{1}{t}\int_0^t |\phi_{s-1}(u)|\,du = O(1).$$

Now

$$\begin{aligned}&\lim_{t\to 0}\frac{1}{t^{s+\alpha}}\int_0^t \phi_s(u)\,u^j(t-u)^{\alpha+s-j-1}\,du \\ &\qquad = \lim_{\lambda\to 1}\lim_{t\to 0}\frac{1}{t^{s+\alpha}}\left(\int_0^{\lambda t}+\int_{\lambda t}^t\right)\phi_s(u)\,u^j(t-u)^{\alpha+s-j-1}\,du\end{aligned}$$

and

$$\begin{aligned}&\lim_{\lambda\to 1}\lim_{t\to 0}\left|\frac{1}{t^{s+\alpha}}\int_{\lambda t}^t \phi_s(u)\,u^j(t-u)^{\alpha+s-j-1}\,du\right| \\ &\qquad \leqslant \lim_{\lambda\to 1}\lim_{t\to 0}\frac{1}{t^{s+\alpha}}\int_{\lambda t}^t u^j(t-u)^{\alpha+s-j-1}\,\frac{du}{u}\int_0^u |\phi_{s-1}(v)|\,dv \\ &\qquad \leqslant \lim_{\lambda\to 1}\lim_{t\to 0}\frac{1}{t^{s+\alpha}}\,O\left[\int_{\lambda t}^t u^j(t-u)^{\alpha+s-j-1}\,du\right] \\ &\qquad = \lim_{\lambda\to 1}\lim_{t\to 0}\frac{1}{t^{s+\alpha}}\,O\left[t^{\alpha+s}(1-\lambda)^{\alpha+s-j}\right] \\ &\qquad = 0.\end{aligned}$$

20. To complete the proof of (40) we must show that

$$\lim_{t\to 0} \frac{1}{t^{s+\alpha}} \int_0^{\lambda t} \phi_s(u)\, u^j (t-u)^{\alpha+s-j-1}\, du = 0.$$

Since $$\phi(t) = o(1) \quad (H, r),$$

we have $$\phi_{s-1}(t) = o(1) \quad (H, r-s+1).$$

Further, $$|\phi_{s-1}(t)| = O(1) \quad (H, 1)$$

by (42). Hence, by Theorem IV,

$$\phi_s(t) = o(1) \quad (H, \delta)$$

for all $\delta > 0$; *i.e.*, by Theorem V,

$$\frac{1}{t^\delta} \int_0^t \phi_s(u)(t-u)^{\delta-1}\, du = o(1). \tag{43}$$

Now

$$\left| \frac{1}{t^{s+\alpha}} \int_0^{\lambda t} \phi_s(u)\, u^j (t-u)^{\alpha+s-j-1}\, du \right| = \left| \frac{(\lambda t)^j}{t^{s+\alpha}} \int_\mu^{\lambda t} \phi_s(u)(t-u)^{\alpha+s-j-1}\, du \right|$$

$$(0 < \mu < \lambda t)$$

$$\leqslant \left| \frac{(\lambda t)^j}{t^{s+\alpha}} \cdot t^{s-j} \int_\mu^{\mu'} \phi_s(u)(t-u)^{\alpha-1}\, du \right|$$

$$+ \left| \frac{(\lambda t)^j}{t^{s+\alpha}} \cdot t^{s-j} \int_{\mu'}^{\lambda t} \phi_s(u)(t-u)^{\alpha-1}\, du \right|$$

$$\leqslant \frac{4(\lambda t)^j t^{s-j}}{t^{s+\alpha}} \operatorname*{Max}_{0 \leqslant \tau \leqslant \mu} \left| \int_0^\tau \phi_s(u)(\tau-u)^{\alpha-1}\, du \right|$$

$$= o\left[\frac{4(\lambda t)^j t^{s-j}}{t^{s+\alpha}} \cdot t^\alpha \right] \text{ by (43)}$$

$$= o(1).$$

This completes the proof of (40).

21. We now turn to the proof of (38). The expression under the limiting signs becomes, on integrating by parts,

$$\frac{1}{t^{1+\delta}} \int_0^{\theta t-\epsilon} \Phi_2(v)\, \frac{d}{dv} \left[\int_{v+\epsilon}^{\theta t} (u-v)^{\alpha-1} \frac{d}{du} \left(\frac{(t-u)^\delta}{u^\alpha} \right) du \right] dv. \tag{44}$$

If p is a positive integer, we find that

$$\frac{d}{dv}\int_{v+\epsilon}^{\theta t}(u-v)^{\alpha-1}\frac{d^p}{du^p}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}du = -(\theta t-v)^{\alpha-1}\left[\frac{d^p}{du^p}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}\right]_{u=\theta t}$$
$$+\int_{v+\epsilon}^{\theta t}(u-v)^{\alpha-1}\frac{d^{p+1}}{du^{p+1}}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}du.$$

By successive integrations by parts, (44) is expressed as the sum of constant multiples of terms of the form

$$\frac{1}{t^{1+\delta}}\left[\frac{d^{s-1}}{du^{s-1}}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}\right]_{u=\theta t}\int_0^{\theta t-\epsilon}\Phi_s(v)(\theta t-v)^{\alpha-1}\,dv \qquad (45)$$
$$(s=2,\ 3,\ r-1)$$

plus a constant multiple of

$$\frac{1}{t^{1+\delta}}\int_0^{\theta t-\epsilon}\Phi_r(v)\,dv\int_{v+\epsilon}^{\theta t}\frac{d^r}{du^r}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}(u-v)^{\alpha-1}\,du. \qquad (46)$$

The contribution of the terms (45) to the first member of (38) is, by (39),

$$\lim_{\theta\to 1}\lim_{t\to 0} o\left[\frac{1}{t^{1+\delta}}.t^{s+\alpha}\left[\frac{d^{s-1}}{du^{s-1}}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}\right]_{u=\theta t}\right]=0.$$

The expression (46) is numerically less than

$$\frac{1}{t^{1+\delta}}\int_0^{\theta t}|\Phi_r(v)|\,dv\int_v^{\theta t}(u-v)^{\alpha-1}\left|\frac{d^r}{du^r}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}\right|du$$
$$=\frac{1}{t^{1+\delta}}\int_0^{\theta t}\left|\frac{d^r}{du^r}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}\right|du\int_0^u|\Phi_r(v)|(u-v)^{\alpha-1}\,du. \qquad (47)$$

Now, since $\phi(u)=o(1)\quad(H,\ r),$

we have $\phi(u)=o(1)\quad(C,\ r),$

i.e. $\Phi_r(v)=o(v^r),$

and (47) is

$$o\left[\frac{1}{t^{1+\delta}}\int_0^{\theta t}\left|\frac{d^r}{du^r}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}\right|du\int_0^u v^r(u-v)^{\alpha-1}\,du\right]$$
$$=o\left[\frac{1}{t^{1+\delta}}\int_0^{\theta t}\left|\frac{d^r}{du^r}\left\{\frac{(t-u)^\delta}{u^\alpha}\right\}\right|u^{r+\alpha}\,du\right]$$
$$=o(1).$$

This completes the proof of Theorem VIII.

22. THEOREM IX.

(*a*) *If*
$$\frac{1}{t}\int_0^t \phi_p(u)\,du = o(1),$$

then
$$\frac{1}{t}\int_0^t \phi_{p+\delta}(u)\,du = o(1)$$

for all $\delta > 0$.

(*b*) *If*
$$\frac{1}{t}\int_0^t |\phi_p(u)|\,du = O(1),$$

then
$$\frac{1}{t}\int_0^t |\phi_{p+\delta}(u)|\,du = O(1)$$

for all $\delta > 0$.

It is sufficient to prove (*a*), the proof of (*b*) being similar. We may clearly suppose that $0 < \delta < 1$. There are two distinct cases to consider; (i) $p = 0$, (ii) $p = \alpha$ $(0 < \alpha < 1)$.

Proof of (i). If $\zeta > 0$, then

$$\int_\zeta^t \frac{du}{u^\delta}\int_\zeta^u \phi(v)\,(u-v)^{\delta-1}\,du = \int_\zeta^t \phi(v)\,dv \int_v^t (u-v)^{\delta-1}\frac{du}{u^\delta}. \tag{48}$$

Further,

$$\begin{aligned}
\int_\zeta^t \frac{du}{u^\delta}\int_0^\zeta \phi(v)\,(u-v)^{\delta-1}\,dv &= \int_0^\zeta \phi(v)\,dv \int_\zeta^t \frac{du}{u^\delta}(u-v)^{\delta-1} \\
&= \int_0^\zeta \frac{d}{dv}\{v\phi_1(v)\}\,dv \int_\zeta^t \frac{du}{u^\delta}(u-v)^{\delta-1} \\
&= o\left[\zeta\int_\zeta^t \frac{du}{u^\delta}(u-\zeta)^{\delta-1}\right] \\
&\qquad + \int_0^\zeta v\phi_1(v)\,dv \,.\, \frac{d}{dv}\int_\zeta^t \frac{du}{u^\delta}(u-v)^{\delta-1} \\
&= o(1) + o\left[\zeta\int_0^\zeta dv\int_\zeta^t \frac{du}{u^\delta}(u-v)^{\delta-2}\right] \\
&\to 0
\end{aligned}$$

as $\zeta \to 0$. Hence, by (48),

$$\frac{1}{t}\int_0^t \frac{du}{u^\delta}\int_0^u \phi(v)\,(u-v)^{\delta-1}\,dv = \frac{1}{t}\int_0^t \phi(v)\,dv \int_v^t (u-v)^{\delta-1}\frac{du}{u^\delta} \tag{49}$$

if the second member exists. Since

$$\frac{d}{dv}\int_v^t (u-v)^{\delta-1}\frac{du}{u^\delta} = -\frac{(t-v)^{\delta-1}}{t^\delta} - \delta\int_v^t (u-v)^{\delta-1}\frac{du}{u^{\delta+1}},$$

the second member of (49) is

$$\frac{1}{t}\int_0^t v\phi_1(v)\,dv\left[\frac{(t-v)^{\delta-1}}{t^\delta} + \delta\int_v^t (u-v)^{\delta-1}\frac{du}{u^{1+\delta}}\right] = o(1).$$

Proof of (ii). We may suppose that $\alpha+\delta \leqslant 1$. Then

$$\begin{aligned}\phi_{\alpha+\delta}(v) &= \frac{1}{v^{\alpha+\delta}}\int_0^v \phi(y)\,(v-y)^{\alpha+\delta-1}\,dy \\ &\simeq \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)\,\Gamma(\delta)}\cdot\frac{1}{v^{\alpha+\delta}}\int_0^v (v-u)^{\delta-1}\,u^\alpha\,\phi_\alpha(u)\,du.\end{aligned}$$

We must show that

$$\frac{1}{y}\int_0^y \frac{du}{u^{\alpha+\delta}}\int_0^u (u-x)^{\delta-1}\,x^\alpha\,\phi_\alpha(x)\,dx = o(1).$$

Write

$$\begin{aligned}\phi_{\alpha 1}(x) &= \frac{1}{x}\int_0^x \phi_\alpha(u)\,du \\ &= o(1).\end{aligned}$$

We have

$$\int_\zeta^y \frac{du}{u^{\alpha+\delta}}\int_\zeta^u x^\alpha\,\phi_\alpha(x)\,(u-x)^{\delta-1}\,dx = \int_\zeta^y x^\alpha\,\phi_\alpha(x)\,dx\int_x^y (u-x)^{\delta-1}\frac{du}{u^{\alpha+\delta}} \qquad (50)$$

and

$$\begin{aligned}&\int_\zeta^y \frac{du}{u^{\alpha+\delta}}\int_0^\zeta x^\alpha\,\phi_\alpha(x)\,(u-x)^{\delta-1}\,dx \\ &\quad= \int_0^\zeta x^\alpha\,\phi_\alpha(x)\,dx\int_\zeta^y (u-x)^{\delta-1}\frac{du}{u^{\alpha+\delta}} \\ &\quad= \int_0^\zeta x^\alpha\frac{d}{dx}\{x\phi_{\alpha 1}(x)\}\,dx\int_\zeta^y (u-x)^{\delta-1}\frac{du}{u^{\alpha+\delta}} \\ &\quad= \zeta^{\alpha+1}\,\phi_{\alpha 1}(\zeta)\int_\zeta^y (u-\zeta)^{\delta-1}\frac{du}{u^{\alpha+\delta}} \\ &\qquad\qquad -\int_0^\zeta x\phi_{\alpha 1}(x)\frac{d}{dx}\left\{x^\alpha\int_\zeta^y (u-x)^{\delta-1}\frac{du}{u^{\alpha+\delta}}\right\}dx \\ &\quad\to 0\end{aligned}$$

as $\zeta\to 0$. Hence, by (50),

$$\frac{1}{y}\int_0^y \frac{du}{u^{\alpha+\delta}}\int_0^u (u-x)^{\delta-1}\,x^\alpha\,\phi_\alpha(x)\,dx = \frac{1}{y}\int_0^y x^\alpha\,\phi_\alpha(x)\,dx\int_x^y (u-x)^{\delta-1}\frac{du}{u^{\alpha+\delta}} \qquad (51)$$

whenever the second member exists. The second member of (51) is

$$-\frac{1}{y}\int_0^y x\phi_{a1}(x)\frac{d}{dx}\left\{x^a\int_x^y(u-x)^{\delta-1}\frac{du}{u^{a+\delta}}\right\}dx$$

$$=o\left[\frac{1}{y}\int_0^y ax^a\,dx\int_x^y(u-x)^{\delta-1}\frac{du}{u^{a+\delta}}\right]$$

$$+o\left[\frac{1}{y}\int_0^y x^{a+1}\,dx\left\{\frac{(y-x)^{\delta-1}}{y^{a+\delta}}+\int_x^y(u-x)^{\delta-1}\frac{(a+\delta)}{u^{a+\delta+1}}\,du\right\}\right]$$

$$=o(1).$$

This proves the theorem.

23. THEOREM X. *If*

$$\frac{1}{t^{1+\delta}}\int_0^t(t-u)^\delta\phi_p(u)\,du=o(1)\quad(p\geqslant 0)$$

for all $\delta>0$, *then*

$$\frac{1}{t}\int_0^t\phi_{p+\delta}(u)\,du=o(1)$$

for all $\delta>0$.

We may, by Theorem IX (*a*), suppose that $\delta<1$. There are two distinct cases to consider: (i) $p=0$; (ii) $p=a$ $(0<a<1)$.

Proof of (i). We are given that

$$\frac{1}{t^{1+\delta}}\int_0^t(t-u)^\delta\phi(u)\,du=o(1).$$

Now

$$\int_0^t(t-u)^\delta\phi(u)\,du=\delta\int_0^t\phi(u)\,du\int_u^t(t-v)^{\delta-1}\,dv$$

$$=\delta\int_0^t(t-v)^{\delta-1}\Phi_1(v)\,dv,$$

so that

$$\int_0^t(t-v)^{\delta-1}\Phi_1(v)\,dv=o(t^{1+\delta}).\qquad(52)$$

Further,

$$\phi(u)=o(1)\quad(C,\,1+\delta)$$

implies

$$\phi(u)=o(1)\quad(C,\,2),$$

i.e.

$$\Phi_2(v)=o(v^2).\qquad(53)$$

We must show that

$$\int_0^t\frac{du}{u^\delta}\int_0^u\phi(v)(u-v)^{\delta-1}\,dv=o(t).$$

By the method used for proving (32) we can show that

$$\int_0^t \frac{du}{u^\delta}\int_0^u \phi(v)\,(u-v)^{\delta-1}\,dv = \lim_{\epsilon\to 0}\int_0^{t-\epsilon}\phi(v)\,dv\int_{v+\epsilon}^t (u-v)^{\delta-1}\,\frac{du}{u^\delta}. \qquad (54)$$

Since $$\frac{d}{dv}\int_{v+\epsilon}^t (u-v)^{\delta-1}\,\frac{du}{u^\delta} = -\frac{(t-v)^{\delta-1}}{t^\delta} - \delta\int_{v+\epsilon}^t \frac{(u-v)^{\delta-1}}{u^{1+\delta}}\,du,$$

the second member of (54) is

$$\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\Phi_1(v)\left[\frac{(t-v)^{\delta-1}}{t^\delta}+\delta\int_{v+\epsilon}^t\frac{(u-v)^{\delta-1}}{u^{1+\delta}}\,du\right]dv$$

$$= o(t)+\lim_{\epsilon\to 0}\delta\int_0^{t-\epsilon}\Phi_1(v)\,dv\int_{v+\epsilon}^t\frac{(u-v)^{\delta-1}}{u^{1+\delta}}\,du,$$

by (52). Again

$$\frac{d}{dv}\int_{v+\epsilon}^t\frac{(u-v)^{\delta-1}}{u^{1+\delta}}\,du = -\frac{(t-v)^{\delta-1}}{t^{1+\delta}} - (1+\delta)\int_{v+\epsilon}^t\frac{(u-v)^{\delta-1}}{u^{2+\delta}}\,du,$$

so that the second member of (54) is

$$o(t)+\lim_{\epsilon\to 0}\delta\int_0^{t-\epsilon}\Phi_2(v)\left[\frac{(t-v)^{\delta-1}}{t^{1+\delta}}+(1+\delta)\int_{v+\epsilon}^t\frac{(u-v)^{\delta-1}}{u^{2+\delta}}\,du\right]$$

$$= o(t)+O\left[\int_0^t|\Phi_2(v)|\,dv\int_v^t\frac{(u-v)^{\delta-1}}{u^{2+\delta}}\,du\right], \text{ by (53)},$$

$$= o(t).$$

Proof of (ii). We may, by Theorem IX (*a*), suppose that $\alpha+\delta<1$. We must show that

$$\int_0^t\frac{du}{u^{\alpha+\delta}}\int_0^u\phi(v)\,(u-v)^{\alpha+\delta-1}\,dv = o(t). \qquad (55)$$

Since $$\int_0^u\phi(v)\,(u-v)^{\alpha+\delta-1}\,dv \simeq \frac{\Gamma(\alpha+\delta)}{\Gamma(\alpha)\,\Gamma(\delta)}\int_0^u(u-v)^{\delta-1}\,\phi_\alpha(v)\,v^\alpha\,dv,$$

the first member of (55) is, neglecting a constant factor,

$$\int_0^t\frac{du}{u^{\alpha+\delta}}\int_0^u(u-v)^{\delta-1}\,\phi_\alpha(v)\,v^\alpha\,dv;$$

and this, by the method used for proving (32), can be proved to be equal to

$$\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\phi_\alpha(v)\,v^\alpha\,dv\int_{v+\epsilon}^t(u-v)^{\delta-1}\,\frac{du}{u^{\alpha+\delta}}. \qquad (56)$$

Write $$\Phi_{\alpha 1}(v) = \int_0^v\phi_\alpha(u)\,du,$$

$$\Phi_{\alpha 2}(v) = \int_0^v\Phi_{\alpha 1}(u)\,du.$$

We have

$$\int_0^t (t-u)^\delta \phi_\alpha(u)\,du = \delta \int_0^t \phi_\alpha(u)\,du \int_u^t (t-v)^{\delta-1}\,dv$$

$$= \delta \int_0^t (t-u)^{\delta-1}\,\Phi_{\alpha 1}(u)\,du,$$

by an argument similar to that of § 9. We thus have

$$\int_0^t (t-u)^{\delta-1}\,\Phi_{\alpha 1}(u)\,du = o(t^{1+\delta}). \tag{57}$$

Next,
$$\left|\int_0^t (t-u)^{\delta-1}\,\Phi_{\alpha 1}(u)\,u^\alpha\,du\right| = \left|\, t^\alpha \int_\xi^t (t-u)^{\delta-1}\,\Phi_{\alpha 1}(u)\,du\,\right|$$

$$\leqslant 2t^\alpha \left|\, \operatorname*{Max}_{0\leqslant\tau\leqslant t} \int_0^\tau (\tau-u)^{\delta-1}\,\Phi_{\alpha 1}(u)\,du\,\right|$$

$$= o(t^{1+\alpha+\delta}) \tag{58}$$

and
$$\Phi_{\alpha 2}(t) = \int_0^t (t-u)^{\delta-1}\,\Phi_{\alpha 1}(u)\,(t-u)^{1-\delta}\,du$$

$$= t^{1-\delta} \int_0^\xi (t-u)^{\delta-1}\,\Phi_{\alpha 1}(u)\,du$$

$$= o(t^2). \tag{59}$$

We now consider (56). We have

$$\lim_{\epsilon\to 0} \int_0^{t-\epsilon} \phi_\alpha(v)\,v^\alpha\,dv \int_{v+\epsilon}^t (u-v)^{\delta-1} \frac{du}{u^{\alpha+\delta}}$$

$$= -\lim_{\epsilon\to 0} \int_0^{t-\epsilon} \Phi_{\alpha 1}(v) \frac{d}{dv}\left\{ v^\alpha \int_{v+\epsilon}^t (u-v)^{\delta-1} \frac{du}{u^{\alpha+\delta}} \right\} dv$$

and
$$\frac{d}{dv}\left\{ v^\alpha \int_{v+\epsilon}^t (u-v)^{\delta-1} \frac{du}{u^{\alpha+\delta}} \right\}$$

$$= \alpha v^{\alpha-1} \int_{v+\epsilon}^t (u-v)^{\delta-1} \frac{du}{u^{\alpha+\delta}} + v^\alpha \frac{d}{dv} \int_{v+\epsilon}^t (u-v)^{\delta-1} \frac{du}{u^{\alpha+\delta}}.$$

There is no difficulty in showing that

$$-\lim_{\epsilon\to 0} \alpha \int_0^{t-\epsilon} \Phi_{\alpha 1}(v)\,v^{\alpha-1}\,dv \int_{v+\epsilon}^t (u-v)^{\delta-1} \frac{du}{u^{\alpha+\delta}}$$

$$= \lim_{\epsilon\to 0} \alpha \int_0^{t-\epsilon} \Phi_{\alpha 2}(v) \frac{d}{dv}\left\{ v^{\alpha-1} \int_{v+\epsilon}^t (u-v)^{\delta-1} \frac{du}{u^{\alpha+\delta}} \right\} dv$$

$$= o(t),$$

by (59); while

$$-\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\Phi_{\alpha 1}(v)\,v^{\alpha}\frac{d}{dv}\left\{\int_{v+\epsilon}^{t}(u-v)^{\delta-1}\frac{du}{u^{\alpha+\delta}}\right\}dv,$$

in virtue of

$$\frac{d}{dv}\int_{v+\epsilon}^{t}(u-v)^{\delta-1}\frac{du}{u^{\alpha+\delta}}=-\frac{(t-v)^{\delta-1}}{t^{\alpha+\delta}}-\int_{v+\epsilon}^{t}(u-v)^{\delta-1}\frac{\alpha+\delta}{u^{\alpha+\delta+1}}\,du,$$

is equal to

$$\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\Phi_{\alpha 1}(v)\,v^{\alpha}\left\{\frac{(t-v)^{\delta-1}}{t^{\alpha+\delta}}+\int_{v+\epsilon}^{t}(u-v)^{\delta-1}\frac{\alpha+\delta}{u^{\alpha+\delta+1}}\,du\right\}dv$$

$$=o(t)+\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\Phi_{\alpha 1}(v)\,v^{\alpha}\,dv\int_{v+\epsilon}^{t}(u-v)^{\delta-1}\frac{\alpha+\delta}{u^{\alpha+\delta+1}}\,du$$

by (58); and there is no difficulty in showing that the second term, which is equal to

$$-\lim_{\epsilon\to 0}\int_0^{t-\epsilon}\Phi_{\alpha 2}(v)\frac{d}{dv}\left\{v^{\alpha}\int_{v+\epsilon}^{t}(u-v)^{\delta-1}\frac{\alpha+\delta}{u^{\alpha+\delta+1}}\,du\right\}dv,$$

is $o(t)$, by (59). This proves the theorem.

24. THEOREM XI. *If*

$$\frac{1}{t}\int_0^t|\phi_p(u)|\,du=O(1)$$

and

$$\phi(t)=o(1)\quad(H,r),$$

where r is a positive integer, then

$$\frac{1}{t}\int_0^t\phi_{p+\delta}(u)\,du=o(1)$$

for all $\delta>0$.

This is a consequence of Theorems VIII and X.

25. We finally prove

THEOREM XII. *If*

$$\frac{1}{t}\int_0^t|\phi_p(u)|^q\,du=O(1)\quad(q>1)$$

and

$$\phi_p(t)=o(1)\quad(H,r)$$

for some positive integral value of r, then

$$\phi_p(t)=o(1)\quad\left(H,\frac{1}{q}+\delta\right)$$

for all $\delta>0$.

We may, without loss of generality, suppose that

$$\frac{1}{q}+\delta<1\,;\quad \delta<\frac{1}{q}.$$

There are two cases to consider : (i) $p=0$, (ii) $p=\alpha$ $(0<\alpha<1)$.

Proof of (i). Since the hypothesis of the theorem implies

$$\frac{1}{t}\int_0^t|\phi(u)|\,du=O(1),$$

we have, by Theorem XI,

$$\begin{aligned}\phi_{1\delta}(t)&=\frac{1}{t}\int_0^t\phi_\delta(u)\,du\\&=o(1).\end{aligned}\tag{60}$$

An examination of the proof of Theorem I shows that

$$\int_0^t\phi(u)\,du\int_u^t(t-v)^{(1/q)-1}(v-u)^{\delta-1}\,du=\int_0^t(t-v)^{(1/q)-1}\,dv\int_0^v\phi(u)\,(v-u)^{\delta-1}\,du,\tag{61}$$

provided that

(*a*) $\int_0^t\phi(u)(t-u)^{\delta+(1/q)-1}\,du$ exists;

(*b*) $\lim_{\epsilon'\to0}\int_\epsilon^t(t-v)^{(1/q)-1}\,dv\int_{v-\epsilon'}^v\phi(u)(v-u)^{\delta-1}\,du=0.$

We have

$$\left|\int_0^t\phi(u)(t-u)^{\delta+(1/q)-1}\,du\right|^q\leqslant\left[\int_0^t|\phi(u)|^q\,du\right]\left[\int_0^t(t-u)^{[\delta+(1/q)-1]\,q/(q-1)}\right]^{q-1},$$

and, since

$$\left(\delta+\frac{1}{q}-1\right)\frac{q}{q-1}>-1,$$

condition (*a*) is satisfied. Further, it is known that, if $\phi(u)$ is of class L^q in $(0,\ t)$, then

$$\int_0^v|\phi(u)|\,(v-u)^{\delta-1}\,du$$

is of class $L^{1/(q^{-1}-\delta)-\epsilon}$ for all $\epsilon>0$ in the same interval. If now P and Q satisfy

$$\frac{1}{P}+\frac{1}{Q}=1\quad(Q>1),$$

and $$Q < \frac{1}{(1/q)-\delta},$$

then $$P > \frac{1}{1-(1/q)+\delta}.$$

We may suppose that P and Q satisfy these inequalities, and that

$$P\left(1-\frac{1}{q}\right) < 1.$$

It follows that

$$\left| \int_0^t (t-v)^{(1/q)-1}\, dv \int_0^v |\phi(u)|\,(v-u)^{\delta-1}\, du \right|$$

$$\leqslant \left[\int_0^t (t-v)^{P[(1/q)-1]}\, dv\right]^{1/P} \left[\int_0^t \left| \int_0^v |\phi(u)|\,(v-u)^{\delta-1}\, du \right|^Q\right]^{1/Q},$$

which is finite for all t in a suitable neighbourhood of the origin. Thus (b) is satisfied.

From (61) we have

$$\int_0^t \phi(u)(t-u)^{(1/q)+\delta-1}\, du = \frac{\Gamma\{(1/q)+\delta\}}{\Gamma(1/q)\,\Gamma(\delta)} \int_0^t (t-v)^{(1/q)-1}\, dv \int_0^v \phi(u)(v-u)^{\delta-1}\, du.$$

Neglecting the constant factor, we have to consider

$$\lim_{t\to 0} \frac{1}{t^{(1/q)+\delta}} \int_0^t (t-v)^{(1/q)-1}\, dv \int_0^v \phi(u)(v-u)^{\delta-1}\, du$$

$$= \lim_{\theta\to 1} \lim_{t\to 0} \frac{1}{t^{(1/q)+\delta}} \left(\int_0^{\theta t} + \int_{\theta t}^t\right)(t-v)^{(1/q)-1}\, dv \int_0^v \phi(u)(v-u)^{\delta-1}\, du.$$

We have

$$\lim_{\theta\to 1} \lim_{t\to 0} \left| \frac{1}{t^{(1/q)+\delta}} \int_{\theta t}^t (t-v)^{(1/q)-1}\, dv \int_0^v \phi(u)(v-u)^{\delta-1}\, du \right|$$

$$\leqslant \lim_{\theta\to 1} \lim_{t\to 0} \frac{1}{t^{(1/q)+\delta}} \int_{\theta t}^t (t-v)^{(1/q)-1}\, dv \left[\int_0^v |\phi(u)|^q\right]^{1/q}$$

$$\times \left[\int_0^v (v-u)^{(\delta-1)[q/(q-1)]}\, du\right]^{(q-1)/q}$$

$$= O\left[\lim_{\theta\to 1} \lim_{t\to 0} \frac{1}{t^{(1/q)+\delta}} \int_{\theta t}^t (t-v)^{(1/q)-1} \cdot v^{1/q} \cdot v^{\delta-(1/q)}\, dv\right]$$

$$= O\left[\lim_{\theta\to 1} (1-\theta)^{1/q}\right]$$

$$= 0.$$

Finally,

$$\lim_{\theta\to 1}\lim_{t\to 0}\left|\frac{1}{t^{(1/q)+\delta}}\int_0^{\theta t}(t-v)^{(1/q)-1}\,dv\int_0^v\phi(u)(v-u)^{\delta-1}\,du\right|$$

$$=\lim_{\theta\to 1}\lim_{t\to 0}\left|\frac{1}{t^{(1/q)-\delta}}\int_0^{\theta t}(t-v)^{(1/q)-1}\,dv\,v^{\delta}\frac{d}{dv}\{v\phi_{1\delta}(v)\}\right|$$

$$=\lim_{\theta\to 1}\lim_{t\to 0}\left|\frac{1}{t^{(1/q)+\delta}}\left[t^{(1/q)-1}(1-\theta)^{(1/q)-1}(\theta t)^{1+\delta}\phi_{1\delta}(\theta t)\right.\right.$$

$$\left.\left.-\int_0^{\theta t}v\phi_{1\delta}(v)\frac{d}{dv}\{v^{\delta}(t-v)^{(1/q)-1}\}\,dv\right]\right|$$

$$=0$$

by (60).

Proof of (ii). Since the hypothesis of the theorem implies

$$\frac{1}{t}\int_0^t|\phi_a(u)|\,du=O(1),$$

we have, by Theorem VIII,

$$\phi_a(u)=o(1)\quad(H,\,1+\delta),$$

so that

$$\phi_a(u)=o(1)\quad(H,\,2).$$

Hence, by the case $p=0$,

$$\phi_a(u)=o(1)\quad\left(H,\,\frac{1}{q}+\delta\right).$$

This proves the theorem.

ON REPRESENTATIONS OF THE S_k's OF S_n AND OF THE GRASSMANN MANIFOLDS $G(k, n)$.

By J. G. SEMPLE.

[Received 21 November, 1929.—Read 12 December, 1929.]

1. In a previous paper* on *Quadric representations of the lines of four-dimensional space*, I indicated a method for obtaining quadric representations of the lines of a [4] S_4 on the points of a [6] S_6; and I showed that the simplest representation of this type, namely that by means of quadrics through a V_3^3 of planes, was obtainable by direct projection of the quintic manifold M_6^5 which represents the lines of S_4. I remarked also that the method employed was capable of immediate generalization, and it is this generalization which is now the subject of the present paper. It leads us to representations of the S_k's of S_n on the points of a space Σ of dimension $(n-k)(k+1)$; and the simplest representation of this type is again found to be obtainable by projection from the Grassmann manifold which represents in a known way the S_k's of S_n. We begin with a short introduction to this Grassmann representation.

2. *The Grassmann manifold* $G(k, n)$. An S_k of S_n has $\binom{n+1}{k+1}$ coordinates called *Grassmann coordinates*†, obtained by extracting all the determinants of order $k+1$ from the matrix of homogeneous coordinates of $k+1$ independent points of S_k. Let these be denoted by p_r, $\left\{r = 1, \ldots, \binom{n+1}{k+1}\right\}$. Considering then, in space of $\binom{n+1}{k+1}-1$ dimensions, all the points whose coordinates are proportional to the Grassmann coordinates p_r of the various S_k's of S_n, we obtain a certain locus of points, one point from each S_k of S_n. The S_k's of S_n will be represented then birationally on the points of this locus, which will thus be of dimension equal to the freedom of S_k's in S_n, *i.e.* $(n-k)(k+1)$. Any locus so

* *Proc. London Math. Soc.* (2), 30 (1930), 500–512.

† *Cf.* Segre, *Encyk. der Math. Wiss.* III, C 7, 791.

obtained is often referred to as a *Grassmann manifold**, and we shall denote it here by $G(k, n)$.

The representation just outlined is, so far as I know, the only known representation of the S_k's of S_n on the points of a manifold†. It will always be the most important one, moreover, on account of its complete freedom from exceptional elements; and indeed the chief importance of representations on a flat space Σ, such as we seek, is that each such representation clearly provides a space representation, and ultimately a parametric representation of $G(k, n)$. This is, in fact, one method of studying $G(k, n)$, and an alternative to finding a geometrical definition of the locus in its own space. We note, moreover, in passing that, since prime sections of $G(k, n)$ represent the linear complexes‡ of S_k's in S_n, *the representation of* $G(k, n)$ *on* Σ, of which we have spoken, *will be by means of those primals in* Σ *which represent the linear* S_k*-complexes of* S_n.

The following investigation falls into three parts. In the first we obtain representations of the S_k's of S_n on a space Σ. In the second we examine specially the simplest of these representations and the associated representation of $G(k, n)$ on Σ, and we show also how to obtain this by projecting $G(k, n)$. In the third, finally, we examine some particular cases for special values of k and n.

N.B.—In virtue of the principle of duality, the theory of the S_k's of S_n is identical with that of the S_{n-k-1}'s of S_n; and in order to introduce a certain simplification, *we shall assume throughout this paper that* $k < \frac{1}{2}n$.

Representation of S_k*'s of* S_n *on points of* Σ.

3. The method which we adopt is an immediate generalization of that used in our previous paper, and we again begin by introducing an intermediate correspondence as follows.

3.1. *The intermediate correspondence.* We take $k+1$ $[n]$'s, $N_0, \ldots, N_k$ say, all related homographically to S_n and to each other, and we denote the space of dimension $(k+1)(n+1)-1$ which contains them all by R. To

* *Cf.* Severi, "Sulla varietà che rappresenta gli spazi subordinati di data dimensione", *Annali di Mat.* (3), 24 (1915), 89.

† Kantor has given many representations for the lines of S_n; *cf. Journal für Math.*, 118 (1897), 74.

‡ "A linear complex of S_k's of S_n" is defined as the system of S_k's of S_n whose coordinates satisfy a fixed linear relation.

any S_k then of S_n there will correspond $k+1$ homographic S_k's in S_n, one in each of the spaces N_μ ($\mu = 0, \ldots, k$); and these $k+1$ S_k's will define a containing space Π of dimension $(k+1)^2-1$. *We define our intermediate correspondence then by making correspond to any S_k of S_n the space Π which joins the $k+1$ homographic S_k's in the N_μ.*

Consider the same matter algebraically. We take for this purpose $n+1$ fundamental points, $X_0, \ldots, X_n$ say, in S_n. To these there will correspond $n+1$ homographic points $Y_{0\mu}, \ldots, Y_{n\mu}$ in each of the spaces N_μ; and we take the homography between S_n and any one of N_μ to be such that to any point $x_\lambda X_\lambda$ of S_n there corresponds the point $x_\lambda Y_{\lambda\mu}$ of N_μ. There are moreover $(k+1)(n+1)$ of the points $Y_{\lambda\mu}$, and we take them therefore as fundamental points of R, any point of R being consequently expressible in the form $y_{\lambda\mu} Y_{\lambda\mu}$ and having the $y_{\lambda\mu}$ as coordinates. Any S_k of S_n then is defined by $k+1$ points of S_n, $x_{\lambda\mu} X_\lambda$ say ($\mu = 0, \ldots, k$), and the corresponding space Π is that defined by all the $(k+1)^2$ homographic points in the N_μ.

We want next to show that through any point P of R there passes one and only one space Π. This depends directly on the fact that through P there can be drawn exactly one $[k]$ to meet each of the spaces N_μ in a point. Let P be the point $y_{\lambda\mu} Y_{\lambda\mu}$. It determines then a unique point P_μ in each of the spaces N_μ, namely, in the space N_{μ_0}, for example, the point $y_{\lambda\mu_0} Y_{\lambda\mu_0}$; and the $k+1$ points P_μ so obtained clearly determine the $[k]$ in question. But any space Π which passes through P contains a $[k]$ of each of the spaces N_μ; and by exactly the same argument as before, one $[k]$ can be drawn through P to meet each of those $[k]$'s. This secant $[k]$ then must be the same as the former, and any Π space through P must therefore contain the points P_μ. But if the space Π contains a point P_μ of N_μ, it must obviously contain all the points homographic with it in the spaces N_μ. Thus Π must contain all the points P_μ and all the homographic points in the spaces N_μ, $(k+1)^2$ points in all, and it is therefore uniquely determined. *Hence through any point of R there passes just one space Π.*

3.2. We want now to find the locus of spaces Π which correspond to S_k's which form a linear complex in S_n. This locus will clearly be of dimension $(k+1)(n+1)-2$, *i.e.* a primal in R. Let the linear complex, Λ say, be given by the linear relation

$$a_r p_r = 0 \qquad (1)$$

connecting the Grassmann coordinates p_r. Then just as the p_r were defined as the determinants of order $k+1$ of the matrix of the coordinates

of $k+1$ independent points of S_k, so we now define functions P_r as the similar determinants of the similar matrix

$$\| y_{\lambda\mu} \| \quad (\lambda = 0, \ldots, n\,;\ \mu = 0, \ldots, k)$$

of all the coordinates of a point P of R. It is easy to see then that the ratios of the functions P_r, determined by any point P or R, are identical with the ratios of the coordinates p_r of the S_k corresponding to the unique space Π which passes through P. It follows then that, as S_k describes the complex Λ, the coordinates of any point of the corresponding space Π satisfy the equation

$$a_r P_r = 0, \tag{2}$$

i.e. the locus of the space Π is the primal given by this equation. *Thus to the linear S_k-complexes of S_n there correspond in R the primals of the linear system*

$$a_r P_r = 0,$$

where the P_r are the determinants of order $k+1$ of the matrix $\| y_{\lambda\mu} \|$.

The primals in question are therefore of order $k+1$, and we shall denote the linear system defined by them by Ψ.

3.3. *The system Ψ.* The algebraical definition of the system Ψ just obtained is very simple and leads immediately to the properties of the system relating to its base manifolds. We are concerned now to determine the character, multiplicity, and algebraical expression of these latter.

The locus of simple base points of the system Ψ is obtained by equating simultaneously to zero all the functions P_r, *i.e.* all the determinants of order $k+1$ of the matrix $\| y_{\lambda\mu} \|$. Denoting the simple base locus of Ψ then by Ψ_1, we can write the equations of Ψ_1 in the form

$$\| y_{\lambda u} \|_{k+1} = 0. \tag{3}$$

Consider next the locus, Ψ_2 say, of base double points of Ψ. By writing down the conditions for a base double point it is easy to see that, for any such point, all the determinants of order k of the matrix $\| y_{\lambda\mu} \|$ must vanish. Thus the equations of Ψ_2 can be written in the form

$$\| y_{\lambda\mu} \|_k = 0 \tag{4}$$

In general, then, if we denote the r-ple base locus of $\Psi (r = 1, \ldots, k)$ by Ψ_r, the equations of Ψ_r are

$$\| y_{\lambda\mu} \|_{k-r+2} = 0. \tag{5}$$

Hence Ψ has base loci $\Psi_1, \ldots, \Psi_k$ of multiplicities $1, \ldots, k$ respectively, given by the vanishing of determinants of orders $k+1, k, \ldots, 2$ respectively of the matrix $\| y_{\lambda\mu} \|$.

It appears now that the base manifolds Ψ_r are all of the very well known type of manifolds which are said to be *projectively generated**. They are each, therefore, known to have two systems of generating spaces and two methods of generation.

In particular, the k-ple base manifold Ψ_k, given by the vanishing of all the second order determinants of $\|y_{\lambda\mu}\|$, can be defined simply as the locus of $[k]$'s which join homographic *points* of the $k+1$ homographic $[n]$'s N_μ, and it has also another generation as the locus of $[n]$'s which join corresponding points of $n+1$ homographic $[k]$'s of itself. The second generating system obviously includes the spaces N_μ.

Similarly Ψ_{k-1} is the locus of $[2(k+1)-1]$'s which join corresponding *lines* of the spaces N_μ; and generally, then, Ψ_{k-r} is the locus of $[(r+1)(k+1)-1]$'s which join corresponding $[r]$'s of the N_μ.

3.4. *Section by space* Σ. *Representation of* S_k*'s of* S_n *on points of* Σ. Consider now a section of the figure in R by a space Σ of dimension $(n-k)(k+1)$. Any space Π meets Σ in a point; and through any point of Σ there passes, as we have shown (§ 3.1), just one space Π. *We obtain, then, in general, a birational representation of the* S_k*'s of* S_n *on* Σ. In certain circumstances, however, this may break down. In fact, if Σ is taken to meet any space N_μ, or indeed any generating $[n]$ of Ψ_k, in an $[n-k]$, every space Π will meet Σ in a point† of that $[n-k]$, and the representation of the S_k's of S_n on points of Σ is destroyed. With this single restriction, however, Σ can be chosen as generally or as specially as we like, and we get in this way a variety of representations.

3.5. *The system* Φ. In any one of these representations the linear S_k-complexes of S_n will be represented in Σ by a certain system of primals which we denote by Φ. Observe that Φ is just the section of Ψ by Σ. It will be composed therefore of primals of order $k+1$, and it will have base manifolds $\Phi_1, \ldots, \Phi_k$, of multiplicities $1, \ldots, k$ respectively, the sections by Σ of the base manifolds $\Psi_1, \ldots, \Psi_k$ of Ψ.

3.6. *The reverse representation.* Consider now the reverse representation, and in particular the system of S_k-complexes, Θ say, corre-

* For a general account of projectively generated manifolds, *cf.* Segre, *Encyk. der Math. Wiss.* III, C 7, 825-830. For a very convenient notation for them, and for a further account of their methods of generation and properties, *cf.* Room, *Journal London Math. Soc.*, 4 (1929), 206. In Room's notation a manifold in $[N]$, given by the vanishing of all minors of order $r+1$ of a matrix of p rows and q columns, is denoted by $\{p, q, r, [N]\}$, and the manifolds $\Psi_1, \ldots, \Psi_k$ are therefore $\{k+1, n+1, k, [N]\}, \ldots, \{k+1, n+1, 1, [N]\}$ respectively, where N is the dimension of R.

† Since every Π space meets every generating $[n]$ of Ψ_k in a $[k]$.

sponding in S_n to primes of Σ. The order of any complex Θ is the number of S_k's which it has in common with a general *pencil* of S_k's, any such pencil being defined as the system of S_k's which lie in a S_{k+1} and pass through a S_{k-1} therein. The order in question then will clearly be equal to that of the curve of Σ which corresponds to a general S_k-pencil.

Consider, then, any such pencil, Λ say, defined by an S_{k+1} and a contained S_{k-1}. To S_k's of Λ there will correspond in R spaces Π which join corresponding S_k's of $k+1$ related pencils in the spaces N_μ. These pencils will lie in spaces S^μ_{k+1} say, in the N_μ and will have vertices S^μ_{k-1} in the S^μ_{k+1}; and each of the spaces Π will contain the S_{k^2+k-1} joining those vertices. Taking a section of the figure then by a space just not meeting this vertex, we get, as section of the spaces Π, $[k]$'s joining corresponding points of $k+1$ homographic lines, and these form a V^{k+1}_{k+1}. Thus the locus of the spaces Π is a $V^{k+1}_{(k+1)^2-1}$, and it is met by Σ, in general, in a rational C^{k+1}. This means that *in the general case the reverse representation is of order* $k+1$. Incidentally the curve C^{k+1} must meet Φ-primals in only one free point, and all its other points of intersection must therefore be at base points of Φ.

We note, however, that the above argument only applies to the general case. In fact, if we take Σ to contain, as is permissible (§ 3.4), $r+1$ homographic $[n-k-1]$'s, $L_0, \ldots, L_r$ say, of the $r+1$ spaces $N_0, \ldots, N_r$, then the spaces $S^0_{k+1}, \ldots, S^r_{k+1}$ meet $L_0, \ldots, L_r$ respectively in homographic points, Σ contains the $[r]$ of $V^{k+1}_{(k+1)^2-1}$ which joins these, and it can easily be shown then that Σ meets that locus residually in a curve C^{k+1-r} of order $k+1-r$ only. In conclusion, then: *If* Σ *be taken to contain homographic* $[n-k-1]$*'s of* $r+1$ *of the spaces* N_μ, *a pencil of* S_k*'s of* S_n *is represented in* Σ *by a rational* C^{k+1-r}, *i.e. the reverse representation is of order* $k+1-r$.

The general representation.

4. Consider first now the case when Σ is of quite general position. Its equations then will be general linear relations on the coordinates $y_{\lambda\mu}$, and the equations of Φ_r will clearly be of exactly the same form as those of Ψ_r, only now the $y_{\lambda\mu}$ are to be considered as linear functions of the coordinates of a point P of Σ. It follows then that

The S_k*'s of* S_n *can be represented on the points of an* $[(n-k)(k+1)]$ Σ *in such a way that to the linear* S_k*-complexes of* S_n *correspond primals* Φ *of order* $k+1$ *of* Σ *which have base manifolds* $\Phi_1, \ldots, \Phi_k$, *of multiplicities* $1, \ldots, k$, *of which any one* Φ_r *is given by equations*

$$\| y_{\lambda\mu} \|_{k+2-r} = 0 \quad (\lambda = 0, \ldots, n;\ \mu = 0, \ldots, k).$$

where the $y_{\lambda\mu}$ are fixed linear functions of the coordinates of a point of Σ. It should be noted, however, that, in general, Σ will not meet all the manifolds Ψ_r; so that usually there is an r_0 such that the manifolds Ψ_r for which $r > r_0$ do not exist.

The reverse representation, by § 3.6, is of order $k+1$. Also, by § 2, $G(k, n)$ is representable on Σ by means of the system of primals Φ defined above.

As examples of the above representation we may mention :

(i) *Lines of* [3]. These are represented on a [4] by means of quadrics through a rational normal C^4. To solids of Σ correspond quadratic complexes of S_3 containing the chords of a cubic curve. These complexes are all tetrahedral.

(ii) *Lines of* [4]. These are represented on S_6 by means of quadrics through a rational quintic scroll F_2^5. To primes of S_6 correspond quadratic complexes of S_4 which contain all the chords of a cubic scroll Γ. Each one of these complexes has a singular rational* normal C^4 on Γ.

(iii) *Planes of* [5]. These are represented on S_9 by means of cubics through a manifold with the symbol (in Room's notation)

$$\{6,\ 3,\ 2,\ [9]\}.$$

This is a V_5^{15} generated by ∞^2 solids.

The intermediate representations.

5. We refer next, very briefly only because they are not of much importance, to the intermediate representations obtained by taking Σ to contain homographic $[n-k-1]$'s, $L_0, \ldots, L_r$ of the spaces $N_0, \ldots, N_r$, where $r = 1, \ldots, k-1$. These are distinguished from the preceding representation chiefly by the fact that their reverses are of orders $k, k-1, \ldots, 2$ respectively (§ 3.6). Their Φ-systems and base systems are defined algebraically in the same way as before, except that the relations connecting the $y_{\lambda\mu}$ are now independent of $(n-k)(r+1)$ of them†; and, furthermore, it is easy to see geometrically that the space joining $N_0, \ldots, N_r$ is a generating space of the second system of Ψ_{k-r}, and it follows then that Φ_{k-r} contains the $[(r+1)(n-k)-1]$ joining $L_0, \ldots, L_r$.

* *Cf.* Semple, *loc. cit.*, 512.

† Σ will be given as the intersection of $(n+1)(k+1)-(n-k)(k+1)-1 = (k+1)^2-1$ primes through $L_0, \ldots, L_r$; and if $L_\alpha (\alpha = 0, \ldots, r)$ is taken to be the space joining the fundamental points $Y_{0,\alpha}, \ldots, Y_{n-k-1,\alpha}$, then the equations of the primes will not involve the coordinates $y_{\lambda\mu} (\lambda = 0, \ldots, n-k-1;\ \mu = 0, \ldots, r)$.

Conversely, it can be shown that, *if the matrix* $\|y_{\lambda\mu}\|$ *in* § 4 *be specialized so that* Φ_{k-r} *contains an* $[(r+1)(n-k)-1]$, *then we get in* Σ *an intermediate representation whose reverse is of order* $k+1-r$.

We pass on now to the simplest representation obtained as an extension of the above by putting $r = k$.

The special representation.

6.1. We take Σ now to contain homographic $[n-k-1]$'s $L_0, \ldots, L_k$ of all the spaces N_μ, and we take any one of these, L_μ say, to be that defined by the points $y_{0,\mu}, \ldots, y_{n-k-1,\mu}$ in N_μ. The spaces L_μ then clearly define an $[(n-k)(k+1)-1]$, *i.e.* a prime, Σ_0 say, of Σ; and Σ can be taken to be any *general* $[(n-k)(k+1)]$ through Σ_0. We note first that Σ meets Ψ_1 (and therefore all the remaining Ψ_r) only in points of Σ_0. This will appear presently from the algebra; and it can also easily be shown geometrically*. It means clearly that, *in this representation, all the base manifolds of* Φ *lie in the prime* Σ_0 *of* Σ. They are, in fact, the sections of the Ψ_r by Σ_0. In particular, Φ_k will be the locus of $[k]$'s joining homographic points of the spaces L_μ; and its complementary system of spaces will consist of $[n-k-1]$'s, such as the L_μ, meeting the $[k]$'s homographically in points. Likewise, Φ_{k-1} will be the locus of $[2(k+1)-1]$'s joining homographic lines of the L_μ; and generally Φ_{k-r} is generated by $[(r+1)(k+1)-1]$'s joining homographic $[r]$'s of the same spaces†; while it has also a second generation as the locus of $[(r+1)(n-k)-1]$'s joining corresponding $[r]$'s of $n-k$ homographic $[k]$'s of Φ_k. We note here also the useful duality which is known to exist between $[n]$'s of Φ_k and the $[k(n-k)-1]$'s of Φ_1; and we observe that just as any join of r of the former is a generating space of the second system of Φ_{k+1-r}, so the intersection of any r of the latter is a generating space, also of the second system, of Φ_r.

6.2. We draw attention next to the fact that the k-ple base manifold Φ_k implies all the others, $\Phi_{k-1}, \ldots, \Phi_1$. For Segre‡ has shown that for

* In fact, let the section of Ψ_1 by Σ_0 be Φ_1', and suppose that a *general* space Σ through Σ_0 meets Ψ_1 in a point P outside Σ_0. P lies then on one generating space of each system of Ψ_1; and since one of these systems consists of $[k(n+1)-1]$'s which meet Σ_0 in the generating $[k(n-k)-1]$'s of one system of Φ_1', clearly Σ must contain the $[k(n-k)]$ joining P to a certain $[k(n-k)-1]$ of Φ_1'. But if we take the totality of such $[k(n-k)]$'s in spaces Σ through Σ_0, we find for the dimension of Ψ_1 the number $N-n+k$; and since the dimension in question is known to be $N-n+k-1$, we infer that Σ contains no point of Ψ_1 outside Σ_0.

† In Room's notation Φ_r is $\{k+1, n-k, k-r+1, [N']\}$, where N' is the dimension of Σ_0.

‡ *Loc. cit.*, 828.

any manifold such as Φ_k, given by equations

$$\| y_{\lambda\mu} \|_2 = 0$$

$$(\lambda = 0, \ldots, n-k-1\,;\ \mu = 0, \ldots, k),$$

in which the $y_{\lambda\mu}$ are all independent, Φ_{k-1} is the locus of chords of Φ_k, Φ_{k-2} is the locus of its trisecant planes, ..., and generally Φ_r is the locus of its $(k-r+1)$-secant $[k-r]$'s. Now, if a primal of order $k+1$ have two k-ple points P, Q, it will have the line joining them as $(k-1)$-ple line; and if it have three such points P, Q, R, it will have the plane PQR as $(k-2)$-ple plane, etc. It follows, then, that primals of order $k+1$ which have Φ_k as k-ple locus have Φ_{k-1}, Φ_{k-2}, ..., Φ_1 as base manifolds of multiplicities $k-1$, ..., 1 respectively.

Thus, in this representation, Φ is completely defined as the system of primals of order $k+1$ in Σ which has a k-ple base manifold Φ_k, lying in a prime Σ_0 of Σ, and generated therein by $[k]$'s which join corresponding points of $k+1$ homographic $[n-k-1]$'s.

6.3. *The equation of* Φ. Considering the same matters now algebraically, we can proceed as follows. The system Φ, being the section of Ψ by Σ, is the linear system of primals defined by the determinants of order $k+1$ of the matrix

$$\| y_{\lambda\mu} \| \quad (\lambda = 0, \ldots, n\,;\ \mu = 0, \ldots, k), \quad \text{(A)}$$

the $y_{\lambda\mu}$ being connected only by the equations defining Σ. Now Σ (*cf.* § 5, Note) is given by $(k+1)^2-1$ equations; and since it contains all those fundamental points which lie in the L_μ, none of those equations will involve any of the coordinates of the matrix

$$\| y^{\lambda\mu} \|$$

$$(\lambda = 0, \ldots, n-k-1\,;\ \mu = 0, \ldots, k). \quad \text{(B)}$$

The equations of Σ then must involve only the remaining $(k+1)^2$ coordinates, and they clearly suffice to express all of these in terms of one of them, y say; and the matrix (A) thereupon takes the form

$$\left\| \begin{matrix} y_{00} & y_{10} & \cdots & y_{n-k-1,\,0} & a_{00}y & a_{10}y & \cdots & a_{k0}y \\ y_{01} & y_{11} & \cdots & y_{n-k-1,\,1} & a_{01}y & a_{11}y & \cdots & a_{k1}y \\ \cdots & & \cdots & & \cdots & & \cdots & \\ y_{0k} & y_{1k} & \cdots & y_{n-k-1,\,k} & a_{0,k}y & a_{1,k}y & \cdots & a_{kk}y \end{matrix} \right\|,$$

the a_{ij} being constants. By a simple manipulation now of the last $k+1$ columns we easily reduce this to the form

$$\left\| \begin{matrix} y_{00} & y_{10} & \cdots & y_{n-k-1,\,0} & 0 & 0 & \cdots & 0 & y \\ y_{01} & y_{11} & \cdots & y_{n-k-1,\,1} & 0 & 0 & \cdots & y & 0 \\ \cdots & & \cdots & & \cdots & & & \cdots & \\ y_{0k} & y_{1k} & \cdots & y_{n-k-1,\,k} & y & 0 & \cdots & 0 & 0 \end{matrix} \right\|, \tag{C}$$

and the system Φ is then obtained by equating to zero the most general linear combination of the determinants of order $k+1$ of this matrix. *Thus the general equation of* Φ *takes the form*

$$\Phi = y^{k+1} + \sum_1^{m_1} y^k \lambda_{1,\,i}\, \Lambda_{1,\,i} + \sum_1^{m_2} y^{k-1} \lambda_{2,\,i}\, \Lambda_{2,\,i} + \cdots$$

$$\cdots + \sum_1^{m_k} y \lambda_{k,\,i}\, \Lambda_{k,\,i} + \sum_1^{m_{k+1}} \lambda_{k+1,\,i}\, \Lambda_{k+1,\,i} = 0, \tag{6}$$

where the $\Lambda_{r,\,i}$ *are the minors, in number* $m_r = \binom{k+1}{r}\binom{n-k}{r}$, *of order* r *of the matrix* $\| y_{\lambda\mu} \|$ ($\lambda = 0, \ldots, n-k-1$; $\mu = 0, \ldots, k$), *and the* $\lambda_{r,\,i}$ *are free numerical coefficients.*

The prime Σ_0 has clearly the equation $y = 0$, and it is easy now to see that all the results stated about $\Phi_1, \ldots, \Phi_k$ follow immediately from the equation of Φ just given.

Note finally, moreover, that $G(k, n)$ is representable on Σ by means of this system Φ.

The parametric representation of $G(k, n)$.

7.1. The simplest parametric representation of $G(k, n)$ follows immediately now, simply by taking coordinates of a space of dimension $\binom{n+1}{k+1} - 1$ proportional to the various terms in the general equation of Φ obtained in § 6.3. Let the space in question be denoted by Z and the various coordinates of a point of it by z's. Adopting, then, an obvious method of grouping the coordinates z, *we obtain the parametric equations of* $G(k, n)$ *in the following form** :

$$\frac{z}{y^{k+1}} = \frac{z_{1,\,i}}{y^k \Lambda_{1,\,i}} = \frac{z_{2,\,i}}{y^{k-1} \Lambda_{2,\,i}} = \cdots = \frac{z_{k+1,\,i}}{\Lambda_{k+1,\,i}}. \tag{7}$$

* The suffixes i are, of course, different in different fractions, and the total number of equal fractions involved is therefore $1 + \sum_1^{k+1} m_r$.

We can express this result much more briefly, however, if we use non-homogeneous coordinates and non-homogeneous parameters; for we can then take both y and z to be unity and express the result as follows:

$G(k, n)$ is the manifold obtained by taking as non-homogeneous coordinates of a point all determinants of all orders of the matrix

$$\| y_{\lambda\mu} \|$$
$$(\lambda = 0, \ldots, n-k-1;\ \mu = 0, \ldots, k),$$

the y's being regarded as independent parameters.*

7.2. *The tangential spaces of $G(k, n)$.* Consider now the point Z, which is quite generally related to the $G(k, n)$, obtained by taking all the parameters $y_{\lambda\mu}$ to be zero. The points of $G(k, n)$ which form the first neighbourhood of Z are obtained by neglecting all but the linear terms in $y_{\lambda\mu}$ (supposed arbitrarily small) in the parametric equations, and these equations reduce then to

$$\frac{z}{y^{k+1}} = \frac{z_{1,i}}{y^k \Lambda_{1,i}},$$

all the remaining coordinates z being zero. Thus all such points lie in a space Π_2 of dimension m_1 (*cf.* § 6.3 for the definition of m_r), and this space is known as the *two-tangential space*† of $G(k, n)$ at Z. Similarly, all the points of the second neighbourhood of Z, obtained by neglecting all terms of order higher than 2 in the $y_{\lambda\mu}$, lie in a space Π_3 of dimension m_1+m_2, known as the *three-tangential space* at Z. Continuing, then, in this way, *we obtain* generally *a series of tangential spaces to $G(k, n)$ at Z, namely* $\Pi_2, \ldots, \Pi_{k+1}$, *of which any one, the*

* This representation can obviously be obtained directly by considering an S_k of S_n as determined by the $k+1$ points in which it meets $k+1$ S_{n-k}'s through a common S_{n-k-1}. The number of parameters entering is clearly just right; and the Grassmann coordinates of the S_k can be taken to be the determinants of the matrix

$$\begin{Vmatrix} y_{00} & y_{10} & \cdots & y_{n-k-1,0} & 0 & 0 & \ldots & 0 & 1 \\ y_{01} & y_{11} & \cdots & y_{n-k-1,1} & 0 & 0 & \ldots & 1 & 0 \\ \cdots & \cdots & & \cdots & \cdots & & \cdots & & \cdots \\ y_{0k} & y_{1k} & \cdots & y_{n-k-1,k} & 1 & 0 & \ldots & 0 & 0 \end{Vmatrix}$$

giving the representation in question.

† See Segre, *Encyk. der Math. Wiss.* III, C 7, 922–23. It is shown there that the r-tangential space Π_r of a manifold V_m is in general of dimension $\binom{m+r-1}{m}-1$, but for special manifolds it may be of much lower dimension, as in fact it is for $G(k, n)$. The 1-tangential space at Z is of course just the point Z itself.

r-tangential space Π_r *for example, has dimension equal to* $\sum_1^{r-1} m_a$. These spaces play an important part in the sequel, and we mention here their most important property, namely that Π_r, for example, is met by any curve of $G(k, n)$ through Z in r coincident points at Z.

The representation of $G(k, n)$ *by projection.*

8.1. The representation of $G(k, n)$ on Σ by means of the *special** system Φ (§ 6) is very remarkable and likely to be of value in any general consideration of the locus. It provides us, as we have seen, with an economical parametric representation, and it gives, among other things, as we shall see, a very simple representation of the generating spaces of $G(k, n)$. The fact, then, that the representation is always obtainable by projecting $G(k, n)$ is of considerable importance, and we proceed now to demonstrate this and the manner of the projection.

8.2. The fact that the representation is equivalent to a projection of some kind arises directly from the fact that all the base loci Φ_r lie in the prime Σ_0 of Σ. For any prime of Σ clearly forms, with Σ_0 counted k times, a primal of Φ, *i.e.* the system of primals residual, with regard to Φ, to the prime Σ_0 counted k times is just the system of all the primes of Σ. This means, then, that the prime sections of $G(k, n)$, which correspond to primes of Σ, form a homaloidal system, and that $G(k, n)$ projects birationally from the space common to all their primes. Let this space-vertex of projection be V. We want to find, then, how V is related to $G(k, n)$.

Consider first the prime Σ_0 and its representation on $G(k, n)$. This is obtained clearly by putting $y = 0$ in the parametric equations (7), and this makes all the z's zero except the $z_{k+1, i}$, while these latter are connected by the equations

$$\frac{z_{k+1, i}}{\Lambda_{k+1, i}} = \frac{z_{k+1, i'}}{\Lambda_{k+1, i'}} \quad (i, i' = 1, \ldots, m_{k+1}).$$

Now these equations are clearly just those obtained by taking coordinates proportional to the $\binom{n-k}{k+1}$ determinants of a matrix of $n-k$ rows and $k+1$ columns; and the locus defined by them is therefore a Grassmann

* By the system Φ we shall always in future be referring to that arising in the special representation (§ 6), whose equation is given in § 6.3.

locus, which we denote by $G_0(k, n-k-1)$. *Thus to the prime* Σ_0 *there corresponds on* $G(k, n)$ *a Grassmann locus* $G_0(k, n-k-1)$.

Consider now those primals Φ which contain Σ_0 doubly and the primes of Z which cut out the corresponding prime sections of $G(k, n)$. The primals in question clearly contain all the points of Σ for which y is a small quantity of the first order*, *i.e.* they contain all the first neighbourhoods of points of Σ_0. One would expect then that the corresponding primes of z should contain all the first neighbourhoods on $G(k, n)$ of points of $G_0(k, n-k-1)$, *i.e.* that they should contain all the two-tangential spaces (*cf.* § 7.2) of $G(k, n)$ at points of $G_0(k, n-k-1)$. This is, in fact, what happens. For to points of Σ for which y is of the first order there correspond the points of $G(k, n)$ for which y^2 and higher powers of y are neglected in the parametric equations, and these points are therefore given by the equations

$$\frac{z_{k,i}}{y\Lambda_{k,i}} = \frac{z_{k+1,i'}}{\Lambda_{k+1,i'}} \quad (i = 1, \ldots, m_k;\ i' = 1, \ldots, m_{k+1}),$$

all the remaining coordinates being zero; and it is easy then to see that these points form the first neighbourhoods† of points of $G(k, n)$ which lie on $G_0(k, n-k-1)$. They all lie, moreover, in the space of dimension $m_{k+1}+m_k-1$ determined by the coordinates which vanish; and this space then must be that containing all the two-tangential spaces of $G(k, n)$ at points of $G_0(k, n-k-1)$, and it must also then be the vertex of the system of primes of Z which correspond to primals of Φ containing Σ_0 doubly.

In an exactly similar way now it follows that, to those primals of Φ which contain Σ_0 triply, there correspond in Z primes which contain all the points of the second neighbourhoods on $G(k, n)$ of points of $G_0(k, n-k-1)$, such points being obtained by neglecting y^3 and higher powers of y in the parametric equations. These points are easily seen to lie in a space of dimension $m_{k+1}+m_k+m_{k-1}-1$, and this is therefore the space containing all the three-tangential spaces of $G(k, n)$ at points of $G_0(k, n-k-1)$, and is likewise the vertex of the system of primes of Z

---

* When we say that a primal, whose equation is $\Phi = 0$, contains all points for which y is a small quantity of the r-th order, we mean, of course, that when y is as stated and all the other coordinates are finite, Φ is an infinitesimal of order $r+1$.

† For we clearly get all the points of all such neighbourhoods by giving all the parameters y and $y_{\lambda,\mu}$ variations δy and $\delta y_{\lambda,\mu}$; and the parametric equations then give

$$\frac{z}{(\delta y)^{k+1}} = \frac{z_{1,i}}{(\delta y)^k \Lambda_{1,i}} = \ldots = \frac{z_{k,i}}{\delta y . \Lambda_{k,i}} = \frac{z_{k+1,i}}{\Lambda_{k+1,i} + \delta\Lambda_{k+1,i}},$$

so that

$$z = z_{1,i} = \ldots = z_{k-1,i} = 0.$$

which correspond to primals Φ which contain Σ_0 triply. Continuing in this way, then, we arrive finally at the result, that to primes of Σ, considered as made up to primals of Φ by the space Σ_0 counted k times, there correspond sections of $G(k, n)$ by primes through a space V which contains all the k-tangential spaces of $G(k, n)$ at points of $G_0(k, n-k-1)$, and whose dimension r is given by

$$r = m_{k+1}+m_k+\ldots+m_2-1$$
$$= \binom{n+1}{k+1}-(n-k)(k+1)-2.$$

It follows then, by what we said earlier, that $G(k, n)$ projects birationally from V on to Σ in such a way that prime sections project into primals of Φ. In fact, if we omit from the parametric equations all the coordinates except z and the $z_{1,i}$ [which is equivalent to projecting $G(k, n)$ from V], we are left with

$$\frac{z}{y} = \frac{z_{1,i}}{\Lambda_{1,i}},$$

where the $\Lambda_{1,i}$ are just the coordinates $y_{\lambda\mu}$; and this shows again that the projection is birational. The theorem proved then is as follows:

$G(k, n)$ projects birationally from the space of dimension

$$\binom{n+1}{k+1}-(n-k)(k+1)-2,$$

which contains all the k-tangential spaces of $G(k, n)$ at points of $G(k, n-k-1)$; and the representation of the manifold arising in this way is the same as that which we have already obtained.

8.3. The above result can also be expressed differently in a rather interesting manner; but in order to prove this we must first make a short digression in order to consider briefly the properties of *rational normal curves on $G(k, n)$*, always assuming, as we did at the beginning of this paper, that $k < \frac{1}{2}n$. The theory of these curves is connected very closely with that of the loci V_r^r of rational simply infinite systems* of $[r-1]$'s in $[2r+1]$. In fact, for $r = 2$, we know already that the quadric regulus is always represented on a Grassmann locus by a conic; and, for $r = 3$, Segre has shown that the V_3^3 of planes is always represented by a twisted

* V_r^r has equations of the form

$$\frac{x_0}{x_0'} = \frac{x_1}{x_1'} = \ldots = \frac{x_r}{x_r'}.$$

cubic curve*. It is true also, in general, that the rational V_r^r of $[r-1]$'s is always represented by a rational normal C^r. From this, then, can be deduced the following result, namely, that *any rational normal C^r on a $G(k, n)$ corresponds either to the conical system of S_k's joining a V_r^r of $[r-1]$'s to a $[k-r]$, or to the section of a V_r^r of $[r-1]$'s by an $[n+k-r+1]$ according as $r \lesseqgtr k+1$.*

8.4. *Significance of the various tangential spaces of $G(k, n)$.* This result enables us now to attach a new significance to the various tangential spaces of $G(k, n)$. Consider, for example, the $(r+1)$-tangential space Π_{r+1} at any point P of $G(k, n)$. Any curve of $G(k, n)$ which passes through P meets Π_{r+1} in $r+1$ coincident points at P; and evidently then, if the curve be a rational normal C^r, it will lie completely in Π_{r+1}. Thus Π_{r+1} contains all rational normal C^r's of $G(k, n)$ which pass through P. Now any such C^r corresponds, by the preceding paragraph, to a conical system of S_n's, M^r say, whose vertex is an S_{k-r} and whose base is a V_r^r of $[r-1]$'s; and since C^r passes through P, M^r will have the S_k corresponding to P, S_k^0 say, as one of its S_k's. Thus the vertex S_{k-r} of M^r must lie in S_k^0, and every S_k of the system will meet S_k^0 in this S_{k-r}. But clearly every S_k which meets S_k^0 in an S_{k-r} belongs to an infinite number of systems such as M^r having that S_{k-r} as vertex. It follows then that Π_{r+1} contains all points of $G(k, n)$, forming a locus W say, which correspond to S_k's of S_n which meet S_k^0 in an S_{k-r}. It is, in fact, determined completely by these points. For if we take $k+1$ of the fundamental points of S_n to lie in S_k^0, and if we take $k-r+1$ of the $k+1$ points which determine an S_k to lie also in S_k^0, then the matrix of the coordinates of those $k+1$ points will be of the form

$$\begin{Vmatrix} x_0^0 & x_1^0 & \dots & x_k^0 & 0 & 0 & \dots & 0 \\ x_0^1 & x_1^1 & \dots & x_k^1 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0^{k-r} & x_1^{k-r} & \dots & x_k^{k-r} & 0 & 0 & \dots & 0 \\ x_0^{k-r+1} & x_1^{k-r+1} & \dots & x_k^{k-r+1} & x_{k+1}^{k-r+1} & x_{k+2}^{k-r+1} & \dots & x_n^{k-r+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0^k & x_1^k & \dots & x_k^k & x_{k+1}^k & x_{k+2}^k & \dots & x_n^k \end{Vmatrix}$$

and it is easy to see that the number, p say, of Grassmann coordinates of

* Segre, *Annali di Mat.* (3), 27 (1918), 84.

S_k which vanish is exactly given by

$$p = \binom{n-k}{k+1} + \binom{n-k}{k}\binom{k+1}{1} + \ldots + \binom{n-k}{r+1}\binom{k+1}{r}.$$

Thus the manifold W lies in a definite space of dimension $N-p$, where N is the dimension of $Z\left[\text{equal to } \binom{n+1}{k+1}-1\right]$; and comparing this with the dimension already found for Π_{r+1} (§ 7.2), it is easy to see that the two dimensions are equal. Thus Π_{r+1} is completely determined by W. In conclusion then :

The $(r+1)$-tangential space at any point P of $G(k, n)$ is that which contains all points of $G(k, n)$ which represent S_k's of S_n which meet the S_k corresponding to P in an S_{k-r}.

In particular, the k-tangential space at P is that containing all points of $G(k, n)$ which represent S_k's which meet S_k^0 in a line, and this leads to the promised restatement of the projection theorem for $G(k, n)$.

8.5. *The projection theorem; second form.* In the form already obtained (§ 8.2), the projection is from a space V containing all the k-tangential spaces of $G(k, n)$ at points of $G_0(k, n-k-1)$. Now the points of $G_0(k, n-k-1)$ clearly represent S_k's of S_n which lie in an $[n-k-1]$, S^0_{n-k-1} say; and by the results obtained in the preceding paragraph, all S_k's which meet S^0_{n-k-1} in a line are clearly represented by the points of the sections of $G(k, n)$ by the k-tangential spaces at points of $G_0(k, n-k-1)$. It follows, then, that V is completely defined as the space containing the locus, Γ say, of points of $G(k, n)$ which correspond to S_k's meeting S^0_{n-k-1} in a line. Hence

$G(k, n)$ projects birationally from the locus Γ of points of itself which correspond to S_k's of S_n which meet a fixed S_{n-k-1} in a line; and this projection is equivalent to the previous one.

Properties of the special representation.

9.1. The second form of the projection theorem for $G(k, n)$ tells us immediately what corresponds in S_n to primes of Σ; for all such primes correspond to prime sections of $G(k, n)$ through Γ, and these represent linear S_k-complexes of S_n which contain all S_k's of S_n which meet S^0_{n-k-1} in a line. All such S_k-complexes, therefore, form a homaloidal system in S_n.

9.2. We proceed now to establish some further properties of the representation, and we begin by introducing some abbreviations as follows:

(i) We shall use the symbol $M(\pi, \nu)$ to denote a manifold, such as Φ_k, generated by $[\pi]$'s joining corresponding points of $\pi+1$ homographic $[\nu-\pi-1]$'s in $[(\nu-\pi)(\pi+1)-1]$.

(ii) We shall use also the symbol $\Phi(\pi, \nu)$ to denote a system of primals of order $\pi+1$ immersed in $[(\nu-\pi)(\pi+1)]$ and defined solely by a π-ple base $M(\pi, \nu)$ therein.

(iii) Lastly we shall use $\Gamma(\pi, \nu)$ to denote a locus which represents on $G(\pi, \nu)$ the system of S_π's of S_ν which meet an $S_{\nu-\pi-1}$ in a line, and from which therefore $G(\pi, \nu)$ projects birationally.

We are able now to prove quite easily what is perhaps the most striking fact about the representation, namely that every $G(k-i, n-j)$ of $G(k, n)$ projects into a flat space of Σ. This we do by showing that it meets the vertex* of projection Γ in a $\Gamma\ (k-i, n-j)$.

9.3. *$G(k-i, n-j)$'s on $G(k, n)$.* All S_k's of S_n which lie in an S_{n-1} are clearly represented† on $G(k, n)$ by a $G(k, n-1)$; and to all those which pass through a point vertex there will correspond, since they meet a prime in all the $[k-1]$'s of that prime, a $G(k-1, n-1)$. We may say briefly then that to the primes and point vertices of S_n correspond $G(k, n-1)$'s and $G(k-1, n-1)$'s of $G(k, n)$; and such manifolds will obviously contain all the $G(k-i, n-j)$ of $G(k, n)$. What corresponds to them, then, in Σ?

Consider first a prime of S_n, S_{n-1} say, and its corresponding $G(k, n-1)$ on $G(k, n)$. In what locus does $G(k, n-1)$ meet the vertex of projection Γ? Clearly in the locus whose points represent S_k's of S_{n-1} which meet S^0_{n-k-1} in a line, *i.e.* which meet the section, S^0_{n-k-2} say, of this space by S_{n-1} in a line. Such a locus, however, is a $\Gamma(k, n-1)$; and $G(k, n-1)$ projects therefore into a flat space, $\Sigma(k, n-1)$ say, of Σ. It must be represented on this space, moreover, by means of the system of primals in which the space is met by Φ; and, since this system must be a $\Phi\ (k, n-1)$, $\Sigma(k, n-1)$ must meet Φ_k in an $M(k, n-1)$, *i.e.* it must

* We refer for brevity to the manifold Γ as the *vertex* of projection. It is really, of course, to the space V containing Γ that the term *vertex* should strictly be applied.

† *Cf.* Severi, *loc. cit.*, 95.

contain a space $\Sigma_0(k, n-1)$ joining homographic $[n-k-2]$'s of the spaces L_μ.

Thus to $G(k, n-1)$'s of $G(k, n)$ correspond spaces $\Sigma(k, n-1)$ of Σ meeting Σ_0 in spaces $\Sigma_0(k, n-1)$ joining homographic $[n-k-2]$'s of the L_μ.

We remark, however, that ∞^k of the primes of S_n contain S^0_{n-k-1}; and it is not difficult to see then that the corresponding $G(k, n-1)$'s project into the ∞^k generating $[n-k-1]$'s of Φ_k.

Consider next a point-vertex P of S_n and its corresponding $G(k-1, n-1)$ on $G(k, n)$. $G(k-1, n-1)$ meets the vertex of projection Γ in a locus which represents all the S_k's which go through P and which meet S^0_{n-k-1} in a line. Projecting these S_k's then from P on to a prime S_{n-1}, we get the system of all S_{k-1}'s of S_{n-1} which meet an S_{n-k-1} (the projection of S^0_{n-k-1}) in a line, and these are represented on $G(k-1, n-1)$ by a $\Gamma(k-1, n-1)$. Thus $G(k-1, n-1)$ meets Γ in a $\Gamma(k-1, n-1)$, and it projects from Γ therefore on to a flat space, $\Sigma(k-1, n-1)$ say, of dimension $(n-k)k$ in Σ. It will be represented on this space, then, by means of the system of primals in which the space is met by Φ, and since this system must be a $\Phi\ (k-1, n-1)$, composed of primals of order k with a $(k-1)$-ple base $M(k-1, n-1)$, $\Sigma(k-1, n-1)$ must meet Σ_0 in a generating $[(n-k)k-1]$ of Φ_1 (such, for example, as that joining k of the spaces L_μ); for it will then meet primals of Φ in that space and, residually, in a system of primals of the kind required.

Thus to $G(k-1, n-1)$'s of $G(k, n)$ correspond spaces $\Sigma(k-1, n-1)$ of Σ, each of which meets Σ_0 in a generating $[(n-k)k-1]$ of Φ_1.

There are ∞^{n-k-1} of the point vertices P, however, which lie in S^0_{n-k-1}; and it is again not difficult to prove that the corresponding $G(k-1, n-1)$'s project into the ∞^{n-k-1} generating $[k]$'s of Φ_k.

It is easy now to see what corresponds to the other $G(k-i, n-j)$ on $G(k, n)$. To any $G(k, n-j)$, for example, corresponding to an $[n-j]$ of S_n, there will correspond a space $\Sigma(k, n-j)$ meeting Σ_0 in a prime of itself joining homographic $[n-k-j-1]$'s of the L_μ; and to any $G(k-i, n-i)$ corresponding to an $[i-1]$-vertex of S_n there will correspond a space $\Sigma(k-i, n-i)$ meeting Σ_0 in a generating space of the second system* of Φ_i. We remark in conclusion, then, that *all Grassmann manifolds of $G(k, n)$ are represented simply on flat spaces of Σ which have appropriate intersections with Φ_k.*

* For any such space is known to be the intersection of i independent generating spaces of the second system of Φ_1 (*cf.* § 6.1); and this corresponds to the fact that any $[i-1]$-vertex of S_n is determined by i independent point vertices.

9.4. *The two systems of generating spaces of* $G(k, n)$. As a deduction from what precedes, we may notice now the representation on Σ of the generating spaces of $G(k, n)$. There are, in fact, two systems of such spaces, corresponding to S_{k+1}'s of S_n and to S_{k-1}-vertices of S_n respectively, and they can be considered as $G(k, k+1)$'s and $G(0, n-k)$'s of $G(k, n)$. They project therefore into $[k+1]$'s and $[n-k]$'s of Σ meeting Σ_0 in the generating $[k]$'s and $[n-k-1]$'s respectively of Φ_k. This could also have been deduced otherwise from the fact that any line of $G(k, n)$ must project into a line of Σ meeting Φ_k; for this proves that any flat space of $G(k, n)$ must project into a flat space of Σ meeting Φ_k in a prime of itself, and this leads to the result in question.

Thus the generating spaces of the two systems of $G(k, n)$, $[k+1]$*'s, and* $[n-k]$*'s respectively, are represented in* Σ *by spaces of the same dimension which meet* Φ_k *in the* $[k]$*'s and* $[n-k-1]$*'s of its two generating systems.*

The results obtained above can now be tabulated as follows:

In S_n	On $G(k, n)$	In Σ
Pencil of S_k's	Line	Line meeting Φ_k
S_{k-1}-vertex...	Generating $[n-k]$	$[n-k]$ through $[n-k-1]$ of Φ_k
S_{k+1}... ...	Generating $[k+1]$	$[k+1]$ through $[k]$ of Φ_k
S_{i-1}-vertex...	$G(k-i, n-i)$ ...	$[(n-k)(k-i+1)]$ meeting Φ_i in a generating $[(n-k)(k-i+1)-1]$
S_{n-j}... ...	$G(k, n-j)$ ...	$[(n-j-k)(k+1)]$ containing corresponding $[n-k-j-1]$'s of the spaces $L\mu$

We pass on now to the last section of this paper, in which we examine some interesting cases of the special representation.

Particular cases.

10.1. *Lines of* S_n. These are represented, according to the general theory, on the points of a $[2n-2]$, Σ, by means of quadrics through a

V_{n-1}^{n-1}, locus of ∞^1 $[n-2]$'s and of ∞^{n-2} lines, and having equations of the form

$$\left\| \begin{matrix} y_{00} & y_{01} & \cdots & y_{0,n-2} \\ y_{10} & y_{1,1} & \cdots & y_{1,n-2} \end{matrix} \right\| = 0 = y.$$

The corresponding $G(1, n)$ contains $[n-1]$'s and planes, represented by similar spaces of Σ meeting V_{n-1}^{n-1} in its $[n-2]$'s and lines respectively; and $G(1, n)$ projects birationally on to Σ from the locus representing all the lines of an $[n-2]$ of S_n, *i.e. from any* $G(1, n-2)$ *of itself.* For $n=4$, for example; this gives the birational projection of M_6^5 from a generating plane of itself*; and for $n=5$ we find that the V_8^9 of [14] dimensions, which represents the lines of [5], projects from any V_4^2 on to an [8], and that the representation so obtained is by quadrics through a V_4^4 of solids.

10.2. *Planes of* S_5. This case deserves special notice, both because it is next in simplicity after that of the lines of S_n and because it is the subject of a very remarkable paper by Segre†, which we have already had occasion to mention. Segre discusses extensively the properties of the $G(2, 5)$ which represents the planes of S_5, and he shows that it is a V_9^{42} of [19], and that it contains two distinct systems of generating solids.

From our point of view V_9^{42} is representable on a [9] Σ by means of the system of cubic primals Φ which have, as simple base locus, that given by the equations

$$\left| \begin{matrix} y_{00} & y_{01} & y_{02} \\ y_{10} & y_{11} & y_{12} \\ y_{20} & y_{21} & y_{22} \end{matrix} \right| = y = 0, \qquad (8)$$

and, as double base locus, that given by

$$\left| \begin{matrix} y_{00} & y_{01} & y_{02} \\ y_{10} & y_{11} & y_{12} \\ y_{20} & y_{21} & y_{22} \end{matrix} \right|_2 = y = 0.$$

These are, respectively, a determinantal cubic primal V_7^3 lying in an [8], Σ_0, and its double locus V_4^6. V_4^6 is just the locus of ∞^2 planes joining corresponding points of three related planes, and it has two such generations and two distinct ∞^2 systems of planes. V_9^{42} *then has two distinct systems of solids represented in* Σ *by solids meeting* V_4^6 *in a plane of one*

* *Cf.* Semple, *loc. cit.*, § 6. Also Todd, *Proc. London Math. Soc.* (2), 30 (1930), 543.

† *Annali di Mat.* (3), 27 (1918), 75.

or other of its two systems. It has also, moreover, two distinct symmetrical systems of M_6^5's representing the points and the primes of S_5 respectively, and these are represented in Σ by [6]'s meeting Φ_1 in generating [5]'s of one or other of its two systems.

From § 8 . 2, too, we see that V_9^{42} *projects on to* Σ *from any tangent* [9] *of itself*, and that this projection gives the above representation; and we see also, from § 8 . 4, that any tangent [9] meets V_9^{42} in a locus representing the planes of S_5 which meet a given plane in a line.

We may notice also the representation in Σ of what Segre calls the *focal* linear complexes of planes in S_5, each of which consists of all the planes which meet a given plane. Consider any one of these, Λ say, fixed by a plane ϖ of S_5, and let P be the point of V_9^{42} corresponding to ϖ. It appears then, in virtue of § 8 . 4, that Λ is represented on V_9^{42} by its intersection with the osculating prime at P, and it is obviously represented on Σ then by the cubic cone whose vertex is P', the projection of P, and whose base is the cubic base locus Φ_1. Moreover, since, according to Segre, any linear plane-complex of S_5 is a linear combination of two focal complexes, so any primal Φ must be a linear combination of two of the cones. Note also that one particular cone of the system consists of the prime Σ_0 counted thrice, and this corresponds to the fundamental plane of the representation, ϖ^0 say, in S_5.

Finally, the parametric representation of V_9^{42}, as deduced from the general case, is as follows. We denote the determinant given in equations (8) by Y and its nine first minors by $Y_{\lambda\mu}$, and the parametric equations then are

$$\frac{z}{y^3} = \frac{z_{\lambda\mu}}{y^2 y_{\lambda\mu}} = \frac{z'_{\lambda\mu}}{y Y_{\lambda\mu}} = \frac{z'}{Y},$$

where the z's and z''s are the twenty coordinates of a point of V_9^{42}. If we use these equations now to find what corresponds on V to the system of ∞^3 planes of a quadric which has the fundamental plane ϖ^0 as a plane of the opposite system, we verify very easily that the locus in question is a V_8^8, and that it is represented in Σ by a solid meeting Φ_1 in a plane *not* lying in a generating [5] of either of its two systems, and that this solid meets Φ residually in the complete system of quadrics of itself*.

10 . 3. *The self-dual* S_k*'s.* Lastly, as a generalization in one way of the preceding case, we may mention the system of self-dual S_k's of an S_{2k+1}, which is in some ways very much simpler than the general case.

* *Cf.* Segre, *Annali di Mat.* (3), 27 (1918), 158.

The corresponding manifold is a $G(k, 2k+1)$, and the matrix from which the equations of $\Phi_1, \ldots, \Phi_k$ are derived is now a *square* matrix. It follows then that, in this case, the simple base locus Φ_1 is a determinantal primal in Σ_0 of order $k+1$. The other base loci have also symmetrical properties of a similar character; and Φ_k is the locus of two distinct symmetrical systems of $[k]$'s.

Thus $G(k, 2k+1)$ has always two symmetrical systems of $[k+1]$'s represented in Σ by $[k+1]$'s which meet Φ_k in a $[k]$ of one or other of its two systems.

We note finally that $G(k, 2k+1)$ projects on to Σ birationally from any one of its k-tangential spaces, and that the focal linear complexes of the preceding case can be immediately generalized for this case. They are evidently again represented in Σ by cones projecting Φ_1.

MATHEMATICAL NOTES (9): ON FUNCTIONS SUBHARMONIC IN A CIRCLE (III)

By J. E. LITTLEWOOD.

[Received 31 December, 1929.—Read 16 January, 1930.]

1. Suppose that $w(r, \theta)$ is a subharmonic function† in $r < 1$ satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |w(r, \theta)|^p \, d\theta = O(1). \tag{1}$$

I proved recently‡ that if $p > 1$ there exists a function $W(\theta)$ which is a "strong limit" of $w(r, \theta)$ for any index $q < p$, *i.e.* that if $p > 1$, then

$$\lim_{r \to 1} \int_{-\pi}^{\pi} |w(r, \theta) - W(\theta)|^q \, d\theta = 0 \quad (q < p). \tag{2}$$

Two functions W satisfying (2) are clearly equivalent (differ in value only at points θ of a set of measure zero).

It is well known that (2) holds with $q = p > 1$ if w is a harmonic function satisfying (1); the question whether it holds also when w is subharmonic I had before to leave unsettled. I can now give the answer; it is, rather unexpectedly perhaps, in the negative.

I showed recently§ that when $p \geqslant 1$ a subharmonic w satisfying (1) has almost always a radial limit, identical with $W(\theta)$ when $p > 1$. It is natural to inquire whether there is necessarily a strong limit, or almost always a radial limit, when $p < 1$. I show in §§ 7–9, what has not been stated explicitly before, that the answers to these questions are in the

† For the general theory of subharmonic functions, see F. Riesz, *Acta Univ. Franc. Jos. Szeged*, 2 (1925), 87–100; *Acta Math.*, 48 (1926), 329–343.

‡ In Note 7 of this series, *Journal London Math. Soc.*, 2 (1927), 192–196.

§ In Note 8 of this series, *Proc. London Math. Soc.* (2), 28 (1929), 383–394. There is an unfortunate misprint in (4.1) (p. 387) and (4.3) (p. 388); in each of which the double integral should be preceded by a *minus* sign. The argument is not affected.

negative, even if we suppose w to be harmonic. I discuss also what happens when $p = 1$, and sum up our knowledge of the subject, which is now complete, in a comprehensive theorem.

2. Let us denote the current point (r, θ) by P, the (parametric) point (ρ, ϕ) by Q. Let $g(P, Q) = g(r, \theta; \rho, \phi)$ be the Green's function for the circle. Its value is

$$g(P, Q) = \tfrac{1}{2} \log \left(1 + \frac{(1-r^2)(1-\rho^2)}{r^2 - 2r\rho \cos(\phi - \theta) + \rho^2}\right). \tag{4}$$

Consider the function w defined by

$$-w = v = \int_0^1 \int_{-\pi}^{\pi} \sigma(\rho, \phi)\, g(r, \theta; \rho, \phi)\, \rho\, d\rho\, d\phi,$$

in which the double integral is taken over the area $\rho < 1$, and the "density" σ is given by

$$\sigma = \sigma(\rho, \phi) = (1-\rho)^{-2} \left(\log \frac{2}{1-\rho}\right)^{-1-\alpha} |\phi|^{-\alpha} \quad (|\phi| \leqslant \pi), \tag{5}$$

where $\alpha = 1/p$. g, *qua* function of Q, has a logarithmic infinity at $Q = P$, and it tends to zero like $1-\rho$ when P is fixed and $\rho \to 1$. The repeated integral is convergent (at $Q = P$ and at $\rho = 1$); v is a positive, w a negative function of P.

$\sigma(Q)$ is continuous at Q's of $\rho < 1$ except for those on the positive real axis. A little consideration shows that $w(P)$ is continuous at P's of $r < 1$. The function $v(P)$, as the potential of a distribution of density σ, satisfies

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -2\pi\sigma(P) < 0,$$

except on the radius vector $\theta = 0$, and there v is continuous. v is therefore superharmonic† and w subharmonic.

---

† This is not entirely trivial. Let $\sigma_N = \text{Min}(\sigma, N)$, $v_N = \iint \sigma_N g \rho\, d\rho\, d\phi$. v_N satisfies $\Delta v_N = -2\pi\sigma_N < 0$ everywhere in $r < 1$; it is therefore superharmonic and satisfies, at each P, or z,

$$v_N(z) \geqslant \frac{1}{2\pi} \int v_N(z + Re^{i\psi})\, d\psi \tag{a}$$

for small values of R. Since the original double integral is convergent, v_N is an increasing sequence of functions with v as limit function, and we may replace v_N in (a) by v, which is therefore superharmonic.

I shall show that

$$\int_{-\pi}^{\pi} v^p \, d\theta = O(1), \tag{6}$$

$$\lim_{r \to 1} \int_{-\pi}^{\pi} v \, d\theta = 0, \tag{7}$$

$$\overline{\lim_{r \to 1}} \int_{-\pi}^{\pi} v^p \, d\theta > 0. \tag{8}$$

The result (7) shows that $W(\theta) \equiv 0$, and (6) and (7) together imply (by Hölder's inequality, and as in Note 7) the strong convergence of w to the limit function 0 for any index $q < p$. A strong limit, if any, with index p, being also a strong limit with index $q < p$, is necessarily equivalent to the function 0. But (8) shows that the strong convergence to the function 0 fails for $q = p$.

3. In what follows I use A's to denote positive absolute constants, B's to denote positive constants depending only on p. A's and B's are not necessarily the same from one occurrence to another. It is convenient to have single symbols for $1-r$ and $\log\{2/(1-r)\}$. Let us write

$$s = 1-r, \quad \lambda = \log \frac{2}{s} = \log \frac{2}{1-r}.$$

With the point (r, θ) we associate four areas, together making up the area of the unit circle:

D_1, the annulus defined by $0 < 1-\rho \leqslant \frac{1}{2}s$;

D_2, the circle (possibly null) defined by $1 \geqslant 1-\rho \geqslant \frac{3}{2}s$;

D_3, the area in which ($\rho \geqslant 0$ and) $|\rho - r| < \frac{1}{2}s$, $|\phi - \theta| < s$;

D_4, the area in which ($\rho \geqslant 0$ and) $|\rho - r| < \frac{1}{2}s$, $|\phi - \theta| \geqslant s$.

We have now

$$v = \sum_1^4 v_n = \sum_1^4 \iint_{D_n} \sigma g \rho \, d\rho \, d\phi. \tag{9}$$

Let us write

$$T = \frac{\lambda^{-\alpha}}{(s+|\theta|)^{\alpha}} = \left(\log \frac{2}{1-r}\right)^{-\alpha} \left(\frac{1}{1-r+|\theta|}\right)^{\alpha}.$$

I shall show that

$$BT < v_1 < BT \quad (|\theta| \leqslant \pi), \tag{10}$$

$$v_n < BT \quad (|\theta| \leqslant \pi,\ n = 2, 3, 4). \tag{11}$$

The truth of (6), (7), (8) follows from (10) and (11). For, in the first place,

$$\lim_{r \to 1} \int_{-\pi}^{\pi} T d\theta \leqslant \lim \lambda^{-\alpha} \int_{-\pi}^{\pi} |\theta|^{-\alpha} d\theta = 0,$$

in virtue of which (7) follows from (10) and (11). In the second,

$$\int_{-\pi}^{\pi} T^p d\theta = 2\lambda^{-1} \int_0^{\pi/s} \frac{dx}{1+x} = 2 \log \left(1 + \frac{\pi}{s}\right) \Big/ \log \frac{2}{s},$$

and so lies between two constants B. Hence (10) and (11) give us

$$B < \int_{-\pi}^{\pi} v_1^p d\theta \leqslant \int_{-\pi}^{\pi} v^p d\theta \leqslant \sum_1^4 B \int_{-\pi}^{\pi} v_n^p d\theta < B$$

for all values of $r < 1$, and these inequalities establish (6) and (8). Everything reduces, therefore, to proving (10) and (11). These are matters of detailed calculation, tedious, but unfortunately hardly trivial enough to be taken for granted.†

4. The v_n are clearly even functions of θ, and we may suppose that $0 \leqslant \theta \leqslant \pi$. The formula (4) shows that for such θ

$$g(r, \theta; \rho, \phi) > g(r, \theta; \rho, -\phi) \quad (0 < \phi < \pi).$$

If then we denote by D_n' the part of D_n that lies in the upper semicircle $0 < \phi < \pi$, we have

$$v_n' \leqslant v_n \leqslant 2v_n',$$

where

$$v_n' = \iint_{D_n'} \sigma g \rho \, d\rho \, d\phi,$$

and it is enough to prove that

$$BT < v_1' < BT \quad (0 < \theta < \pi) \tag{12}$$

and

$$v_n' < BT \qquad (0 < \theta < \pi,\ n = 2, 3, 4). \tag{13}$$

† Since v behaves much like T, it is natural to try whether some explicit expression such as T can serve as our *Gegenbeispiel*. T itself, however, is not superharmonic throughout $r < 1$, nor have I succeeded in finding a suitable modification that is.

5. The following approximations hold for $g(P, Q)$. We suppose always that $0 < \theta < \pi$. Then

$$\frac{As(1-\rho)}{s^2+(\phi-\theta)^2} < g < \frac{As(1-\rho)}{s^2+(\phi-\theta)^2} \quad (Q \text{ in } D_1'). \tag{14}$$

$$g < \frac{As(1-\rho)}{(1-\rho)^2+(\phi-\theta)^2} \quad (Q \text{ in } D_2'). \tag{15}$$

$$g < \log\left(1+\frac{As^2}{(\phi-\theta)^2}\right) \quad (Q \text{ in } D_3'). \tag{16}$$

$$g < \frac{As^2}{s^2+(\phi-\theta)^2} \quad (Q \text{ in } D_4'). \tag{17}$$

These are matters of straightforward calculations from (4) (details may be extracted from Note 8).

We require also the following

LEMMA. *We have*

$$\frac{B}{(1+t)^\alpha} < J = \int_0^{\pi/s} \frac{y^{-\alpha}\,dy}{1+(y-t)^2} < \frac{B}{(1+t)^\alpha} \quad \left(0 \leqslant t \leqslant \frac{\pi}{s}\right); \tag{18}$$

$$L = \int_{t-1}^{t+1} |y|^{-\alpha} \log\left(1+\frac{A}{(y-t)^2}\right) dy < \frac{B}{(1+t)^\alpha} \quad \left(0 \leqslant t \leqslant \frac{\pi}{s}\right). \tag{19}$$

If $t < 1$, then

$$\int_1^\pi \frac{y^{-\alpha}\,dy}{1+y^2} < J < \int_0^1 y^{-\alpha}\,dy + \int_1^\infty \frac{y^{-\alpha}\,dy}{1+(y-1)^2},$$

or

$$B < J < B,$$

which is equivalent to (18) for $t < 1$.

If $1 \leqslant t \leqslant \pi/s$, then

$$J > \int_0^{\frac{1}{2}} \frac{(t-u)^{-\alpha}\,du}{1+u^2} > \int_0^{\frac{1}{2}} \frac{(t-\frac{1}{2}t)^{-\alpha}\,du}{1+u^2} = Bt^{-\alpha} > \frac{B}{(1+t)^\alpha},$$

and

$$J < \left(\int_0^{\frac{1}{2}t} + \int_{\frac{1}{2}t}^\infty\right) \frac{y^{-\alpha}\,dy}{1+(y-t)^2} < \frac{A}{t^2}\int_0^{\frac{1}{2}t} y^{-\alpha}\,dy + \int_{\frac{1}{2}t}^\infty \frac{t^{-\alpha}\,dy}{1+(y-t)^2}$$

$$\leqslant Bt^{-1-\alpha} + \int_{-\infty}^\infty \frac{t^{-\alpha}\,dy}{1+(y-t)^2} = Bt^{-1-\alpha} + Bt^{-\alpha} < \frac{B}{(1+t)^\alpha},$$

which together give (18).

In (19) suppose first that $t \geqslant 2$. Then $|y| > \frac{1}{2}t$ in the range of integration, and

$$L < \int_{t-1}^{t+1} Bt^{-\alpha} \log\left(1+\frac{A}{(y-t)^2}\right) dy = Bt^{-\alpha} \int_{-1}^{1} \log\left(1+\frac{A}{u^2}\right) du < \frac{B}{(1+t)^\alpha}.$$

Suppose then that $t < 2$. Then, if u is positive, the integrand is less for $y = t+u$ than for $y = t-u$; hence

$$\tfrac{1}{2}L < \int_{t-1}^{t} y^{-\alpha} \log\left(1+\frac{A}{(y-t)^2}\right) dy = \int_0^1 |u-t|^{-\alpha} \log\left(1+\frac{A}{u^2}\right) du$$

$$\leqslant L_1+L_2+L_3,$$

where $$L_1 = \int_0^{\frac{1}{2}t}, \quad L_2 = \int_{\frac{1}{2}t}^{t}, \quad L_3 = \int_{,}^{1}$$

(and L_3 is to be omitted if $t > 1$). In L_1 $|u-t|^{-\alpha} < Bt^{-\alpha}$, and

$$L_1 < Bt^{-\alpha} \int_0^{\frac{1}{2}t} \log\left(1+\frac{B}{u^2}\right) du < Bt^{1-\alpha} \log\frac{A}{t} < B.$$

Next, $$L_2 < A \log\frac{A}{t} \int_{\frac{1}{2}t}^{t} |u-t|^{-\alpha} du < Bt^{1-\alpha} \log\frac{A}{t} < B.$$

Finally

$$L_3 = \int_0^{1-t} v^{-\alpha} \log\left(1+\frac{A}{(v+t)^2}\right) dv \leqslant \int_0^1 v^{-\alpha} \log\left(1+\frac{A}{v^2}\right) dv < B.$$

Hence $$L < B < \frac{B}{(1+t)^\alpha},$$

and the proof of (19), and so of the lemma, is completed.

6. Consider now the v'_n, and let us write

$$\rho = 1-xs, \quad \phi = ys, \quad \theta = ts.$$

By (14) v'_1 lies between multiples A of

$$\int_{1-\frac{1}{2}s}^{1} d\rho \int_0^{\pi} (1-\rho)^{-2} \left(\log\frac{2}{1-\rho}\right)^{-1-\alpha} \frac{\phi^{-\alpha} s(1-\rho)\, d\phi}{s^2+(\phi-\theta)^2}$$

$$= s^{-\alpha} \left\{\int_0^{\frac{1}{2}} dx\, x^{-1} \left(\lambda+\log\frac{1}{x}\right)^{-1-\alpha}\right\} \int_0^{\pi/s} \frac{y^{-\alpha}\, dy}{1+(y-t)^2},$$

or, using (18) of the lemma for the last factor, between multiples B of

$$s^{-\alpha}\lambda^{-\alpha}\frac{1}{(1+t)^{\alpha}}=\frac{\lambda^{-\alpha}}{(s+\theta)^{\alpha}}=T,$$

which establishes (12).

Next, v_2' is zero unless $\frac{3}{2}s<1$. If this condition is satisfied we have, by (15),

$$v_2'\leqslant A\int_0^{1-\frac{3}{2}s}d\rho(1-\rho)^{-2}\left(\log\frac{2}{1-\rho}\right)^{-1-\alpha}\int_0^{\pi}\frac{\phi^{-\alpha}s(1-\rho)\,d\phi}{(1-\rho)^2+(\phi-\theta)^2}$$

$$=As^{-\alpha}\int_{\frac{3}{2}}^{s^{-1}}dx\,x^{-1}\left(\log\frac{2}{sx}\right)^{-1-\alpha}\int_0^{\pi/s}\frac{y^{-\alpha}\,dy}{x^2+(y-t)^2}.$$

We divide the range of the outer integration into

$$\left\{\tfrac{3}{2},\ (\tfrac{3}{2}s)^{-\frac{3}{4}}\right\},\quad \left\{(\tfrac{3}{2}s)^{-\frac{3}{4}},\ s^{-1}\right\}.$$

The contribution of the first to v_2' does not exceed

$$As^{-\alpha}\int_{\frac{3}{2}}^{(\frac{3}{2}s)^{-\frac{3}{4}}}dx\,x^{-1}(A\lambda)^{-1-\alpha}\int_0^{\pi/s}\frac{y^{-\alpha}\,dy}{1+(y-t)^2}$$

$$<As^{-\alpha}A\lambda(A\lambda)^{-1-\alpha}\frac{B}{(1+t)^{\alpha}}=BT, \tag{20}$$

by (18). The contribution of the second does not exceed

$$As^{-\alpha}\int_{(\frac{3}{2}s)^{-\frac{3}{4}}}^{s^{-1}}dx\,x^{-1}B\int_0^{\pi/s}\frac{y^{-\alpha}\,dy}{x^2} \tag{21}$$

$$<Bs^{-\alpha}(s^{-\frac{3}{4}})^{-2}(\pi/s)^{1-\alpha}=Bs^{\frac{1}{2}}<B\lambda^{-\alpha}<BT.$$

From (20) and (21) we have

$$v_2'<BT. \tag{22}$$

Next, by (16),

$$v_3'\leqslant\int_{1-\frac{1}{2}s}^{1-\frac{3}{2}s}d\rho(1-\rho)^{-2}\left(\log\frac{2}{1-\rho}\right)^{-1-\alpha}\int_{\theta-s}^{\theta+s}|\,\phi\,|^{-\alpha}\log\left(1+\frac{As^2}{(\phi-\theta)^2}\right)dy$$

$$=s^{-\alpha}\int_{\frac{1}{2}}^{\frac{3}{2}}dx\,x^{-2}\left(\log\frac{2}{xs}\right)^{-1-\alpha}\int_{t-1}^{t+1}|\,y\,|^{-\alpha}\log\left(1+\frac{A}{(y-t)^2}\right)dy$$

$$<s^{-\alpha}A(B\lambda)^{-1-\alpha}\frac{B}{(1+t)^{\alpha}}<BT, \tag{23}$$

by (19) of the lemma.

Finally, using (17) and once more (18), we have

$$v_4' < \int_{1-\frac{3}{2}s}^{1-\frac{1}{2}s} d\rho(1-\rho)^{-2}\left(\log\frac{2}{1-\rho}\right)^{-1-\alpha}\int_0^{\pi}\frac{\phi^{-\alpha}As^2\,d\phi}{s^2+(\phi-\theta)^2}$$

$$= s^{-\alpha}\int_{\frac{1}{2}}^{\frac{3}{2}} dx\,x^{-2}\left(\lambda+\log\frac{1}{x}\right)^{-1-\alpha}\int_0^{\pi/s}\frac{y^{-\alpha}\,dy}{1+(y-t)^2}$$

$$(24)\qquad < s^{-\alpha}A(A\lambda)^{-1-\alpha}\frac{B}{(1+t)^{\alpha}} < BT.$$

The results (22), (23), (24) establish (13), and our calculations are completed.

7. I take up now the outstanding questions of the cases $p \leqslant 1$. Let us take first the case $p = 1$. By my result (*cf.* Note 8) quoted in § 1, there is a $W(\theta)$, such that

$$(25)\qquad \lim_{r\to 1} w(r,\theta) = W(\theta)$$

for almost all θ. By a theorem of Fatou

$$\int_{-\pi}^{\pi} |\,W(\theta)\,|\,d\theta = \int_{-\pi}^{\pi} \lim |\,w\,|\,d\theta \leqslant \underline{\lim}\int_{-\pi}^{\pi} |\,w\,|\,d\theta < K,$$

and so

$$\int_{-\pi}^{\pi} |\,w-W\,|\,d\theta < \int |\,w\,|\,d\theta + \int |\,W\,|\,d\theta < K.$$

By Egoroff's theorem $|\,w-W\,|$ converges uniformly to its limit 0, except in a set E of arbitrarily small measure η. If now $q = 1/r < p = 1$, we have

$$\lim\int_{CE} |\,w-W\,|^q\,d\theta = 0,$$

$$\int_E |\,w-W\,|^q\,d\theta \leqslant (mE)^{1-1/r}\left(\int_E |\,w-W\,|^{qr}\,d\theta\right)^{1/r} \leqslant \eta^{1-1/r}\left(\int_{-\pi}^{\pi} |\,w-W\,|\,d\theta\right)^{1/r}$$

$$\leqslant K\eta^{1-1/r},$$

$$\overline{\lim}\int_{-\pi}^{\pi} |\,w-W\,|^q\,d\theta = \overline{\lim}\int_E |\,w-W\,|^q\,d\theta \leqslant K\eta^{1-1/r},$$

$$\lim\int_{-\pi}^{\pi} |\,w-W\,|^q\,d\theta = 0.$$

Thus the result about strong convergence for index $q < p$ extends to the case $p = 1$.

The failure of the strong convergence for $q = p$ occurs here also, but for a different reason, and now even for harmonic functions. The (positive) function

$$u(r, \theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u|\, d\theta = 1. \tag{26}$$

It has almost always (indeed for θ other than $\theta = 0$) the radial limit 0, and this is the strong limit for index $q < 1$. If U is a strong limit for index 1, it is one also for index $q < 1$, and we must have $U \equiv 0$; strong convergence to 0 with index 1, is, however, incompatible with (26).

8. I shall prove next that there exists a harmonic function w satisfying (1) for every positive $p < 1$, and without the property of possessing a radial limit for almost all θ. Let

$$f_1(z) = \log k^2(\tau), \qquad f_2(z) = \log k'^2(\tau), \qquad \tau = \frac{1+z}{1-z},$$

where k^2, k'^2 have their usual meaning in elliptic function theory, and let $u_1 = \Re f_1$, $u_2 = \Re f_2$. u_1 and u_2 are harmonic in $|z| < 1$. I have shown elsewhere† that

$$\int_{-\pi}^{\pi} |u_i|^p\, d\theta = O(1) \quad (i = 1, 2).$$

The u_i have, in point of fact, radial limits only in a null set of θ, but it is simpler, and sufficient for our purpose, to prove that both u's cannot have almost always a radial limit.

Suppose that, on the contrary,

$$\lim_{r\to 1} u_i = l_i(\theta) \quad (i = 1, 2)$$

almost always. Then

$$\int_{-\pi}^{\pi} |l_i|^p\, d\theta = \int \lim |u_i|^p\, d\theta \leqslant \underline{\lim} \int_{-\pi}^{\pi} |u_i|^p\, d\theta < \infty,$$

and $l_i(\theta)$ must be finite almost everywhere. Hence, almost always,

$$\lim_{r\to 1} |e^{f_i(z)}| = c_i(\theta) = e^{l_i(\theta)},$$

† J. E. Littlewood, "On inequalities in the theory of functions", *Proc. London Math. Soc.* (2), 23 (1925), 481–519 (517). The result is stated there (with a parameter τ_0, here equal to i), only for u_1, but the proof implicitly establishes it for u_2 also.

where $c_i \neq 0, \infty$ $(i = 1, 2)$. Now $\zeta = e^{f_i(z)}$ is an automorphic function whose fundamental regions are zero-angled curvilinear triangles, with corners on $|z| = 1$, whose sides correspond to the intervals $(0, 1)$, $(1, +\infty)$, $(-\infty, 0)$ of the real axis of ζ. Except for a null set of θ, the *radius vector* to $e^{\theta i}$ crosses two sides of each of an infinity of triangles, and in a sub-infinity of cases crosses two sides meeting at one particular kind of vertex, corresponding to $\zeta = w$, say, where w is one of $0, 1, \infty$. $|e^{f_i}|$ cannot tend, on such a *radius vector*, to a radial limit different from w. For almost all θ, 0 and ∞ are impossible as limits, and we must have

$$\lim |e^{f_1}| = 1, \quad \lim |e^{f_2}| = 1.$$

But $e^{f_1} + e^{f_2} = 1$ (and f_1, f_2 vary continuously on a radius vector). Hence we must have, for almost all θ,

$$\lim_{r \to 1} e^{f_1} = \pm e^{\frac{1}{3}\pi i} \tag{27}$$

(the sign being a function of θ). But a glance back at the argument by fundamental triangles shows that a radial limit of e^{f_1} itself must almost always be one of $0, 1, \infty$. This contradicts (27) and establishes our conclusion.

9. I shall now show finally that a harmonic function w exists, satisfying (1), for every positive $p < 1$, and possessing a strong limit for no positive index μ. Such a function is supplied by our $u_1 = \Re f_1$ above. It is convenient to change our notation. Let

$$\kappa(\tau) = \log k^2(\tau),$$

and

$$w(r, \theta) = w(z) = \Re \kappa(\tau), \qquad \tau = \frac{1+z}{1-z}\,\dagger,$$

Let

$$r_1 = 1-\eta, \qquad r_2 = 1-\tfrac{1}{2}\eta,$$

where η is small. I shall show that for any positive μ, which we may suppose less than 1,

$$\lim_{\eta \to 0} \int_{-\pi}^{\pi} |w(r_1, \theta) - w(r_2, \theta)|^{\mu}\, d\theta > 0. \tag{28}$$

This shows that w does not possess a strong limit for index μ, and completes our task.

---

† w is u_1, and the determination of the logarithm does not affect its value.

I do not see how to prove (28) without heavy calculations, but the material is for the most part available in my paper quoted in § 8. Since we are proving only a negative result, and one very plausible, I shall reduce my account to the minimum necessary to convince a reader acquainted with that paper. I adopt its notation, and write further,

$$z_1 = r_1 e^{\theta i}, \quad z_2 = r_2 e^{\theta i}, \quad w_1 = w(r_1, \theta), \quad w_2 = w(r_2, \theta), \quad \Delta = |w_1 - w_2|;$$

$$\tau = u + iv, \quad \tau_1 = \frac{1+z_1}{1-z_1}, \quad \tau_2 = \frac{1+z_2}{1-z_2}.$$

We have first†

$$\int_{-\pi}^{\pi} \Delta^\mu d\theta > \int_{K'} \Delta^\mu d\theta = \sum_m \int_{K_m} \Delta^\mu d\theta > \int_{K_1} \Delta^\mu d\theta > \int_{K_1} \Delta^\mu \left|\frac{dz_1}{d\tau_1}\right| \left|\frac{d\tau_1}{du_1}\right| du_1.$$

For θ of K_1

$$\left|\frac{d\tau_1}{du_1}\right| > A, \quad \left|\frac{dz_1}{d\tau_1}\right| = A|1-z_1|^2 > A.$$

Hence

$$\int_{-\pi}^{\pi} \Delta^\mu d\theta > A \int_{-1}^{1} \Delta^\mu du_1 > A \int_0^1 \Delta^\mu du_1.$$

We now employ the Farey dissection of the interval $0 \leqslant u_1 \leqslant 1$ of order $N = [\eta^{-\frac{1}{2}}]+1$, and denote its intervals by $\xi_{p,q}$. Let c be a small constant, to be chosen (as an A) later, and in any case less than 1; and let us denote by $c\xi_{p,q}$ the interval $\xi_{p,q}$ reduced by the factor c about its origin p/q. We integrate only through the $c\xi_{p,q}$, and further take the q summation only to cN (instead of the full N), obtaining

$$\int_{-\pi}^{\pi} \Delta^\mu d\theta > A \sum_{q<cN} \sum_p \int_{c\xi_{p,q}} \Delta^\mu du_1. \tag{29}$$

Let us now suppose, as a rough approximation to the truth, that all the substitutions $S_{p,q}$ leave $\kappa(\tau)$ invariant (they actually leave invariant a set of six functions, all on the same footing, of which κ is one). If $S_{p,q}$ changes τ_1, τ_2 to T_1, T_2, we have for a certain q',

$$T = S_{p,q}(\tau) = \frac{q'}{q} - \frac{1}{q^2(\tau - p/q)},$$

$$\Im T_i = \frac{v_i}{q^2\{(u_i - p/q)^2 + v_i^2\}}.$$

† The association of a suffix 1 or 2 with the variable z_1 or z_2 is violated in K_1, which means, of course, "K_m with $m = 1$".

Calculation shows that

$$A\eta < v_1 < A\eta, \quad A\eta < v_2 < A\eta, \quad |u_1 - u_2| < A\eta, \quad v_1 - v_2 > A\eta.$$

Hence for u_1 of $c\xi_{p,q}$, we have

$$\mathfrak{J}\mathrm{T}_i > \frac{A\eta}{q^2\{(A\eta + cq^{-1}N^{-1})^2 + A\eta^2\}} = \frac{A\eta}{(Aq\eta + cN^{-1})^2 + Aq^2\eta^2},$$

which, when we substitute the worst possible value cN of q, gives

$$\mathfrak{J}\mathrm{T}_i > \frac{A}{c^2} > 1, \tag{30}$$

the last inequality being secured by a suitable choice of c. On $c\xi_{p,q}$, we have further

$$|\mathfrak{J}(\mathrm{T}_1 - \mathrm{T}_2)| = \frac{|\tau_1 - \tau_2|}{q^2|\tau_1 - p/q||\tau_2 - p/q|} > \frac{A\eta}{q^2\{(B\eta + cq^{-1}N^{-1})^2 + A\eta^2\}} \tag{31}$$

$$> \frac{A}{c^2}$$

as before. Now

$$\Delta = |\mathfrak{J}(\mathrm{T}_1 - \mathrm{T}_2) + \mathfrak{R}P(e^{\pi i \mathrm{T}_1}) - \mathfrak{R}P(e^{\pi i \mathrm{T}_2})|,$$

where $P(x)$ is a power series depending on no parameter and with radius of convergence unity. By (30) and (31), we have on $c\xi_{p,q}$

$$\Delta > \frac{A}{c^2} - Ae^{-(A/c^2)}$$

$$> A,$$

if c is a sufficiently small A. Then

$$\sum_{q<cN} \sum_{p} \int_{c\xi_{p,q}} \Delta^\mu \, du_1 > \sum_{q<cN} \sum_{p} A \frac{c}{qN} A = \frac{A}{N} \sum_{q<cN} \frac{\phi(q)}{q} > \frac{A}{N} A(cN) = A \tag{32}$$

by a known property of Euler's function†.

(29) and (32) show that we should obtain

$$\int_{-\pi}^{\pi} \Delta^\mu \, d\theta > A,$$

† We may also argue that

$$\sum_{q \leq v} \frac{\phi(q)}{q} = v \sum_{q \leq v} \sum_{p} \frac{1}{qv} > v \sum_{q \leq v} \sum_{p} \int_{\xi_{p,q}} A \, du = v \int_0^1 A \, du = Av.$$

if the simplifying assumption about the $S_{p,q}$ were true. Actually it is false. But the intervals $\xi^*_{p,q}$ for which it is true are roughly one-sixth of the total number of $\xi_{p,q}$, and their distribution is uniform and explicit enough for us to be able to adapt the argument above to prove that

$$\int_{c\xi^*_{p,q}} \Delta^\mu \, du_1 > A,$$

and so to arrive at the desired conclusion.

10. We may sum up the state of our knowledge as follows.

THEOREM. *Suppose that w is subharmonic in $r<1$ and satisfies* (1). *When $p \geqslant 1$ there is almost always a radial limit $W(\theta) = \lim w(r, \theta)$, and W is a strong limit of w for any index $q<p$. When $p>1$ there need be no strong limit of w with index p, but W is one under the stronger hypothesis that w is harmonic. When $p=1$ there need be no strong limit of w with index $p(=1)$, even when w is harmonic. When $p<1$ there need be no strong limit for any index, nor need a radial limit exist almost everywhere (it need not exist outside a null set of θ).*

FERMAT'S EQUATION IN REAL QUATERNIONS

By D. E. LITTLEWOOD *and* A. R. RICHARDSON.

[Received 13 November, 1929.—Read 14 November, 1929.]

1. From the various ways in which the theorem concerning Fermat's equation may be proved we shall select one suggesting further extensions.

Let x denote a real quaternion integer, which, for the purpose of this note, may be considered to have real integral coefficients. It satisfies the rank equation

$$x^2 = 2x_0x-(x_0^2+x_1^2+x_2^2+x_3^2), \tag{1}$$

where $$x = x_0+x_1i+x_2j+x_3k.$$

We shall prove that, if N be a real prime, x satisfies the equation

$$x^2(x^{N(N^2-1)}-1) \equiv 0 \pmod{N}. \tag{2}$$

THEOREM 1. *In the equation* $x^2 \equiv \lambda x+M$, λ *and* M *can have any of the values* $0, 1, 2, \ldots, N-1 \pmod{N}$.

It is well known that every positive integer, not of the form $8n+7$ or $4n$, is a sum of three squares*.

Further, the equation $$8n+1 \equiv \mu \pmod{N}$$

can always be solved for n. Hence any residue mod N can be expressed as the sum of three squares. In order to prove the theorem, we note that λ may take any value (mod N), and that when λ is assigned M may take any value, since any residue may be expressed as the sum of three squares. A proof of (2) follows, for

$$\prod_{M=0}^{N-1}\prod_{\lambda=0}^{N-1}(x^2-\lambda x-M) \equiv 0 \pmod{N},$$

giving $$x^{N+1}(x^{N^2-1}-1)(x^{N(N-1)}-1) \equiv 0 \pmod{N},$$

in which x^{N+1} may be replaced by x^2.

* L. E. Dickson, *History of the theory of numbers*, 2 (1920), 261.

THEOREM 2. *All real quaternion integers satisfy equation* (2).

By Fermat's theorem

$$M^N \equiv M \quad \text{and} \quad \lambda^N \equiv \lambda \pmod{N}.$$

Hence equation (1) may be written

$$(x^N - x)(x^N - \bar{x}) \equiv 0 \pmod{N}.$$

Therefore, either $x^N - x \equiv 0 \pmod{N}$ (3)

or $x^N - \bar{x} \equiv 0 \pmod{N}$,

in which case $\bar{x}^N - x \equiv 0 \pmod{N}$,

and therefore $x^{N^2} - x \equiv 0 \pmod{N}$, (4)

or the norms of $x^N - x$ and $x^N - \bar{x}$ are both zero (mod N).

Now $$x + \bar{x} = \lambda,$$

therefore $$x^N + \bar{x}^N \equiv \lambda^N \equiv \lambda \equiv x + \bar{x} \pmod{N}.$$

Hence $$\text{norm}\,(x^N - x) \equiv (x^N - x)(\bar{x}^N - \bar{x}) \equiv -(x^N - x)^2 \pmod{N}$$

and $$\text{norm}\,(x^N - \bar{x}) \equiv (x^N - \bar{x})(\bar{x}^N - x) \equiv -(x^N - \bar{x})^2 \pmod{N}.$$

If both are zero, then

$$(x^N - x)^N \equiv x^{N^2} - x^N \equiv 0 \pmod{N}$$

and $$(x^N - \bar{x})^N \equiv x^{N^2} - \bar{x}^N \equiv 0 \pmod{N},$$

so that $$x^N \equiv \bar{x}^N \pmod{N},$$

and hence $$x^{N^2} \equiv \bar{x}^{N^2} \equiv x^N \pmod{N}, \quad (5)$$

i.e., $$x^N(x^{N^2-N} - 1) \equiv 0 \pmod{N}.$$

Hence, either $x^N \equiv 0$, in which case $x^2 \equiv 0 \pmod{N}$, (6)

or $$x^{N^2-N} - 1 \equiv 0 \pmod{N}, \quad (7)$$

or the norm of $x \equiv 0 \pmod{N}$ and the norm of $(x^{N^2-N} - 1) \equiv 0 \pmod{N}$. But norm $x \equiv 0$ is compatible with $(x^N - x)^2 \equiv (x^N - \bar{x})^2 \equiv 0 \pmod{N}$ only if $x^2 \equiv 0$; hence all possible relations are given by (3), (4), (5), (6), (7), which may be incorporated in the one formula

$$x^2(x^{N(N^2-1)} - 1) \equiv 0 \pmod{N}. \quad (8)$$

Another method of proof, peculiar however to quaternions, is obtained as the converses of the following results:

THEOREM 3. *If* $x^N \equiv x \pmod N$ *and* V *be the vector part of* x, *then the negative of the norm of* V *is a quadratic residue* (mod N).

Let $x = S+V$, then $\quad S^N+V^N \equiv S+V \pmod N$,

therefore $\quad V^N \equiv V \pmod N$,

therefore $\quad V[(-M)^{\frac{1}{2}(N-1)}-1] \equiv 0 \pmod N$,

where $V^2 \equiv -M$.

Hence $(-M)^{\frac{1}{2}(N-1)} \equiv 1$, and the result is proved.

THEOREM 4. *If* $x^N \equiv \bar{x} \pmod N$, *then the negative of the norm of* V *is a non-quadratic residue* (mod N).

For $\quad x^N \equiv \bar{x} \equiv S-V \pmod N$,

therefore $\quad S^N+V^N \equiv S-V \pmod N$,

therefore $\quad V^N \equiv -V \pmod N$,

therefore $\quad V[(-M)^{\frac{1}{2}(N-1)}+1] \equiv 0 \pmod N$,

therefore $\quad (-M)^{\frac{1}{2}(N-1)} \equiv -1 \pmod N$,

and the result is proved.

THEOREMS. *If* $(x^N-x)^2 \equiv 0$ *and* $(x^N-\bar{x})^2 \equiv 0 \pmod N$, *then the norm of* V *is zero.*

For $\quad (x^N-x)^2 \equiv (S+V^n-S-V)^2 \equiv (V^n-V)^2 \equiv 0 \pmod N$

requires $\quad M[(-M)^{\frac{1}{2}(N-1)}-1]^2 \equiv 0 \pmod N$,

while $(x^N-\bar{x})^2 \equiv 0$ requires

$$M[(-M)^{\frac{1}{2}(N-1)}+1]^2 \equiv 0 \pmod N.$$

Hence $M \equiv 0 \pmod N$, and the result is proved.

These results enable us to estimate the numbers of quaternions (mod N) which satisfy equations of the various types.

When the modulus is taken to be a real *prime* quaternion the same result is true, N being now the norm of m.

The importance of Fermat's equation in commutative algebra lies in the fact that it is the only equation satisfied by all the numbers in the algebra. In non-commutative systems, however, in addition to the

appropriate generalization of Fermat's equation, there are other identical relations satisfied by all the numbers in the algebra.

2. *Primitive roots**. Since $N^3-N<N^4$, there is no quaternion residue the powers of which represent all the non-zero residues (mod N). The structure of the algebra is shown by the following theorems, in which x will denote a quaternion residue (mod N), y its vector part, and α and β will represent scalars.

THEOREM 6. *If y^2 is not a real quadratic residue* (mod N), *then the quaternion residues $\alpha+\beta y$ form a Galois field of order N^2.*

THEOREM 7. *If $\theta(m)$ is the number of integers less than m and prime to it, the Galois field of Theorem* 6 *has $\theta(N^2-1)$ primitive roots.*

THEOREM 8. *If y^2 is a real quadratic residue* (mod N), *then the quaternion residues $\alpha+\beta y$ satisfy the equation $x^N\equiv x$.*

For $$(\alpha+\beta y)^N\equiv\alpha^N+\beta^N y^N\equiv\alpha+\beta y(y^2)^{\frac{1}{2}(N-1)}\equiv\alpha+\beta y.$$

THEOREM 9. *If $\phi(m)$ denote the number of pairs of integers p, q less than m, such that p, q, and m have no common factor, then the set of residues of Theorem* 8 *contains $\phi(N-1)$ primitive roots of $x^{N-1}\equiv 1$, and $2\theta(N-1)$ primitive roots of $x^N\equiv x$.*

The condition that $\alpha+\beta y$ is a factor of zero is that $\alpha^2\equiv\beta^2y^2$. Then $\alpha-\beta y$ is also a factor of zero. For a given value of α, β may be chosen uniquely, save for sign, so that $\alpha+\beta y$ is a factor of zero. Choose $\alpha=\frac{1}{2}(N+1)$, and let $\xi=\alpha+\beta y$, $\eta=\alpha-\beta y$ be factors of zero.

Then
$$\xi\eta=\alpha^2-\beta^2y^2\equiv 0 \pmod N$$
$$\xi+\eta \equiv 1$$
$$\xi^2\equiv\alpha^2+2\alpha\beta y+\beta^2y^2\equiv 2\alpha(\alpha+\beta y)\equiv 2\alpha\xi\equiv\xi \pmod N$$
$$\eta^2 \equiv \eta.$$

Choose a scalar S, a primitive root of $S^{N-1}\equiv 1$.

Then the set of residues can be expressed in the form
$$S^p\xi,\quad S^q\eta,\quad S^p\xi+S^q\eta$$
$$(0<p\leqslant N-1,\quad 0<q\leqslant N-1).$$

* Added at the suggestion of a referee. The primitive roots mod p, p being a prime quaternion, have been considered by A. R. Richardson, *Proc. London Math. Soc.* (2), 31 (1930), 117–122.

If p is prime to $N-1$, then $S^p\xi$ is a primitive root of $x^N \equiv x$.

If q is prime to $N-1$, then $S^q\xi$ is a primitive root of $x^N \equiv x$.

Also, since $$(S^p\xi+S^q\eta)^r \equiv S^{rp}\xi+S^{rq}\eta,$$

if p, q, and $N-1$ have no common factor, $S^p\xi+S^q\eta$ is a primitive root of $x^{N-1} \equiv 1$. The theorem follows immediately.

THEOREM 10. *If* $y^2 \equiv 0 \pmod N$ *the residues* ay *satisfy* $x^2 \equiv 0 \pmod N$, *and the residues* $\alpha+\beta y$, *where* $\alpha \neq 0$, *satisfy* $x^{N^2-N} \equiv 1$.

For $$(ay)^2 = a^2y^2 \equiv 0 \pmod N.$$

If $x = \alpha+\beta y$, then $x^r \equiv \alpha^r + r\alpha^{N-1}\beta y$, and if r is a multiple of N, $x^r \equiv \alpha^r$.

Thus $$x^{N^2-N} \equiv 1 \pmod N.$$

THEOREM 11. *The set of residues of Theorem* 10 *contains* $N-1$ *residues satisfying* $x^2 \equiv 0$, *and* $\theta(N^2-N)$ *primitive roots of* $x^{N^2-N} \equiv 1 \pmod N$.

There are $N-1$ non-zero residues of the form ay.

There remain N^2-N residues of the form $\alpha+\beta y$, satisfying $x^{N^2-N} \equiv 1$.

If a is a primitive root of $a^{N-1} \equiv 1$, and p is not zero, then the index of the lowest power of x congruent to unity must be divisible by N, otherwise the vector part would be non-zero, and must be divisible by $N-1$, otherwise the scalar part would not be congruent to unity. Hence x is a primitive root of $x^{N^2-N} \equiv 1 \pmod N$.

If there is a primitive root, there will be $\theta(N^2-N)$ primitive roots, namely, those powers of the primitive root with index prime to N^2-N.

To estimate the total number of primitive roots of $x^{N^2-1} \equiv 1$, $x^{N^2-N} \equiv 1$, $x^{N-1} \equiv 1$, $x^N \equiv x$, and $x^2 \equiv 0 \pmod N$, we have only to find the number of vector residues whose squares are real quadratic residues, non-quadratic residues, or congruent to zero (mod N).

There are $(N+1)(N^2-1)$ factors of zero*. Of these N^2-1 are vectors, and will satisfy $x^2 \equiv 0$.

There remain $N(N^2-1)$ factors of zero of the form $\alpha+y$, where α is a scalar and y a vector.

For every vector whose square is a quadratic residue, there correspond two factors of zero, obtained by adding scalars. There are thus $\frac{1}{2}(N^3-N)$

* D. E. Littlewood, *Proc. London Math. Soc.* (2), 32 (1931), 115–128.

vector residues whose squares are real quadratic residues, or $\frac{1}{2}(N^2+N)$ independent ones, the others being multiples of these. There are $N+1$ independent vectors with zero norm (mod N). The number of independent vector residues whose squares are non-quadratic residues is therefore

$$N^2+N+1-(N+1)-\tfrac{1}{2}(N^2+N) = \tfrac{1}{2}(N^2-N).$$

THEOREM 12. *Amongst the quaternion residues with a prime modulus N there are $\frac{1}{2}(N^2-N)\,\theta(N^2-1)$ primitive roots of $x^{N^2-1} \equiv 1$; $\frac{1}{2}(N^2+N)\,\phi(N-1)$ primitive roots of $x^{N-1} \equiv 1$; $(N^2+N)\,\theta(N-1)$ primitive roots of $x^N \equiv x$; $(N+1)\,\theta(N^2-1)$ primitive roots of $x^{N^2-N} \equiv 1$; and N^2-1 residues satisfying $x^2 \equiv 0$.*

A CHARACTERISTIC FUNCTION IN RIEMANNIAN SPACE AND ITS APPLICATION TO THE SOLUTION OF GEODESIC TRIANGLES

By J. L. SYNGE.

[Received 10 January, 1930.—Read 16 January, 1930.]

1. *Introduction.*

In a Riemannian space V_N the distance from a point $P'(x'^i)$ to a point $P(x^i)$, measured along the geodesic joining these points, is a function of the $2N$ quantities x'^i, x^i. This function may be regarded as a characteristic function of V_N, since from it the geometrical properties of V_N could be deduced. It is the direct analogue to the characteristic function $\int v ds$ employed by Hamilton in geometrical optics*. We shall show how a modification of this characteristic function may be usefully employed in Riemannian space, particularly in connection with the solution of geodesic triangles.

The direct analogue to Hamilton's characteristic function, namely the geodesic distance mentioned above, does not prove to be satisfactory for our purposes. For if we denote this function by V, it is easily seen that

$$\lim_{P \to P'} \frac{\partial V}{\partial x^i}$$

depends on the path by which P approaches P'; the second derivatives become infinite in a similar limit, as can be seen by considering a flat space.

We shall avoid this inconvenience by taking as our characteristic function *half the square of the geodesic distance from P' to P*†.

Although approached from a different point of view, some of the formal results of this paper are related to results given by Veblen and

* *Trans. Royal Irish Acad.*, 17 (1837), 1–144 ; *Mathematical Papers*, I, 1618.

† [*Added in proof, February* 28, 1931.] I have been anticipated in regarding this function as fundamental by H. S. Ruse in two recent papers, these *Proceedings*, 31 (1930), 225–230, and 32 (1931), 87–92.

Thomas*. A fundamental difference of method consists in the fact that normal coordinates are not employed in the present paper, every expression being a tensor expressed in terms of a general coordinate system.

One of the advantages of employing the squared geodesic distance is that, even though the line-element is indefinite, this squared distance is always real. Up to the end of § 4, the results are valid for an indefinite line-element, but, with the introduction of angles in § 5, it is assumed that the line-element is positive definite.

2. *The characteristic function.*

Let there be a space V_N with a line-element

$$ds^2 = g_{ij}\,dx^i\,dx^j. \tag{1}$$

Through two assigned points $P'(x'^i)$, $P(x^i)$ there will in general pass a unique shortest geodesic, provided that the points are sufficiently close to one another. Let s denote the length of this geodesic. We define the *characteristic function of* V_N to be†

$$F(x',\, x) = \tfrac{1}{2}s^2. \tag{2}$$

Holding P' fixed, let us displace P to $x^i + dx^i$; by a well known result we have

$$dF = s\,ds = s\lambda_i\,dx^i, \tag{3}$$

where λ^i is the unit tangent vector at P to the geodesic $P'P$. Thus‡

$$F_i \equiv \frac{\partial F}{\partial x^i} = s\lambda_i, \quad F^i = s\lambda^i, \tag{4}$$

so that F^i is a vector field, defined when the point P' is given§. Taking

---

* O. Veblen and T. Y. Thomas, "The geometry of paths", *Trans. American Math. Soc.*, 25 (1923), 551–608. *Cf.* Veblen, *Invariants of quadratic differential forms* (Cambridge, 1927) ch. 6.

† F is analytic in the neighbourhood of an ordinary point of V_N: *cf.* Darboux, *Théorie générale des surfaces*, 3e Partie, 163.

‡ We shall denote covariant derivatives by subscripts, without commas or other special marks.

§ I we wished to develop results for the geometry of paths, we would adopt as definition of this vector field F^i, with respect to the base point P', the equation

$$F^i = t\,\frac{dx^i}{dt},$$

where t is one of the fundamental parameters for which the equations of the paths are

$$\frac{d^2x^i}{dt} + \Gamma^i_{jk}\,\frac{dx^j}{dt}\,\frac{dx^k}{dt} = 0,$$

and $t = 0$ at P' (*cf.* Veblen and Thomas, *loc. cit.*, 556). We would then be led to (6).

contravariant (or absolute) derivatives along the geodesic, we obtain

$$F^i_{\cdot j}\lambda^j = \lambda^i, \tag{5}$$

or, multiplying by s,

$$\boxed{F^i_{\cdot j}F^j = F^i.} \tag{6}$$

We may note that, since F_i is a gradient,

$$F_{ij} = F_{ji}, \tag{7}$$

and also that, by (4),

$$2F = s^2 = F_i F^i. \tag{8}$$

3. *General equations satisfied by the covariant derivatives of the characteristic function.*

Repeatedly differentiating (6) covariantly, we obtain

$$F_{i_1 j i_2}F^j + F_{i_1 j}F^j_{\cdot i_2} = F_{i_1 i_2}, \tag{9}$$

$$F_{i_1 j i_2 i_3}F^j + F_{i_1 j i_2}F^j_{\cdot i_3} + F_{i_1 j i_3}F^j_{\cdot i_2} + F_{i_1 j}F^j_{\cdot i_2 i_3} = F_{i_1 i_2 i_3}, \tag{10}$$

$$\begin{aligned} F_{i_1 j i_2 i_3 i_4}F^j + F_{i_1 j i_2 i_3}F^j_{\cdot i_4} + F_{i_1 j i_2 i_4}F^j_{\cdot i_3} + F_{i_1 j i_3 i_4}F^j_{\cdot i_2} \\ + F_{i_1 j i_2}F^j_{\cdot i_3 i_4} + F_{i_1 j i_3}F^j_{\cdot i_2 i_4} + F_{i_1 j i_4}F^j_{\cdot i_2 i_3} \\ + F_{i_1 j}F^j_{\cdot i_2 i_3 i_4} = F_{i_1 i_2 i_3 i_4}, \end{aligned} \tag{11}$$

and generally

$$\Sigma F_{i_1 j a_2 \dots a_k}F^j_{\cdot a_{k+1} \dots a_l} = F_{i_1 i_2 \dots i_l}, \tag{12}$$

where the summation extends over all expressions for which $a_2 \dots a_k$ is a set selected in order from $i_2 \dots i_l$, and $a_{k+1} \dots a_l$ is the remainder of the set $i_2 \dots i_l$ in order, and k takes all values from 1 to l.

We have also the fundamental equation*

$$F_{i_1 i_2 i_3} - F_{i_1 i_3 i_2} = -R_{i_1 j i_2 i_3}F^j, \tag{13}$$

where $R_{i_1 i_2 i_3 i_4}$ is the curvature tensor of V_N; hence, by covariant differentiation,

$$F_{i_1 i_2 i_3 i_4} - F_{i_1 i_3 i_2 i_4} = -R_{i_1 j i_2 i_3 i_4}F^j - R_{i_1 j i_2 i_3}F^j_{\cdot i_4}, \tag{14}$$

and generally

$$F_{i_1 i_2 i_3 i_4 \dots i_l} - F_{i_1 i_3 i_2 i_4 \dots i_l} = -\Sigma R_{i_1 j i_2 i_3 b_4 \dots b_k}F^j_{\cdot b_{k+1} \dots b_l}, \tag{15}$$

* Eisenhart, *Riemannian geometry*, 30. The sign of the curvature tensor employed in the present paper is that of Eisenhart.

where the summation extends over all expressions for which $b_4 \dots b_k$ is a set selected in order from $i_4 \dots i_l$, and $b_{k+1} \dots b_l$ is the remainder of the set $i_4 \dots i_l$ in order, and k takes all values from 3 to l.

4. *Limiting values of the covariant derivatives of the characteristic function when the point P moves into coincidence with P'.*

Let us adopt the notation

$$X' = \lim_{P \to P'} X(x', x), \tag{16}$$

where X is any function of the coordinates of P' and P. We shall also mark with an accent functions of position at P'.

We have at once, from (2) and (4),

$$\boxed{F' = 0, \quad F'_i = 0.} \tag{17}$$

Also (5) may be written

$$(F^i_{.j} - \delta^i_j)\lambda^j = 0, \tag{18}$$

and, since λ^i is an arbitrary unit vector at P', we deduce that*

$$\boxed{F'^i_{.j} = \delta^i_j, \quad F'_{ij} = g'_{ij}.} \tag{19}$$

From (10) and (13) we have

$$F'_{i_1 i_3 i_2} + F'_{i_1 i_2 i_3} = 0 = F'_{i_1 i_2 i_3} - F'_{i_1 i_3 i_2}, \tag{20}$$

and hence† $$\boxed{F'_{i_1 i_2 i_3} = 0.} \tag{21}$$

For the next order we have, by (11) and (14),

$$F'_{i_1 i_4 i_2 i_3} + F'_{i_1 i_3 i_2 i_4} + F'_{i_1 i_2 i_3 i_4} = 0, \tag{22}$$

$$F'_{i_1 i_2 i_3 i_4} - F'_{i_1 i_3 i_2 i_4} = -R'_{i_1 i_4 i_2 i_3}. \tag{23}$$

By (23) we convert (22) into

$$F'_{i_1 i_2 i_4 i_3} + 2F'_{i_1 i_2 i_3 i_4} + R'_{i_1 i_3 i_2 i_4} + R'_{i_1 i_4 i_2 i_3} = 0\,; \tag{24}$$

* It is perhaps well to point out that F'_{ij} is not the covariant derivative of F'_i with respect to x'^j; covariant differentiation of F' has been carried out with respect to x^j, and these variables have then been replaced by x'^j.

† This was to be expected, since Riemannian space has no intrinsic tensor of the third order.

interchanging 3 and 4, and subtracting, we get

$$F'_{i_1 i_2 i_3 i_4} = F'_{i_1 i_2 i_4 i_3}, \tag{25}$$

and thus (24) gives*

$$\boxed{F'_{i_1 i_2 i_3 i_4} = -\tfrac{1}{3}(R'_{i_1 i_3 i_2 i_4} + R'_{i_1 i_4 i_2 i_3}),} \tag{26}$$

or

$$F'_{i_1 i_2 i_3 i_4} = -\tfrac{1}{3} \underset{3,4}{P} R'_{i_1 i_3 i_2 i_4}, \tag{26 a}$$

where $\underset{3,4}{P}$ denotes the sum of expressions obtained by permutation of 3, 4.

For the fifth order we have, by (12) and (15),

$$F'_{i_1 i_5 i_2 i_3 i_4} + F'_{i_1 i_4 i_2 i_3 i_5} + F'_{i_1 i_3 i_2 i_4 i_5} + F'_{i_1 i_2 i_3 i_4 i_5} = 0, \tag{27}$$

$$F'_{i_1 i_2 i_3 i_4 i_5} - F'_{i_1 i_3 i_2 i_4 i_5} = -R'_{i_1 i_4 i_2 i_3 i_5} - R'_{i_1 i_3 i_2 i_3 i_4}. \tag{28}$$

By (28) we convert (27) into

$$\begin{aligned} F'_{i_1 i_2 i_3 i_3 i_4} + F'_{i_1 i_2 i_4 i_3 i_5} + 2F'_{i_1 i_2 i_3 i_4 i_5} + R'_{i_1 i_4 i_2 i_3 i_5} + R'_{i_1 i_3 i_2 i_3 i_4} \\ + R'_{i_1 i_3 i_2 i_4 i_5} + R'_{i_1 i_3 i_2 i_4 i_3} \\ + R'_{i_1 i_3 i_2 i_3 i_4} + R'_{i_1 i_4 i_2 i_5 i_3} = 0. \end{aligned} \tag{29}$$

Interchanging 4 and 5, and subtracting, we get

$$F'_{i_1 i_2 i_3 i_4 i_5} = F'_{i_1 i_2 i_3 i_5 i_4}; \tag{30}$$

interchanging 3 and 4 in (29), subtracting, and using (30), we get

$$F'_{i_1 i_2 i_3 i_4 i_5} = F'_{i_1 i_2 i_4 i_3 i_5}; \tag{31}$$

thus (29) becomes

$$\boxed{\begin{aligned} F'_{i_1 i_2 i_3 i_4 i_5} = -\tfrac{1}{4}(R'_{i_1 i_3 i_2 i_4 i_5} + R'_{i_1 i_3 i_2 i_5 i_4} \\ + R'_{i_1 i_4 i_2 i_3 i_5} + R'_{i_1 i_4 i_2 i_5 i_3} \\ + R'_{i_1 i_5 i_2 i_3 i_4} + R'_{i_1 i_5 i_2 i_4 i_3}), \end{aligned}} \tag{32}$$

or

$$F'_{i_1 i_2 i_3 i_4 i_5} = -\tfrac{1}{4} \underset{3,4,5}{P} R'_{i_1 i_3 i_2 i_4 i_5}, \tag{32 a}$$

---

* Thus F'_{ijkl} is equal to $g_{ij,kl}$, "the second extension" of the fundamental tensor. *Cf.* Veblen, *Invariants of quadratic differential forms*, 97: he uses a curvature tensor (*loc. cit.*, 41) with a sign opposite to that used in the present paper.

where $P_{3,4,5}$ denotes the sum of expressions obtained by permutation of 3, 4, 5.

For the sixth order we have, by (12),

$$F'_{i_1 i_6 i_2 i_3 i_4 i_5}+F'_{i_1 i_5 i_2 i_3 i_4 i_6}+F'_{i_1 i_4 i_2 i_3 i_5 i_6}+F'_{i_1 i_3 i_2 i_4 i_5 i_6}+F'_{i_1 i_2 i_3 i_4 i_5 i_6}$$
$$+H'_{i_1 i_2 i_3 i_4 i_5 i_6}=0, \quad (33)$$

where

$$H'_{i_1 i_2 i_3 i_4 i_5 i_6}=F'_{i_1 j i_2 i_3}F'^{j}_{.i_4 i_5 i_6}+F'_{i_1 j i_2 i_4}F'^{j}_{.i_3 i_5 i_6}+F'_{i_1 j i_2 i_5}F'^{j}_{.i_3 i_4 i_6}+F'_{i_1 j i_2 i_6}F'^{j}_{.i_3 i_4 i_5}$$
$$+F'_{i_1 j i_3 i_4}F'^{j}_{.i_2 i_5 i_6}+F'_{i_1 j i_3 i_5}F'^{j}_{.i_2 i_4 i_6}+F'_{i_1 j i_3 i_6}F'^{j}_{.i_2 i_4 i_5}$$
$$+F'_{i_1 j i_4 i_5}F'^{j}_{.i_2 i_3 i_6}+F'_{i_1 j i_4 i_6}F'^{j}_{.i_2 i_3 i_5}$$
$$+F'_{i_1 j i_5 i_6}F'^{j}_{.i_2 i_3 i_4}. \quad (34)$$

The value of this tensor is known by (26). From (15) we have

$$F'_{i_1 i_2 i_3 i_4 i_5 i_6}-F'_{i_1 i_3 i_2 i_4 i_5 i_6}$$
$$=-R'_{i_1 i_4 i_2 i_3 i_5 i_6}-R'_{i_1 i_5 i_2 i_3 i_4 i_6}-R'_{i_1 i_6 i_2 i_3 i_4 i_5}-R'_{i_1 j i_2 i_3}F'^{j}_{.i_4 i_5 i_6}. \quad (35)$$

By means of this equation we convert (33) into

$$F'_{i_1 i_2 i_6 i_3 i_4 i_5}+F'_{i_1 i_2 i_5 i_3 i_4 i_6}+F'_{i_1 i_2 i_4 i_3 i_5 i_6}+2F'_{i_1 i_2 i_3 i_4 i_5 i_6}+H'_{i_1 i_2 i_3 i_4 i_5 i_6}$$
$$+R'_{i_1 i_4 i_2 i_3 i_5 i_6}+R'_{i_1 i_5 i_2 i_3 i_4 i_6}+R'_{i_1 i_6 i_2 i_3 i_4 i_5}+R'_{i_1 j i_2 i_3}F'^{j}_{.i_4 i_5 i_6}$$
$$+R'_{i_1 i_3 i_2 i_4 i_5 i_6}+R'_{i_1 i_5 i_2 i_4 i_3 i_6}+R'_{i_1 i_6 i_2 i_4 i_3 i_5}+R'_{i_1 j i_2 i_4}F'^{j}_{.i_3 i_5 i_6}$$
$$+R'_{i_1 i_3 i_2 i_5 i_4 i_6}+R'_{i_1 i_4 i_2 i_5 i_3 i_6}+R'_{i_1 i_6 i_2 i_5 i_3 i_4}+R'_{i_1 j i_2 i_5}F'^{j}_{.i_3 i_4 i_6}$$
$$+R'_{i_1 i_3 i_2 i_6 i_4 i_5}+R'_{i_1 i_4 i_2 i_6 i_3 i_5}+R'_{i_1 i_5 i_2 i_6 i_3 i_4}+R'_{i_1 j i_2 i_6}F'^{j}_{.i_3 i_4 i_5}=0. \quad (36)$$

If we employ the symbol I to denote the operation of interchanging numerals, so that $I(3, 4)$, for example, denotes an interchange of 3 and 4, (36) may be written

$$[2+I(3,4)+I(4,5)\,I(3,4)+I(5,6)\,I(4,5)\,I(3,4)]\,F'_{i_1 i_2 i_3 i_4 i_5 i_6}$$
$$+[1+I(3,4)+I(4,5)\,I(3,4)+I(5,6)\,I(4,5)\,I(3,4)]\,L'_{i_1 i_2 i_3 i_4 i_5 i_6}$$
$$+H'_{i_1 i_2 i_3 i_4 i_5 i_6}=0, \quad (36\,a)$$

where $L'_{i_1 i_2 i_3 i_4 i_5 i_6}$ denotes the second line of (36).

Operating with $[1-I(5, 6)]$ on (36), and using (25), we get

$$[2+I(3, 4)][1-I(5, 6)]\,F'_{i_1 i_2 i_3 i_4 i_5 i_6} + [1+I(3, 4)][1-I(5, 6)]\,R'_{i_1 i_3 i_2 i_4 i_5 i_6} = 0. \quad (37)$$

Since
$$I(3, 4)\,I(3, 4) = 1, \quad (38)$$

we have

$$\left.\begin{aligned} [2-I(3, 4)][2+I(3, 4)] &= 3, \\ [2-I(3, 4)][1+I(3, 4)] &= 1+I(3, 4), \end{aligned}\right\} \quad (39)$$

and therefore operation on (37) with $[2-I(3, 4)]$ gives

$$[1-I(5, 6)]\,F'_{i_1 i_2 i_3 i_4 i_5 i_6} = -\tfrac{1}{3}[1+I(3, 4)][1-I(5, 6)]\,R'_{i_1 i_3 i_2 i_4 i_5 i_6}. \quad (40)$$

Operation on (36) with $[1-I(4, 5)]$ gives

$$[1-I(4, 5)]\,F'_{i_1 i_2 i_6 i_3 i_4 i_5} + 2[1-I(4, 5)]\,F'_{i_1 i_2 i_3 i_4 i_5 i_6} + [1-I(4, 5)]\,H'_{i_1 i_2 i_3 i_4 i_5 i_6}$$
$$+[1+I(3, 6)][1-I(4, 5)]\,R'_{i_1 i_3 i_2 i_6 i_4 i_5} + [1-I(4, 5)]\,R'_{i_1 j i_2 i_3}\,F'^{j}_{\cdot i_4 i_5 i_6} = 0, \quad (41)$$

in which the first term may be evaluated by applying to (40) the substitution

$$1\ 2\ 3\ 4\ 5\ 6$$
$$1\ 2\ 6\ 3\ 4\ 5,$$

while the term in H' is, by (34),

$$[1-I(4, 5)]\,F'_{i_1 j i_2 i_3}\,F'^{j}_{\cdot i_4 i_5 i_6}.$$

Thus (41) may be written

$$[1-I(4, 5)]\,F'_{i_1 i_2 i_3 i_4 i_5 i_6} = -\tfrac{1}{3}[1+I(3, 6)][1-I(4, 5)]\,R'_{i_1 i_3 i_2 i_6 i_4 i_5} - \tfrac{1}{2}[1-I(4, 5)]\,(R'_{i_1 j i_2 i_3} + F'_{i_1 j i_2 i_3})\,F'^{j}_{\cdot i_4 i_5 i_6}. \quad (42)$$

Operation on (36) with $[1-I(3, 4)]$ gives

$$[1+I(5, 6)][1-I(3, 4)]\,F'_{i_1 i_2 i_5 i_3 i_4 i_6} + [1-I(3, 4)]\,F'_{i_1 i_2 i_3 i_4 i_5 i_6}$$
$$+[1-I(3, 4)]\,H'_{i_1 i_2 i_3 i_4 i_5 i_6}$$
$$+[1+I(5, 6)][1-I(3, 4)]\,R'_{i_1 i_5 i_2 i_6 i_3 i_4}$$
$$+[1+I(5, 6)][1-I(3, 4)]\,R'_{i_1 j i_2 i_5}\,F'^{j}_{\cdot i_3 i_4 i_6} = 0. \quad (43)$$

The first term may be evaluated by applying to (42) the substitution

$$1\ 2\ 3\ 4\ 5\ 6$$
$$1\ 2\ 5\ 3\ 4\ 6$$

and it becomes

$$-\tfrac{1}{3}[1+I(5,6)][1+I(5,6)][1-I(3,4)]R'_{i_1i_5i_2i_6i_3i_4}$$
$$-\tfrac{1}{2}[1+I(5,6)][1-I(3,4)](R'_{i_1ji_2i_5}+F'_{i_1ji_2i_5})F'^{ij}_{\cdot i_3i_4i_6}, \quad (44)$$

in which the first term can be simplified, since

$$[1+I(5,6)][1+I(5,6)] = 2[1+I(5,6)]. \quad (45)$$

The H' term in (43) is, by (34),

$$[1+I(5,6)][1-I(3,4)]F'_{i_1ji_2i_5}F'^{ij}_{\cdot i_3i_4i_6}. \quad (46)$$

Thus (43) may be written

$$[1-I(3,4)]F'_{i_1i_2i_3i_4i_5i_6} = -\tfrac{1}{3}[1+I(5,6)][1-I(3,4)]R'_{i_1i_5i_2i_6i_3i_4}$$
$$-\tfrac{1}{2}[1+I(5,6)][1-I(3,4)](R'_{i_1ji_2i_5}+F'_{i_1ji_2i_5})F'^{ij}_{\cdot i_3i_4i_6}. \quad (47)$$

We are now to apply (40), (42), (47) to the solution of (36) or (36a). If we represent $i_1i_2i_3i_4i_5i_6$ by ..., the first term in (36a) is

$$[2+\{1+I(4,5)+I(5,6)I(4,5)\}I(3,4)]F'\ldots$$
$$= [3+\{1+I(5,6)\}I(4,5)]F'\ldots$$
$$-[1+I(4,5)+I(5,6)I(4,5)][1-I(3,4)]F'\ldots; \quad (48)$$

but

$$[3+\{1+I(5,6)\}I(4,5)]F'\ldots$$
$$= [4+I(5,6)]F'\ldots-[1+I(5,6)][1-I(4,5)]F'\ldots,$$

and $$[4+I(5,6)]F'\ldots = 5F'\ldots-[1-I(5,6)]F'\ldots.$$

Adding these three equations, and using (40), (42), (47), we find that the first term of (36a) is

$$5F'_{i_1i_2i_3i_4i_5i_6}+\tfrac{1}{3}[1+I(3,4)][1-I(5,6)]R'_{i_1i_3i_2i_4i_5i_6}$$
$$+\tfrac{1}{3}[1+I(5,6)][1+I(3,6)][1-I(4,5)]R'_{i_1i_3i_2i_6i_4i_5}$$
$$+\tfrac{1}{2}[1+I(5,6)][1-I(4,5)](R'_{i_1ji_2i_3}+F'_{i_1ji_2i_3})F'^{ij}_{\cdot i_4i_5i_6}$$
$$+\tfrac{1}{3}[1+I(4,5)+I(5,6)I(4,5)][1+I(5,6)][1-I(3,4)]R'_{i_1i_5i_2i_6i_3i_4}$$
$$+\tfrac{1}{2}[1+I(4,5)+I(5,6)I(4,5)][1+I(5,6)][1-I(3,4)]$$
$$(R'_{i_1ji_2i_5}+F'_{i_1ji_2i_5})F'^{ij}_{\cdot i_3i_4i_6}. \quad (49)$$

Substituting this expression for the first term in (36a), we get

$$
\begin{aligned}
5F'_{i_1 i_2 i_3 i_4 i_5 i_6} = &-R'_{i_1 i_4 i_2 i_3 i_5 i_6} - R'_{i_1 i_5 i_2 i_3 i_4 i_6} - R'_{i_1 i_6 i_2 i_3 i_4 i_5} - R'_{i_1 j i_2 i_3} F'^{j}_{\cdot i_4 i_5 i_6} \\
&- R'_{i_1 i_3 i_2 i_4 i_5 i_6} - R'_{i_1 i_5 i_2 i_4 i_3 i_6} - R'_{i_1 i_6 i_2 i_4 i_3 i_5} - R'_{i_1 j i_2 i_4} F'^{j}_{\cdot i_3 i_5 i_6} \\
&- R'_{i_1 i_3 i_2 i_5 i_4 i_6} - R'_{i_1 i_4 i_2 i_5 i_3 i_6} - R'_{i_1 i_6 i_2 i_5 i_3 i_4} - R'_{i_1 j i_2 i_5} F'^{j}_{\cdot i_3 i_4 i_6} \\
&- R'_{i_1 i_3 i_2 i_6 i_4 i_5} - R'_{i_1 i_4 i_2 i_6 i_3 i_5} - R'_{i_1 i_5 i_2 i_6 i_3 i_4} - R'_{i_1 j i_2 i_6} F'^{j}_{\cdot i_3 i_4 i_5} \\
&- \tfrac{1}{3}(R'_{i_1 i_3 i_2 i_4 i_5 i_6} - R'_{i_1 i_3 i_2 i_4 i_6 i_5} + R'_{i_1 i_3 i_2 i_6 i_4 i_5} - R'_{i_1 i_3 i_2 i_6 i_5 i_4} \\
&\qquad + R'_{i_1 i_4 i_2 i_3 i_5 i_6} - R'_{i_1 i_4 i_2 i_3 i_6 i_5} + R'_{i_1 i_6 i_2 i_3 i_4 i_5} - R'_{i_1 i_6 i_2 i_3 i_5 i_4} \\
&\qquad + R'_{i_1 i_3 i_2 i_5 i_4 i_6} - R'_{i_1 i_3 i_2 i_5 i_6 i_4} + R'_{i_1 i_5 i_2 i_6 i_3 i_4} - R'_{i_1 i_5 i_2 i_6 i_4 i_3} \\
&\qquad + R'_{i_1 i_5 i_2 i_3 i_4 i_6} - R'_{i_1 i_5 i_2 i_3 i_6 i_4} + R'_{i_1 i_6 i_2 i_5 i_3 i_4} - R'_{i_1 i_6 i_2 i_5 i_4 i_3} \\
&\qquad + R'_{i_1 i_4 i_2 i_6 i_3 i_5} - R'_{i_1 i_4 i_2 i_6 i_5 i_3} + R'_{i_1 i_4 i_2 i_5 i_3 i_6} - R'_{i_1 i_4 i_2 i_5 i_6 i_3} \\
&\qquad + R'_{i_1 i_6 i_2 i_4 i_3 i_5} - R'_{i_1 i_6 i_2 i_4 i_5 i_3} + R'_{i_1 i_5 i_2 i_4 i_3 i_6} - R'_{i_1 i_5 i_2 i_4 i_6 i_3}) \\
&- \tfrac{1}{2}(R'_{i_1 j i_2 i_3} + F'_{i_1 j i_2 i_3})(F'^{j}_{\cdot i_4 i_5 i_6} - F'^{j}_{\cdot i_5 i_4 i_6} + F'^{j}_{\cdot i_4 i_6 i_5} - F'^{j}_{\cdot i_6 i_4 i_5}) \\
&- \tfrac{1}{2}(R'_{i_1 j i_2 i_4} + F'_{i_1 j i_2 i_4})(F'^{j}_{\cdot i_3 i_5 i_6} - F'^{j}_{\cdot i_5 i_3 i_6} + F'^{j}_{\cdot i_3 i_6 i_5} - F'^{j}_{\cdot i_3 i_5 i_6}) \\
&- \tfrac{1}{2}(R'_{i_1 j i_2 i_5} + F'_{i_1 j i_2 i_5})(F'^{j}_{\cdot i_3 i_4 i_6} - F'^{j}_{\cdot i_4 i_3 i_6} + F'^{j}_{\cdot i_3 i_6 i_4} - F'^{j}_{\cdot i_6 i_3 i_4}) \\
&- \tfrac{1}{2}(R'_{i_1 j i_2 i_6} + F'_{i_1 j i_2 i_6})(F'^{j}_{\cdot i_3 i_4 i_5} - F'^{j}_{\cdot i_4 i_3 i_5} + F'^{j}_{\cdot i_3 i_5 i_4} - F'^{j}_{\cdot i_5 i_3 i_4}) \\
&- H'_{i_1 i_2 i_3 i_4 i_5 i_6}. \qquad (50)
\end{aligned}
$$

The last term in the first line, the terms in the eleventh line, and the first term in $-H'_{i_1 i_2 i_3 i_4 i\ i_6}$ [see (34)], together make up [by use of (25) and (26)]

$$
\begin{aligned}
&[R'_{i_1 j i_2 i_3} + F'_{i_1 j i_2 i_3}]\,[-2F'^{j}_{\cdot i_4 i_5 i_6} + \tfrac{1}{2}(F'^{j}_{\cdot i_5 i_4 i_6} + F'^{j}_{\cdot i_6 i_4 i_5})] \\
&\quad = [-R'_{i_2 i_3 j i_1} - \tfrac{1}{3}(R'_{i_1 i_2 j i_3} + R'_{i_1 i_3 j i_2})]\,[\tfrac{2}{3}(R'^{j}_{\cdot i_5 i_4 i_6} + R'^{j}_{\cdot i_6 i_4 i_5}) - \tfrac{1}{6}(R'^{j}_{\cdot i_6 i_5 i_4} + R'^{j}_{\cdot i_5 i_6 i_4})] \\
&\quad = -\tfrac{5}{6}(R'_{i_2 i_3 j i_1} + \tfrac{1}{3}R'_{i_1 i_2 j i_3} + \tfrac{1}{3}R'_{i_1 i_3 j i_2})(R'^{j}_{\cdot i_5 i_4 i_6} + R'^{j}_{\cdot i_6 i_4 i_5}) \\
&\quad = -\tfrac{5}{6}g'^{jk}(R'_{i_2 i_3 j i_1} + \tfrac{1}{3}R'_{i_1 i_2 j i_3} + \tfrac{1}{3}R'_{i_1 i_3 j i_2})(R'_{i_4 i_5 k i_6} + R'_{i_4 i_6 k i_5}). \qquad (51)
\end{aligned}
$$

But
$$R'_{i_2 i_3 j i_1} + R'_{i_2 j i_1 i_3} + R'_{i_2 i_1 i_3 j} = 0, \qquad (52)$$

and therefore
$$R'_{i_2 i_3 j i_1} = R'_{i_1 i_3 j i_2} - R'_{i_1 i_2 j i_3}, \qquad (53)$$

so that (51) is equal to

$$-\tfrac{5}{9}g^{rjk}(2R'_{i_1i_3ji_2}-R'_{i_1i_2ji_3})(R'_{i_4i_5ki_6}+R'_{i_4i_6ki_5}). \tag{54}$$

The first of the remaining terms in $-H'_{i_1i_2i_3i_4i_5i_6}$ which are not used up in forming (54) and the three expressions similar to it, is [see (34)] equal to

$$-\tfrac{1}{9}(R'_{i_1i_3ji_4}+R'_{i_1i_4ji_3})(R'^{rj}_{\cdot\cdot i_5i_2i_6}+R'^{rj}_{\cdot\cdot i_6i_2i_5})$$
$$=-\tfrac{1}{9}g^{rjk}(R'_{i_1i_3ji_4}+R'_{i_1i_4ji_3})(R'_{i_2i_5ki_6}+R'_{i_2i_6ki_5}). \tag{55}$$

Thus we see that (50) may be written

$$\begin{aligned}
F'_{i_1i_2i_3i_4i_5i_6}=&-\tfrac{4}{15}\left\{\begin{array}{r}R'_{i_1i_3i_2i_4i_5i_6}+R'_{i_1i_3i_2i_5i_4i_6}+R'_{i_1i_3i_2i_6i_4i_5}\\+R'_{i_1i_4i_2i_3i_5i_6}+R'_{i_1i_4i_2i_5i_3i_6}+R'_{i_1i_4i_2i_6i_3i_5}\\+R'_{i_1i_5i_2i_3i_4i_6}+R'_{i_1i_5i_2i_4i_3i_6}+R'_{i_1i_5i_2i_6i_3i_4}\\+R'_{i_1i_6i_2i_3i_4i_5}+R'_{i_1i_6i_2i_4i_3i_5}+R'_{i_1i_6i_2i_5i_3i_4}\end{array}\right\}\\
&+\tfrac{1}{15}\left\{\begin{array}{r}R'_{i_1i_3i_2i_4i_6i_5}+R'_{i_1i_3i_2i_5i_6i_4}+R'_{i_1i_3i_2i_6i_5i_4}\\+R'_{i_1i_4i_2i_3i_6i_5}+R'_{i_1i_4i_2i_5i_6i_3}+R'_{i_1i_4i_2i_6i_5i_3}\\+R'_{i_1i_5i_2i_3i_6i_4}+R'_{i_1i_5i_2i_4i_6i_3}+R'_{i_1i_5i_2i_6i_4i_3}\\+R'_{i_1i_6i_2i_3i_5i_4}+R'_{i_1i_6i_2i_4i_5i_3}+R'_{i_1i_6i_2i_5i_4i_3}\end{array}\right\}\\
&-\tfrac{1}{9}g^{rjk}(2R'_{i_1i_3ji_2}-R'_{i_1i_2ji_3})(R'_{i_4i_5ki_6}+R'_{i_4i_6ki_5})\\
&-\tfrac{1}{9}g^{rjk}(2R'_{i_1i_4ji_2}-R'_{i_1i_2ji_4})(R'_{i_3i_5ki_6}+R'_{i_3i_6ki_5})\\
&-\tfrac{1}{9}g^{rjk}(2R'_{i_1i_5ji_2}-R'_{i_1i_2ji_5})(R'_{i_3i_4ki_6}+R'_{i_3i_6ki_4})\\
&-\tfrac{1}{9}g^{rjk}(2R'_{i_1i_6ji_2}-R'_{i_1i_2ji_6})(R'_{i_3i_4ki_5}+R'_{i_3i_5ki_4})\\
&-\tfrac{1}{45}g^{rjk}(R'_{i_1i_3ji_4}+R'_{i_1i_4ji_3})(R'_{i_2i_5ki_6}+R'_{i_2i_6ki_5})\\
&-\tfrac{1}{45}g^{rjk}(R'_{i_1i_3ji_5}+R'_{i_1i_5ji_3})(R'_{i_2i_4ki_6}+R'_{i_2i_6ki_4})\\
&-\tfrac{1}{45}g^{rjk}(R'_{i_1i_3ji_6}+R'_{i_1i_6ji_3})(R'_{i_2i_4ki_5}+R'_{i_2i_5ki_4})\\
&-\tfrac{1}{45}g^{rjk}(R'_{i_1i_4ji_5}+R'_{i_1i_5ji_4})(R'_{i_2i_3ki_6}+R'_{i_2i_6ki_3})\\
&-\tfrac{1}{45}g^{rjk}(R'_{i_1i_4ji_6}+R'_{i_1i_6ji_4})(R'_{i_2i_3ki_5}+R'_{i_2i_5ki_3})\\
&-\tfrac{1}{45}g^{rjk}(R'_{i_1i_5ji_6}+R'_{i_1i_6ji_5})(R'_{i_2i_3ki_4}+R'_{i_2i_4ki_3}).
\end{aligned} \tag{56}$$

We may observe, as a check, that the above expression is not altered in value by interchange of 1 and 2.

We have thus evaluated the limiting values, as P tends to P', of the covariant derivatives with respect to P, up to the sixth order, of the characteristic function for P', P. The sixth order derivatives are of interest because they involve the contraction between two curvature

tensors of the type

$$g'^{jk} R'_{i_1 i_2 j i_3} R'_{i_4 i_5 k i_6}.$$

On account of the complexity of the calculation, we shall not pursue the evaluation to a higher order; it is clear that the evaluation could be carried out to any order by means of (12) and (15).

5. *Application of the method to the solution of geodesic triangles. Preliminary.*

Let ABC be a triangle in V_N, its sides being geodesics. We suppose this triangle to be finite, but sufficiently small to ensure the validity of the expansion involved. Our object is to show how a, B, C may be

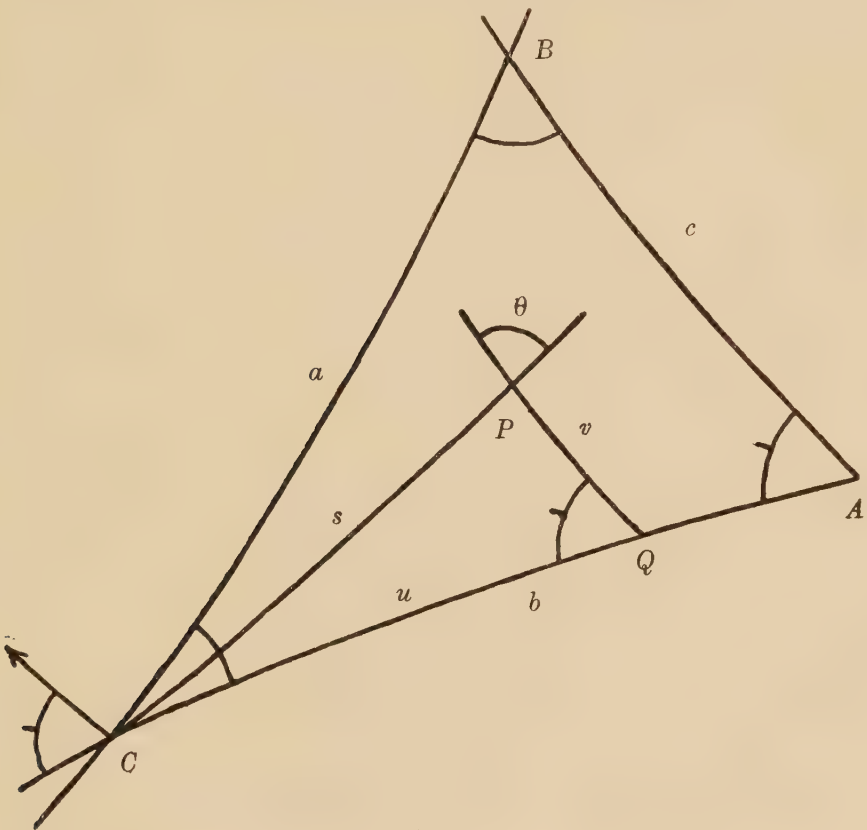


Fig. 1.

expressed as functions of b, c, A, and curvature properties of V_N at A, and to evaluate, in terms of the curvature tensor, the coefficients of the series obtained, as far as they are given by the results of the preceding section.

Let us propagate parallelly along AC the unit tangent vector to AB at A, and let us draw the geodesics to which the vectors thus obtained are tangents. In this way we construct a V_2, containing the points A, B, C, but not in general for $N>2$ the geodesic CB. Let P be any point on V_2, and let Q be that point on AC from which the generating geodesic passes through P; let us assign to P the coordinates

$$u = CQ, \quad v = QP, \tag{57}$$

so that V_2 has the equations $x^i = x^i(u, v)$. The coordinates of A are $u=b$, $v=0$, and those of B are $u=b$, $v=c$.

We note here the following formulae, in which X^i is any vector field defined over V_2:

$$\frac{\delta X^i}{\delta u} = \frac{\partial X^i}{\partial u} + \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\} X^j \frac{\partial x^k}{\partial u}, \quad \frac{\delta X^i}{\delta v} = \frac{\partial X^i}{\partial v} + \left\{ \begin{matrix} i \\ j\ k \end{matrix} \right\} X^j \frac{\partial x^k}{\partial v}, \tag{58}$$

$$\frac{\delta}{\delta v}\frac{\partial x^i}{\partial u} = \frac{\delta}{\delta u}\frac{\partial x^i}{\partial v}. \tag{59}$$

By virtue of the manner in which V_2 has been constructed, we have

$$g_{ij}\frac{\partial x^i}{\partial v}\frac{\partial x^j}{\partial v} = 1, \quad \left(g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial u}\right)_{v=0} = 1, \quad \left(g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}\right)_{v=0} = -\cos A, \tag{60}$$

$$\frac{\delta}{\delta v}\frac{\partial x^i}{\partial v} = 0, \quad \left(\frac{\delta}{\delta u}\frac{\partial x^i}{\partial u}\right)_{v=0} = 0, \quad \left(\frac{\delta}{\delta u}\frac{\partial x^i}{\partial v}\right)_{v=0} = 0. \tag{61}$$

Now it is easily seen, by the first of (61) and the first of (60), that

$$\frac{\partial}{\partial v}\left(g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}\right) = \tfrac{1}{2}\frac{\partial}{\partial u}\left(g_{ij}\frac{\partial x^i}{\partial v}\frac{\partial x^j}{\partial v}\right) = 0; \tag{62}$$

therefore the last of (60) has the more general form

$$g_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v} = -\cos A. \tag{63}$$

Let s be the length of the geodesic CP, and θ the angle at P between CP and QP; these are functions of u, v, such that

$$\left.\begin{aligned} (s)_{u=b,\ v=c} &= a, \quad (s)_{u=b,\ v=0} = b, \\ (\theta)_{u=b,\ v=c} &= B, \quad (\theta)_{u=b,\ v=0} = \pi - A. \end{aligned}\right\} \tag{64}$$

Let us take C as base point x'^i, and let P be x^i; let $F(x', x)$ be the corresponding characteristic function. F may be expressed as a function of

x'^i, u, v, and we have

$$\frac{\partial F}{\partial v} = F_i \frac{\partial x^i}{\partial v} = s\lambda_i \frac{\partial x^i}{\partial v} = s\cos\theta, \tag{65}$$

where λ^i is the unit tangent vector to CP at P. We have, then,

$$\tfrac{1}{2}s^2 = F, \quad s\cos\theta = \frac{\partial F}{\partial v}, \tag{66}$$

and therefore

$$\boxed{\tfrac{1}{2}a^2 = (F)_{u=b,\, v=c}, \quad a\cos B = \left(\frac{\partial F}{\partial v}\right)_{u=b,\, v=c}.} \tag{67}$$

The solution of the triangle will be given by expanding F as a power series in u, v.

6. *The expansion of F.*

We have

$$\boxed{F = \sum_{m,\, n=0}^{\infty} \frac{1}{m!\, n!} M'_{m,\, n} u^m v^n,} \tag{68}$$

where (the accent denoting evaluation at C) we write generally

$$M_{m,\, n} = \frac{\partial^{m+n} F}{\partial u^m \partial v^n}. \tag{69}$$

Thus, using (61), we have

$$M_{m,\, n} = \frac{\partial^m}{\partial u^m}\left(F_{i_1 \ldots i_n} \frac{\partial x^{i_1}}{\partial v} \cdots \frac{\partial x^{i_n}}{\partial v}\right), \tag{70}$$

and hence

$$\boxed{M'_{m,\, n} = F'_{i_1 \ldots i_{n+m}} \frac{\partial x^{i_1}}{\partial v} \cdots \frac{\partial x^{i_n}}{\partial v} \frac{\partial x^{i_{n+1}}}{\partial u} \cdots \frac{\partial x^{i_{n+m}}}{\partial u},} \tag{71}$$

in which the partial derivatives are to be evaluated at C. From (69) we have

$$(M_{m,\, 0})_{v=0} = \frac{d^m}{du^m}(\tfrac{1}{2}u^2), \quad (M_{0,\, n})_{u=0} = \frac{d^n}{dv^n}(\tfrac{1}{2}v^2),$$

so that

$$\boxed{\begin{array}{ll} M'_{m,\, 0} = 0 \ \ (m \neq 2); & M'_{0,\, n} = 0 \ \ (n \neq 2); \\ M'_{2,\, 0} = 1; & M'_{0,\, 2} = 1. \end{array}} \tag{72}$$

Also $$(M_{m,1})_{v=0} = \frac{d^m}{du^m}\left(\frac{\partial F}{\partial v}\right)_{v=0} = \frac{d^m}{du^m}(-u\cos A),$$

$$(M_{1,n})_{u=0} = \frac{d^n}{dv^n}\left(\frac{\partial F}{\partial u}\right)_{u=0} = \frac{d^n}{dv^n}\left(v\lambda_i\frac{\partial x^i}{\partial u}\right)_{u=0} = \frac{d^n}{dv^n}\left(vg_{ij}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}\right)_{u=0}$$

$$= \frac{d^n}{dv^n}(-v\cos A),$$

by (63); thus

$$\boxed{\begin{gathered} M'_{m,1} = 0 \quad (m \neq 1); \qquad M'_{1,n} = 0 \quad (n \neq 1); \\ M'_{1,1} = -\cos A. \end{gathered}} \tag{73}$$

By (72) and (73), we see that (68) may be written

$$\boxed{\begin{gathered} F = \tfrac{1}{2}(u^2 - 2uv\cos A + v^2) + u^2v^2\phi, \\ \text{where} \\ \phi = \sum_{m,n=2}^{\infty} \frac{1}{m!\,n!} M'_{m,n} u^{m-2} v^{n-2}. \end{gathered}} \tag{74}$$

So far we have not made use of the results of § 4.

It will be convenient to adopt the following notation:

$$\left.\begin{aligned} S &= R_{i_1 i_2 i_3 i_4}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial v}\frac{\partial x^{i_4}}{\partial u}, \\ S_{(u)} &= R_{i_1 i_2 i_3 i_4 i_5}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial v}\frac{\partial x^{i_4}}{\partial u}\frac{\partial x^{i_5}}{\partial u}, \\ S_{(v)} &= R_{i_1 i_2 i_3 i_4 i_5}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial v}\frac{\partial x^{i_4}}{\partial u}\frac{\partial x^{i_5}}{\partial v}, \\ S_{(u,u)} &= R_{i_1 i_2 i_3 i_4 i_5 i_6}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial v}\frac{\partial x^{i_4}}{\partial u}\frac{\partial x^{i_5}}{\partial u}\frac{\partial x^{i_6}}{\partial u}, \\ S_{(v,v)} &= R_{i_1 i_2 i_3 i_4 i_5 i_6}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial v}\frac{\partial x^{i_4}}{\partial u}\frac{\partial x^{i_5}}{\partial v}\frac{\partial x^{i_6}}{\partial v}, \\ S_{(u,v)} &= R_{i_1 i_2 i_3 i_4 i_5 i_6}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial v}\frac{\partial x^{i_4}}{\partial u}\frac{\partial x^{i_5}}{\partial u}\frac{\partial x^{i_6}}{\partial v}, \\ S_{(v,u)} &= R_{i_1 i_2 i_3 i_4 i_5 i_6}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial v}\frac{\partial x^{i_4}}{\partial u}\frac{\partial x^{i_5}}{\partial v}\frac{\partial x^{i_6}}{\partial u}, \\ U_j &= R_{i_1 i_2 j i_3}\frac{\partial x^{i_1}}{\partial u}\frac{\partial x^{i_2}}{\partial v}\frac{\partial x^{i_3}}{\partial u} = -R_{i_1 i_2 j i_3}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial u}, \\ V &= R_{i_1 i_2 j i_3}\frac{\partial x^{i_1}}{\partial v}\frac{\partial x^{i_2}}{\partial u}\frac{\partial x^{i_3}}{\partial v}. \end{aligned}\right\} \tag{75}$$

By virtue of the identity of Ricci and Levi-Civita* we have

$$R_{i_1 i_2 i_3 i_4 i_5 i_6} - R_{i_1 i_2 i_3 i_4 i_6 i_5} = R^{j}_{.i_1 i_5 i_6} R_{j i_2 i_3 i_4} + R^{j}_{.i_2 i_5 i_6} R_{i_1 j i_3 i_4} + R^{j}_{.i_3 i_5 i_6} R_{i_1 i_2 j i_4} + R^{j}_{.i_4 i_5 i_6} R_{i_1 i_2 i_3 j}, \quad (76)$$

from which we see that

$$S_{(u,\,v)} = S_{(v,\,u)}. \quad (77)$$

We shall mark with an accent the symbols in (75), when calculated at C.

Using (71), (26), (32), (56), we find

$$\boxed{\begin{aligned} &M'_{2,2} = -\tfrac{2}{3} S'; \\ &M'_{3,2} = -\tfrac{3}{2} S'_{(u)}, \quad M'_{2,3} = -\tfrac{1}{2} S'_{(v)}; \\ &M'_{4,2} = -\tfrac{12}{5} S'_{(u,\,u)} - \tfrac{8}{15} U'_j U'^j, \quad M'_{3,3} = -\tfrac{6}{5} S'_{(u,\,v)} + \tfrac{8}{5} U'_j V'^j, \\ &\qquad M'_{2,4} = -\tfrac{2}{5} S'_{(v,\,v)} - \tfrac{8}{15} V'_j V'^j. \end{aligned}} \quad (78)$$

The values are to be substituted in (74).

7. *The solution of the triangle*†.

From (67), (74) we have

$$\boxed{a^2 = b^2 - 2bc\cos A + c^2 + b^2 c^2 \psi,} \quad (79)$$

where

$$\psi = \tfrac{1}{2} M'_{2,2} + \tfrac{1}{6}(M'_{3,2}\, b + M'_{2,3}\, c) + \tfrac{1}{24} M'_{4,2}\, b^2 + \tfrac{1}{18} M'_{3,3}\, bc + \tfrac{1}{24} M'_{2,4}\, c^2 + O_3, \quad (80)$$

where O_3 stands for a quantity of the third order in b, c; we have also

$$\boxed{a\cos B = -b\cos A + c + b^2 c\chi,} \quad (81)$$

where

$$\chi = \psi + \tfrac{1}{2} c \frac{\partial \psi}{\partial c}$$

$$= \tfrac{1}{2} M'_{2,2} + \tfrac{1}{6} M'_{3,2}\, b + \tfrac{1}{4} M'_{2,3}\, c + \tfrac{1}{24} M'_{4,2}\, b^2 + \tfrac{1}{12} M'_{3,3}\, bc + \tfrac{1}{12} M'_{2,4}\, c^2 + O_3. \quad (82)$$

---

* "Méthodes de calcul différentiel absolu", *Math. Annalen*, 54 (1900), 143. *Cf.* Levi-Civita, *Absolute differential calculus*, 186. The identity referred to might have been employed to solve for $F'_{i_1 i_2 i_3 i_4 i_5 i_6}$, but the method adopted in § 4 appears to be as convenient.

† Geodesic triangles for the case $N = 2$ were discussed by Darboux, *Théorie générale des surfaces*, livre 6, ch. 8.

In these formulae the coefficients involve curvature tensors at the point C; these will now be transformed to the point A. Marking with a double accent quantities evaluated at A, we have

$$\left.\begin{aligned} S' &= S'' - bS''_{(u)} + \tfrac{1}{2}b^2 S''_{(u,u)} - \ldots, \\ S'_{(u)} &= S''_{(u)} - bS''_{(u,u)} + \ldots, \\ S'_{(v)} &= S''_{(v)} - bS''_{(u,v)} + \ldots, \end{aligned}\right\} \tag{83}$$

and thus (80) and (82) may be written

$$\boxed{\begin{aligned} \psi &= \tfrac{1}{2}\bar{M}_{2,2} + \tfrac{1}{6}(\bar{M}_{3,2}\,b + \bar{M}_{2,3}\,c) + \tfrac{1}{24}\bar{M}_{4,2}\,b^2 + \tfrac{1}{18}\bar{M}_{3,3}\,bc + \tfrac{1}{24}\bar{M}_{2,4}\,c^2 + O_3, \\ \chi &= \tfrac{1}{2}\bar{M}_{2,2} + \tfrac{1}{6}\bar{M}_{3,2}\,b + \tfrac{1}{4}\bar{M}_{2,3}\,c + \tfrac{1}{24}\bar{M}_{4,2}\,b^2 + \tfrac{1}{12}\bar{M}_{3,3}\,bc + \tfrac{1}{12}\bar{M}_{2,4}\,c^2 + O_3, \end{aligned}} \tag{84}$$

in which

$$\boxed{\begin{aligned} &\bar{M}_{2,2} = -\tfrac{2}{3}S''; \\ &\bar{M}_{3,2} = \tfrac{1}{2}S''_{(u)}, \quad \bar{M}_{2,3} = -\tfrac{1}{2}S''_{(v)}; \\ &\bar{M}_{4,2} = -\tfrac{2}{5}S''_{(u,u)} - \tfrac{8}{15}U''_j U''^j, \quad \bar{M}_{3,3} = \tfrac{3}{10}S''_{(u,v)} + \tfrac{8}{5}U''_j V''^j, \\ &\bar{M}_{2,4} = -\tfrac{2}{5}S''_{(v,v)} - \tfrac{8}{15}V''_j V''^j. \end{aligned}} \tag{85}$$

Equations (79), (81), (84), (85) contain the solution of the triangle, the quantities appearing on the right-hand sides in (85) being calculated by the formulae (75) at the point A.

Let us now make the coefficients (85) explicit in the angle A. Let ξ^i be a unit vector at A, pointing in the reversed direction of the internal bisector of the angle A; let η^i be a unit vector, pointing in the direction of the external bisector of the angle A, to that side on which B lies, the vectors ξ^i, η^i, $\partial x^i/\partial u$, $\partial x^i/\partial v$ lying in an elementary two-space at A. We have

$$\left.\begin{aligned} \frac{\partial x^i}{\partial u} &= \xi^i \cos\tfrac{1}{2}A + \eta^i \sin\tfrac{1}{2}A, \\ \frac{\partial x^i}{\partial v} &= -\xi^i \cos\tfrac{1}{2}A + \eta^i \sin\tfrac{1}{2}A. \end{aligned}\right\} \tag{86}$$

Substituting these values in (75) we obtain

$$S'' = K'' \sin^2 A, \tag{87}$$

where

$$K'' = R_{i_1 i_2 i_3 i_4}\,\xi^{i_1}\eta^{i_2}\xi^{i_3}\eta^{i_4},$$

the Riemannian curvature of V_N corresponding to the element of the triangle at A; also, if we define $S''_{(\xi)}$, $S''_{(\eta)}$, $S''_{(\xi,\xi)}$, $S''_{(\eta,\eta)}$, $S''_{(\xi,\eta)}$, X''_j, Y''_j as the

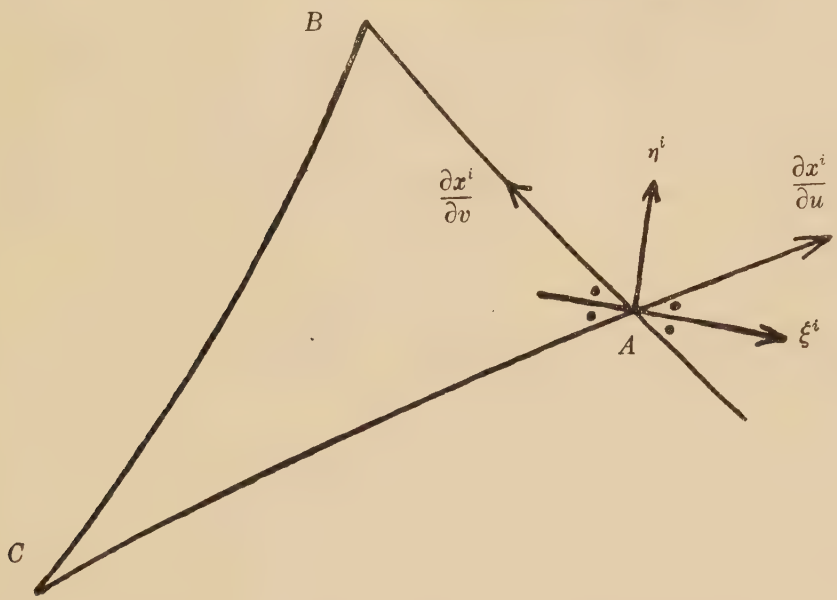


Fig 2.

results of writing ξ, η for $\partial x^i/\partial u$, $\partial x^i/\partial v$ in the formulae (75) for $S_{(u)}$, $S_{(v)}$, $S_{(u,u)}$, $S_{(v,v)}$, $S_{(u,v)}$, U_j, V_j, respectively, evaluated at the point A, we find

$$\left.\begin{aligned}
S''_{(u)} &= \sin^2 A\,(S''_{(\xi)} \cos \tfrac{1}{2}A + S''_{(\eta)} \sin \tfrac{1}{2}A),\\
S''_{(v)} &= \sin^2 A\,(-S''_{(\xi)} \cos \tfrac{1}{2}A + S''_{(\eta)} \sin \tfrac{1}{2}A),\\
S''_{(u,u)} &= \sin^2 A\,(S''_{(\xi,\xi)} \cos^2 \tfrac{1}{2}A + 2S''_{(\xi,\eta)} \cos \tfrac{1}{2}A \sin \tfrac{1}{2}A + S''_{(\eta,\eta)} \sin^2 \tfrac{1}{2}A),\\
S''_{(v,v)} &= \sin^2 A\,(S''_{(\xi,\xi)} \cos^2 \tfrac{1}{2}A - 2S''_{(\xi,\eta)} \cos \tfrac{1}{2}A \sin \tfrac{1}{2}A + S''_{(\eta,\eta)} \sin^2 \tfrac{1}{2}A),\\
S''_{(u,v)} &= \sin^2 A\,(-S''_{(\xi,\xi)} \cos^2 \tfrac{1}{2}A + S''_{(\eta,\eta)} \sin^2 \tfrac{1}{2}A),\\
U''_j &= \sin A\,(X''_j \cos \tfrac{1}{2}A - Y''_j \sin \tfrac{1}{2}A),\\
V''_j &= \sin A\,(X''_j \cos \tfrac{1}{2}A + Y''_j \sin \tfrac{1}{2}A).
\end{aligned}\right\} \quad (88)$$

The quantities $S''_{(\xi)}$, $S''_{(\eta)}$, $S''_{(\xi,\xi)}$, $S''_{(\eta,\eta)}$, $S''_{(\xi,\eta)}$, X''_j, Y''_j are independent of the angle A, depending only on the direction of the bisector of the angle and on the two-element in which the angle lies.

When the above values are substituted in (85), and the results substituted in (84), (79), we get the following explicit expression for the square of the side of a geodesic triangle in terms of the other two sides

and the contained angle :

$$\begin{aligned} a^2 = b^2 - 2bc\cos A + c^2 \\ -\tfrac{1}{3}b^2c^2\sin^2 A\,[K'' - \tfrac{1}{4}(b+c)\,S''_{(\xi)}\cos\tfrac{1}{2}A - \tfrac{1}{4}(b-c)\,S''_{(\eta)}\sin\tfrac{1}{2}A \\ + \tfrac{1}{20}(b^2+bc+c^2)\,S''_{(\xi,\,\xi)}\cos^2\tfrac{1}{2}A \\ + \tfrac{1}{20}(b^2-bc+c^2)\,S''_{(\eta,\,\eta)}\sin^2\tfrac{1}{2}A \\ + \tfrac{1}{10}(b^2-c^2)\,S''_{(\xi,\,\eta)}\cos\tfrac{1}{2}A\sin\tfrac{1}{2}A \\ + \tfrac{1}{15}(b^2-4bc+c^2)\,X''_j X''^j\cos^2\tfrac{1}{2}A \\ + \tfrac{1}{15}(b^2+4bc+c^2)\,Y''_j Y''^j\sin^2\tfrac{1}{2}A \\ - \tfrac{2}{15}(b^2-c^2)\,X''_j Y''^j\cos\tfrac{1}{2}A\sin\tfrac{1}{2}A] + O_7. \end{aligned} \tag{89}$$

This gives a in terms of b, c, A ; for B, we have

$$a\cos B = \frac{\partial}{\partial c}(\tfrac{1}{2}a^2). \tag{90}$$

The angle C will be given from this last result by changing B, b, c into C, c, b, at the same time changing the sign of η^i in the coefficients.

It is interesting to note that (89) contains terms which disappear or are simplified when the manifold is of two dimensions ; for when $N = 2$ it is easy to see that

$$X''_j X''^j = Y''_j Y''^j = (K'')^2, \quad X''_j Y''^j = 0. \tag{91}$$

THE COMPLETE SYSTEM OF THE BINARY (2, 1, 1) FORM

By C. W. GILHAM.

[Received 12 March, 1930.—Read 13 March, 1930.]

1. The complete irreducible system of concomitants is known for each of the double binary (1, 1), (2, 1), and (2, 2) forms, but the only system evaluated for forms in more than two sets of variables is that of the (1, 1, 1) form, obtained by Saddler* and Schwartz† independently. The general method developed by the author‡ is applied in the present communication to the triple binary form which is next in ascending order, *i.e.* the (2, 1, 1) form. A comparison of the complete system for the binary quartic, containing five concomitants, with that for the binary (2, 2) form, containing eighteen members, exhibits the rapidity with which the complexity of the system increases when the number of distinct variables is increased without alteration of the total order, so that it is not surprising that the system here obtained consists of sixty-five forms. It is not proved that every concomitant retained is irreducible, since the method of Gordan and its generalizations will only ensure that all concomitants can be expressed rationally and integrally in terms of those retained.

The Clebsch-Aronhold symbolism is used throughout, and, following the usual practice, only the bracket factors of concomitants are given. The sign of congruence indicates equality, save for reducible forms.

The variables x, y, z are assumed to be independent, and to undergo independent linear transformations, so that the triple binary form gives a relation between three binary fields.

---

* Saddler, **5**.
† Schwartz, **6**.
‡ Gilham, **1**.

2. Let $f = a_x^2 a_y' a_z'' = b_x^2 b_y' b_z'' = \ldots$ be the triple binary (2, 1, 1) form.

Let

$$J_1 = (a'b')(a''b'')\, a_x^2 b_x^2,$$

$$J_2 = (ab)(a''b'')\, a_x b_x a_y' b_y',$$

$$J_3 = (ab)(a'b')\, a_x b_x a_z'' b_z'',$$

and let j_1, j_2, j_3 denote the bracket factors of these covariants.

If $(A)_1$ is the complete system of f considered as a (2, 1) form in x and y, then

$$(A)_1 \equiv c \pmod{j_1, j_2}.$$

If $(B)_1$ is the relatively complete system of J_2 for the modulus j_1, and if $(B)_2$ is the complete system of J_1, then

$$(A)_2 \equiv [(A)_1, (B)_1] \equiv c \pmod{j_1},$$

$$(A)_3 \equiv [(A)_2, (B)_2] \equiv c \pmod{\Delta_2},$$

where Δ_2 is the invariant $(ab)^2(a'b')(a''b'')$*. Since $(A)_3$ contains f the complete system required will consist of $(A)_3$ and Δ_2.

In constructing $(B)_1$ and $(B)_2$, single terms may be taken from the copied transvectants [which are formed on the model of the complete systems for the binary quartic and (2, 2) forms respectively], and, furthermore, we may reject from $(B)_1$ any transvectant which, by transformation, can be made to contain the factor j_1.

3. *The system* $(A)_1$.

The complete system of a (2 . 1) form ϕ consists of†

$$\phi, \quad (\phi, \phi)^{2.0} = H, \quad (\phi, \phi)^{1.1} = J, \quad (H, \phi)^{0.1} = T, \quad (T, \phi)^{2.1} = Q.$$

Taking single terms from the copied transvectants, we obtain

$$(A)_1 \equiv f, P_2, J_3, T_1, Q_1,$$

where

$$P_2 = (ab)^2, \quad T_1 = (ab)^2(b'c'), \quad Q_1 = (ab)^2(cd)^2(a'c')(b'd').$$

4 . 1. *The system* $(B)_1$.

The system for a (2, 2) form contains eighteen members‡. It will be shown that all but two of the copied transvectants contain terms which

---

* Gilham, **1**.
† Peano, **4**.
‡ Turnbull, **8**.

may be transformed so as to contain the factor j_1. As the third index of transvection is zero in every case, it is omitted.

The copied system is

$$J_2, \quad L_1 = (J_2, J_2)^{1.1}, \quad L_2 = (J_2, J_2)^{2.0}, \quad L_3 = (J_2, J_2)^{0.2}, \quad L_4 = (J_2, J_2)^{2.2},$$

$$L_5 = (L_3, J_2)^{2.0}, \ (L_1, J_2)^{1.0}, \ (L_3, J_2)^{1.0}, \ (L_1, J_2)^{2.2}, \ (L_5, J_2)^{1.0}, \ (L_5, J_2)^{0.1},$$

$$L_6 = (L_1, L_1)^{2.0},$$

$$L_7 = (L_1, L_1)^{0.2}, \ (L_3, L_3)^{4.0}, \ (L_6, J_2)^{0.1}, \ (L_7, J_2)^{1.0}, \ (L_6, L_2)^{0.1}, \ (L_7, L_3)^{1.0}.$$

4.2. $L_1 = \{(ab)(a''b'')\,a_x b_x a'_y b'_y, \ (cd)(c''d'')\,c_x d_x c'_y d'_y\}^{1.1}$.

A transvectant of two forms is equal to any one of its terms, plus transvectants with lower indices of forms derived by convolution from the original ones. Since the only non-vanishing form derived by convolution from J_2 is Δ_2, it is found that

$$L_1 = (ab)(ac)(cd)(b'c')(a''b'')(c''d'')\,b_x d_x a'_y d'_y + \lambda \Delta_2 J_2$$
$$\equiv (ab)(ac)(cd)(b'c')(a''b'')(c''d'') \pmod{j_1}.$$

By use of the three-term identity we obtain

$$L_1 \equiv (ab)(ac)(cd)(b'c')[(a''c'')(b''d'')-(a''d'')(b''c'')]$$
$$\equiv (ab)(ac)(cd)(b'c')(a''c'')(b''d'') \pmod{j_1}$$
$$= \tfrac{1}{2}(ab)(ac)(cd)(a''c'')(b''d'')[(b'c')+(a'd')]$$

by interchange of the equivalent symbols, a with c and b with d. But

$$(b'c')\,a'_y d'_y + (a'd')\,b'_y c'_y = (b'd')\,a'_y c'_y + (d'c')\,a'_y b'_y + (a'd')\,b'_y c'_y$$
$$= (b'd')\,a'_y c'_y + (a'c')\,b'_y d'_y.$$

Hence $L_1 \equiv \tfrac{1}{2}(ab)(ac)(cd)(a''c'')(b''d'')[(b'd')+(a'c')] \equiv 0 \pmod{j_1}$.

4.3. $L_3 = \{(ab)(a''b''), \ (cd)(c''d'')\}^{0.2} = (ab)(cd)(a'c')(b'd')(a''b'')(c''d'')$.

Let
$$M_3 = (ac)(bd)(a'b')(c'd')(a''b'')(c''d''),$$
$$N_3 = (ab)(cd)(a'b')(c'd')(a''c'')(b''d''),$$
$$\theta = (ab)(cd)(a'c')(b'd')(a''d'')(b''c'').$$

$$L_3 = (ab)(cd)(a'c')(b'd')[(a''c'')(b''d'')-(a''d'')(b''c'')] = M_3 - \theta.$$

Similarly,
$$M_3 = N_3 - \theta,$$
$$N_3 = L_3 - \theta.$$

Hence $\theta = 0$, $L_3 = M_3 = N_8$, and therefore

$$L_3 \equiv 0 \pmod{j_1}.$$

All the members of $(B)_1$ are derived by transvection from L_1 or L_3, except L_2 and L_4.

L_4 is the invariant

$$\Delta_4 = (ab)^2(cd)^2(a'c')(b'd')(a''c'')(b''d'') \equiv 0 \pmod{j_1}.$$

Hence $$(B)_1 \equiv J_2, \qquad Q_2 \equiv c \pmod{j_1},$$

where $$Q_2 = (ac)^2(bd)^2(a''b'')(c''d'').$$

The reductions of L_1 and L_3 have been given in detail, as similar transformations are used in many of the reductions obtained in the following paragraphs.

5.1. *The system* $(A)_2$.

Since f and T_1 are (2, 1) forms, all possible products arising from $(A)_1$ are included in

$$\Theta = f_1^\alpha J_3^\beta P_2^\gamma Q_1^\delta \quad \text{where} \quad f_1 = f + \lambda T_1.$$

The system $(A)_2$ will therefore consist of all transvectants

$$(\Theta,\ J_2^\epsilon Q_2^\zeta)^{\rho\cdot\sigma}$$

which do not contain any reducible terms $(\text{mod}\, j_1)$. Q_1 does not contain x or y, and may therefore be omitted.

It will first be shown that the values of α, β, etc., can be limited, after which the reduction of individual transvectants will be carried out.

5.2. J_3 and P_2 are quadratics in x and y respectively, and J_2 is a (2, 2) form in x and y, so that if none of β, γ, ϵ are zero, the transvectant is reduced by means of

$$(J_3 P_2,\ J_2)^{\rho\cdot\sigma}.$$

Also, since Q_2 is a quartic in y, the transvectant is reduced by means of

$$(P_2^2,\ Q_2)^{0\cdot\sigma} \quad (\zeta \neq 0,\ \gamma \geqslant 2).$$

If $\epsilon = 0$, the transvectant is reducible unless $\beta = 0$, $\zeta = 1$. There remain

(i) $(f_1^\alpha J_3^\beta,\ J_2^\epsilon Q_2^\zeta)^{\rho\cdot\sigma}$ $\quad (\epsilon \neq 0)$,

(ii) $(f_1^\alpha P_2^\gamma,\ J_2^\epsilon)^{\rho\cdot\sigma}$ $\quad (\gamma \neq 0)$,

(iii) $(f_1^\alpha P_2,\ J_2^\epsilon Q_2^\zeta)^{\rho\cdot\sigma}$ $\quad (\epsilon \neq 0,\ \zeta \neq 0)$,

(iv) $(f_1^\alpha P_2^\gamma,\ Q_2)^{0\cdot\sigma}$ $\quad (\gamma \leqslant 1)$.

5.3. (i) The transvectant is reducible unless $\zeta = 0$, so that the irreducibles are

$$(f_1,\ J_2)^{\rho\cdot\sigma},\quad (f_1^2,\ J_2)^{\rho\cdot 2},\quad (J_3,\ J_2)^{\rho\cdot 0}.$$

(ii) The transvectant is reduced by $(f_1^2 P_2,\ J_2^2)^{\rho\cdot\sigma}$ if $\alpha \geqslant 2$, $\gamma \geqslant 1$. If $\gamma = 0$, it has been considered under (i). There remain

$$(P_2,\ J_2)^{0\cdot\sigma},\quad (f_1 P_2,\ J_2)^{\rho\cdot 2}.$$

(iii) $\alpha < 2$ for irreducibles [from (ii)], and the transvectant is immediately reducible if $\alpha < 2$.

(iv) $\alpha + 2\gamma \leqslant 4$. There remain

$$(f_1 P_2,\ Q_2)^{0\cdot 3},\quad (f_1^2 P_2,\ Q_2)^{0\cdot 4},\quad (P_2,\ Q_2)^{0\cdot\sigma},\quad (f_1^\alpha,\ Q_2)^{0\cdot\alpha}\quad (\alpha = 1,\ 2,\ 3,\ 4).$$

Some of the transvectants retained possess terms which immediately break up into factors. T_1 is a Jacobian and

$$T_1^2 = -\tfrac{1}{2} P_2 J_3 - Q_1 f^2.$$

Any transvectant containing f_1^α will therefore give rise to two others, one containing f^α, the other containing $f^{\alpha-1} T_1$, since powers of T_1 may be neglected.

Any x Jacobian of T_1 is reducible, being a Jacobian of a Jacobian of forms of order two at least.

There remains a system of forty-seven forms:

$$(f,\ Q_2)^{0.1},\quad (f^2,\ Q_2)^{0.2},\quad (f^3,\ Q_2)^{0.3},\quad (f^4,\ Q_2)^{0.4},\quad (fP_2,\ Q_2)^{0.3},\quad (f^2 P_2,\ Q_2)^{0.4},$$

$$(T_1,\ Q_2)^{0.1},\quad (fT_1,\ Q_2)^{0.2},\quad (f^2 T_1,\ Q_2)^{0.3},\quad (f^3 T_1,\ Q_2)^{0.4},\quad (T_1 P_2,\ Q_2)^{0.3},$$

$$(fT_1 P_2,\ Q_2)^{0.4},\quad (P_2,\ Q_2)^{0.1},\quad (P_2,\ Q_2)^{0.2},\quad (P_2^2,\ Q_2)^{0.3},\quad (P_2^2,\ Q_2)^{0.4},\quad (f,\ J_2)^{1.0},$$

$$(f,\ J_2)^{2.0},\quad (f,\ J_2)^{0.1},\quad (f,\ J_2)^{1.1},\quad (f,\ J_2)^{2.1},\quad (f^2,\ J_2)^{0.2},\quad (f^2,\ J_2)^{1.2},\quad (f^2,\ J_2)^{2.2},$$

$$(T_1,\ J_2)^{2.0},\quad (T_1,\ J_2)^{0.1},\quad (T_1,\ J_2)^{1.1},\quad (T_1,\ J_2)^{2.1},\quad (fT_1,\ J_2)^{0.2},\quad (fT_1,\ J_2)^{1.2},$$

$$(fT_1,\ J_2)^{2.2},\quad (J_3,\ J_2)^{1.0},\quad (J_3,\ J_2)^{2.0},\quad (P_2,\ J_2)^{0.1},\quad (P_2,\ J_2)^{0.2},\quad (J_3 P_2,\ J_2)^{1.1},$$

$$(J_3 P_2,\ J_2)^{2.1},\ (J_3 P_2,\ J_2)^{1.2},\ (J_3 P_2,\ J_2)^{2.2},\ (fP_2,\ J_2)^{1.2},\ (fP_2,\ J_2)^{2.2},\ (f^2 P_2,\ J_2^2)^{3.4},$$

$$(f^2 P_2,\ J_2^2)^{4.4},\quad (T_1 P_2,\ J_2)^{1.2},\quad (T_1 P_2,\ J_2)^{2.2},\quad (fT_1 P_2,\ J_2^2)^{3.4},\quad (fT_1 P_2,\ J_2^2)^{4.4}.$$

It will next be shown that all but two of these transvectants are reducible (mod j_1), it being observed that if a form contains $(ab)^2 (a'b')$ or $(ab)^2 (a''b'')$ it is reducible, since it can be transformed to contain Δ_2.

5 . 4.

(i) $(Q_2, f)^{0.1} = \{(ab)^2(cd)^2(a''c'')(b''d''),\ e_x^2 e_y' e_z''\}^{0.1}$

$\equiv (ab)^2(cd)^2(a'e')(a''c'')(b''d'') \equiv (ab)^2(cd)^2(a'e')(e''c'')(b''d'')$

$= (ab)^2(cd)^2[(a'b')+(b'c')+(c'e')](e''c'')(b''d'')$

$\equiv (ab)^2(cd)^2(b'c')[(e''b'')(c''d'')-(e''d'')(c''b'')] \equiv 0 \pmod{j_1}$.

Hence $(Rf, Q_2)^{0.\rho} \equiv \{R, (f, Q_2)^{0.1}\}^{0.\rho-1} \equiv 0 \pmod{j_1}$.

(ii) $(Q_2, P_2)^{0.1} \equiv (ab)^2(cd)^2(ef)^2(a'e')(a''c'')(b''d'')$

$= \{(ab)^2(cd)^2(a'e')(a''c'')(b''d''),\ f\}^{2.0} \equiv 0 \pmod{j_1}$ from (i).

$(Q_2, P_2)^{0.2} \equiv (ab)^2(cd)^2(ef)^2(a'e')(c'f')(a''c'')(b''d'')$

$= (ab)^2(cd)^2(ef)^2(a'e')(c'f')(b''d'')[(a''e'')+(e''f'')+(f''c'')]$

$\equiv 0 \pmod{j_1}$.

Hence $(RP_2, Q_2)^{0.\sigma} = \{R, (P_2, Q_2)^{0.2}\}^{0.\sigma-2} \equiv 0 \pmod{j_1}$.

(iii) $(Q_2, T_1)^{0.1} \equiv (ab)^2(cd)^2(ef)^2(a'e')(f'g')(a''c'')(b''d'')$

$= \{(ab)^2(cd)^2(ef)^2(a'e')(a''c'')(b''d''),\ g\}^{0.1} + kR\,.f,$

where R is derived from $(Q_2, P_2)^{0.1}$ by convolution. Hence, from (ii), $(Q_2, T_1)^{0.1} \equiv 0 \pmod{j_1}$.

Thus all transvectants containing Q_2 may be omitted from $(A)_2$.

5 . 5.

(i) $(P_2, J_2)^{0.2} = (ab)^2(cd)(a'c')(b'd')(c''d'') \equiv (ab)^2(cd)(a'c')(b'd')(a''d'')$

$\equiv (ab)^2(cd)(a'd')(b'c')(a''d'') \equiv 0 \pmod{j_1}$.

Hence every transvectant of the form $(RP_2, SJ_2)^{\rho.\sigma}$ $\sigma \geqslant 2$ is reducible.

(ii) $(f, J_2)^{0.1} = (bc)(a'b')(b''c'') = (ab)(b'c')(b''c'') \equiv 0 \pmod{j_1}$.

Hence $(f^2, J_2)^{0.2} = \{f, (fJ_2)^{0.1}\}^{0.1} \equiv 0,$

$(fT_1, J_2)^{0.2} = \{T_1, (fJ_2)^{0.1}\}^{0.1} \equiv 0 \pmod{j_1}$.

(iii) $(f, J_2)^{1.1} = (ab)(bc)(a'c')(b''c'') = \frac{1}{2}(ab)(bc)(a'c')(a''c'') \equiv 0 \pmod{j_1}$.

Hence $(f^2, J_2)^{1.2} \equiv \{f, (fJ)^{1.1}\}^{0.1} \equiv 0,$

$(fT_1, J_2)^{1.2} \equiv \{T_1, (fJ_2)^{1.1}\}^{0.1} \equiv 0 \pmod{j_1}$.

(iv) $$(f^2, J_2)^{2\cdot 2} \equiv (ac)(bd)(cd)(a'c')(b'd')(c''d'')$$
$$\equiv (ac)(bd)(cb)(a'c')(b'd')(c''d'')$$
$$\equiv -\tfrac{1}{2}(ac)(bd)(bc)(a'd')(b'c')[(c''d'')+(a''b'')]$$
$$= -\tfrac{1}{2}(ac)(bd)(bc)(a'd')(b'c')[(c''b'')+(a''d'')] \equiv 0 \pmod{j_1}.$$

(v) $$(P_2, J_2)^{0\cdot 1} = (ab)^2(cd)(a'c')(c''d'')$$
$$= (\psi, f)^{2\cdot 0} + \Sigma k(\bar{\psi}, f)^{1\cdot 0} + k'Uf,$$

where $\psi = (cd)(a'c')(c''d'')$, $\bar{\psi}$ is derived from ψ by convolution, and $U = (ac)(ad)(cd)(a'c')(c''d'') = (f, J_2)^{2\cdot 1}$. $\bar{\psi}$ is either

$$(ac)(cd)(a'c')(c''d'') \equiv 0 \pmod{j_1}$$ by 5.5 (iii),

or $$(cd)^2(a'c')(c''d'') \equiv 0 \pmod{j_1}.$$

$\psi \equiv 0 \pmod{j_1}$, by 5.5 (ii), and U is already included in the system.

Therefore $(P_2, J_2)^{0\cdot 1} \equiv kU.f \pmod{j_1}$ and may be omitted. Also

$$(P_2J_3, J_2)^{1\cdot 1} = \{J_3, (P_2, J_2)^{0\cdot 1}\}^{1\cdot 0} \equiv kU(J_3, f)^{1\cdot 0},$$

and $$(P_2J_3, J_2)^{2\cdot 1} = \{J_3, (P_2J_2)^{0\cdot 1}\}^{2\cdot 0} \equiv kU(J_3, f)^{2\cdot 0} \pmod{j_1},$$

so that both these transvectants may be omitted.

$$(T_1, J_2)^{0\cdot 1} = (ab)^2(de)(a'd')(b'c')(d''e'') \equiv \{(ab)^2(de)(a'd')(d''e''), f\}^{0\cdot 1}$$
$$= \{(P_2, J_2)^{0\cdot 1}, f\}^{0\cdot 1} = (kUf, f)^{0\cdot 1} = k(Uf)^{0\cdot 1}f \equiv 0 \pmod{j_1},$$

since $$(U, f)^{0\cdot 1} \equiv (f^2, J_2)^{2\cdot 2} \equiv 0$$ by 5.5 (iv).

(vi) $$(T_1, J_2)^{2\cdot 1} \equiv (ab)^2(cd)(ce)(de)(a'd')(b'c')(d''e'')$$
$$\equiv (ab)^2(cd)(ce)(de)(a'd')(b'c')(a''e'')$$
$$= -\tfrac{1}{2}(ab)^2(cd)(ce)(de)(a'b')(c'd')(a''e'') \equiv 0 \pmod{j_1}.$$

Hence $$(fT_1, J_2)^{2\cdot 2} = \{f, (T_1, J_2)^{2\cdot 1}\}^{0\cdot 1} \equiv 0.$$

(vii) $$(J_3, J_2)^{1\cdot 0} = (ab)(bc)(cd)(a'b')(c''d'')$$
$$= (ad)(bc)(cd)(a'b')(c''d'') - (bd)(bc)(cd)(a'b')(c''d'')$$
$$= \tfrac{2}{3}Uf \equiv 0.$$

$$(J_3, J_2)^{2\cdot 0} = (ab)(cd)(ac)(bd)(a'b')(c''d'')$$
$$= (ac)^2(bd)^2(a'b')(c''d'')$$
$$\equiv (ac)^2(bd)^2(a'b')(b''d'') - (ac)^2(bd)^2(a'b')(a''c'')$$
$$\equiv 0 \pmod{j_1}.$$

(viii) $$(f, J_2)^{2.0} = (ab)(ac)(bc)(b''c'') = 0.$$

(ix) $$(T_1, J_2)^{2.0} = (ab)^2(cd)(ce)(de)(b'c')(d''e'')$$
$$= \{(ab)^2, (cd)(ce)(de)(d''e'')\}^{0.1} + kUP_2 \equiv 0 \pmod{j_1}.$$
$$(T_1, J_2)^{1.1} \equiv (ab)^2(cd)(de)(b'c')(a'd')(d''e'')$$
$$= -\tfrac{1}{2}(ab)^2[(cd)^2+(de)^2-(ce)^2](b'c')(a'd')(d''e'')$$
$$\equiv (ab)^2(ce)^2(b'c')(a'd')[(d''a'')+(a''b'')+(b''c'')+(c''e'')]$$
$$\equiv 0 \pmod{j_1}.$$

5.6. It follow that the only additional transvectants which do not contain reducible terms are T_2 and U, where

$$U = (f, J_2)^{2.1},$$
$$T_2 = -(f, J_2)^{1.0} = (ab)^2(b''c'').$$

Thus $$(A)_2 \equiv f,\ P_2,\ J_2,\ J_3,\ T_1,\ T_2,\ Q_1,\ Q_2,\ U \equiv c \pmod{j_1}.$$

6. *The system* $(B)_2$.

The system is copied from that of the binary quartic. Single terms form a system complete for the modulus Δ_2, so we may take

$$(B)_2 \equiv J_1,\ Q_4,\ P_6,\ \Delta_4,\ \Delta_6,$$

where

$$Q_4 = (bc)^2(a'b')(c'd')(a''b'')(c''d''),$$
$$P_6 = (ae)(bc)^2(a'b')(c'd')(e'f')(a''b'')(c''d'')(e''f''),$$
$$\Delta_4 = (bc)^2(ad)^2(a'b')(c'd')(a''b'')(c''d''),$$
$$\Delta_6 = (bc)^2(de)^2(fa)^2(a'b')(c'd')(e'f')(a''b'')(c''d'')(e''f''),$$

Δ_4 and Δ_6 are invariants.

7.1. *The system* $(A)_3$.

Since $(A)_3 \equiv [(A)_2, (B)_2]$, it is unnecessary to include in the products derived from $(A)_2$ those forms which do not contain x. There remain f, J_2, J_3, T_1, T_2, all of which are quadratics in x. The transvectants to be considered are of the form

$$(f^\alpha J_2^\beta J_3^\gamma T_1^\delta T_2^\epsilon,\ J_1^\zeta Q_4^\eta P_6^\theta)^\rho,$$

where the index ρ refers to x transvection only. Since J_1, Q_4, P_6 are all of even order, and the concomitants in the first product are quadratics, it is unnecessary to consider powers or products of J_1, Q_4, P_6. The types to be considered are therefore

(*a*) $(q_1,\ R)^\rho$, $(\rho = 1, 2)$,

(*b*) $(q_1^2,\ R)^\rho$, $(q_1 q_2,\ R)^\rho$ $(\rho = 3, 4)$,

(*c*) $(q_1^3,\ P_6)^\rho$, $(q_1^2 q_2,\ P_6)^\rho$, $(q_1 q_2 q_3,\ P_6)^\rho$ $(\rho = 5, 6)$,

where q_1, q_2, q_3 are any of f, J_2, J_3, T_1, T_2, and R is one of J_1, Q_4, P_6.

Of the one hundred and ninety transvectants, a large number can be ignored in consequence of the reductions contained in the following sections of this paragraph.

7.2.

$$T_1 = (ab)^2(b'c') = -(f,\ J_3)^{1\cdot 0}, \qquad T_2 = (ab)^2(b''c'') = -(f,\ J_2)^{1\cdot 0}.$$

Hence T_1^2 and T_2^2 are connected with other forms of the system by syzygys. T_1 and T_2 need therefore be included only raised to the first power.

Also $T_1 T_2$ is a quadratic function of f, J_2, J_3, and may therefore be omitted.

Further, since T_1, T_2, P_6 are Jacobians, all transvectants of index one and containing any of them are reducible. *Any* transvectant of odd index containing P_6 is reducible*. It will be shown in the next section of this paragraph that, if the index of transvection is odd, and T_1 or T_2 is contained, the transvectant is reducible.

7.3. For suppose that q_1, q_2, q_3, q_4 are quadratics.

(i) $$\{\xi_x^\lambda,\ q_{1x}^2 (q_2 q_3)\, q_{2x} q_{3x}\}^3 \quad (\lambda \geqslant 4)$$

contains the term

$$(\xi q_1)^2 (\xi q_2)(q_2 q_3) = \tfrac{1}{2}(\xi q_1)^2 [(\xi q_3)^2 - (\xi q_2)^2 - (q_2 q_3)^2] \equiv 0.$$

(ii) $$(\xi_x^\lambda,\ q_{1x}^2 q_{2x}^2 (q_3 q_4)\, q_{3x} q_{4x})^5 \quad (\lambda \geqslant 6)$$

contains the term

$$(\xi q_1)^2 (\xi q_2)^2 (\xi q_3)(q_3 q_4) = \tfrac{1}{2}(\xi q_1)^2 (\xi q_2)^2 [(\xi q_4)^2 - (\xi q_3)^2 - (q_3 q_4)^2] \equiv 0.$$

* Grace and Young, 2.

It will be shown, in addition, that all transvectants of even order which contain P_6 and either T_1 or T_2 are reducible.

$$P_6 = (Q_4, J_1)^1 = (\theta\phi)\,\theta_x^3\phi_x^3, \quad \text{say.}$$

(iii) $$\{(q_1q_2)\,q_{1x}q_{2x}q_{3x}^2,\ (\theta\phi)\,\theta_x^3\phi_x^3\}^4$$

contains the term

$$(q_3\theta)^2(q_1q_2)\,(\theta\phi)\,(q_1\theta)\,(q_2\phi)$$
$$= \tfrac{1}{2}(q_3\theta)^2\left[(q_1q_2)^2(\theta\phi)^2+(q_1\theta)^2(q_2\phi)^2-(q_2\theta)^2(q_1\phi)^2\right] \equiv 0.$$

Similarly, $$\{(q_1q_2)\,q_{1x}q_{2x}q_{3x}^2q_{4x}^2,\ P_6\}^6$$

contains the term

$$(q_3\theta)^2(q_4\phi)^2(q_1q_2)\,(\theta\phi)\,(q_1\theta)\,(q_2\phi) \equiv 0.$$

7.4. Two other syzygys assist in simplifying the system.

(i) $$J_2J_3 = (ab)\,(cd)\,(a'b')\,(c''d'') = 2(ab)\,(cd)\,(a'c')\,(c''d'')$$
$$= (ab)\,(cd)\,(a'c')\,[c''d'')-(a''b'')]$$
$$= -(ab)\,(cd)\,(a'c')\,(a''c'')+\tfrac{1}{2}(a'c')\,(b''d'')\,[(ab)\,(cd)-(ad)\,(cb)]$$
$$= -(ab)\,(cd)\,(a'c')\,(a''c'')+\tfrac{1}{2}J_2J_3.$$

Hence

$$J_2J_3 = (a'c')\,(a''c'')\,[(ac)^2+(bd)^2-(ad)^2-(bc)^2]$$
$$= f^2\Delta_2+P_2J_1-2f.(fJ_1)^2.$$

It may easily be verified that, by use of this syzygy, all transvectants containing J_2J_3 are reducible.

(ii) $$T_1J_2 = -(ab)\,(bc)\,(b'c')\,(de)\,(d''e'')$$
$$= -(ad)\,(bc)\,(b'c')\,(de)\,(d''e'')+(bd)\,(bc)\,(de)\,(b'c')\,(d''e'').f$$
$$= +T_2J_3+(bd)\,(bc)\,(ce)\,(b'c')\,(d''e'').f+(bd)\,(bc)\,(dc)\,(b'c')\,(d''e'')f$$
$$= T_2J_3+\tfrac{1}{3}(bd)\,(bc)\,(dc)\,[(b'c')\,(d''e'')+(c'd')\,(b''e'')+(d'b')\,(c''e'')].f$$
$$= T_2J_3-\tfrac{2}{3}U.f^2.$$

Hence all transvectants containing T_2J_3 may be omitted.

7.5. A few further reductions are given below.

(i) $$(fJ_2,\ J_1)^3 \equiv (ab)\,(ad)\,(bd)\,(ce)\,(d'e')\,(a''b'')\,(d''e'')$$
$$= \{(ab)\,(ad)\,(bd)\,(a''b''),\ (ce)\}^{0.1.1}+\lambda UJ_2 = \lambda UJ_2.$$

Similarly $$(fJ_3,\ J_1)^3 \equiv \lambda U J_3.$$

(ii) $$(J_2,\ J_2)^{1.1} = (ab)(ac)(cd)(b'c')(a''b'')(c''d'') + k\Delta_2 J_2$$
$$= (ab)ac)(cd)(b'c')(a''c'')(b''d'')$$
$$-(ab)(ac)(cd)(b'c')(a''d'')(b''c'') + k\Delta_2 J_2$$
$$= \tfrac{1}{2}(ab)(ac)(cd)(a''c'')(b''d'')[(b'c')+(a'd')]$$
$$-\tfrac{1}{2}(ac)(ad)(cd)(b'c')(a''d'')(b''c'') + k\Delta_2 J_2$$
$$= (ab)(cd)(ac)(a''c'')(b'd')(b''d'') + k'\Delta_2 J_2$$
$$= (J_2,\ J_1)^{2.0} + k''\Delta_2 J_2.$$

$$(J_2,\ J_1)^{2.0} \equiv (J_2,\ J_2)^{1.1.0} \pmod{\Delta_2}.$$

Similarly $$(J_3,\ J_1)^{2.0} \equiv (J_3,\ J_3)^{1.0.1} \pmod{\Delta_2}.$$

Let $$J_2 = A_x^2 A_y'^2 = B_x^2 B_y'^2 = \ldots.$$

$$(J_2^2,\ J_1)^4 = \{J_2,\ (J_2 J_1)^2\}^2 = \{A_x^2 A_y'^2,\ (BC)(B'C')B_x C_x B_y' C_y'\}^{2.0}.$$
$$= (AB)(AC)(BC)(B'C') = 0.$$

Also $$(J_3^2,\ J_1)^4 = 0.$$

(iii) $$T_2 = (J_2 f)^{1.0} = (Aa) \quad \text{and} \quad (Aa)^2 = (J_2, f)^2 = 0.$$

Hence

$$(J_2 T_2,\ J_1)^4 = \{(Aa),\ (BC)(B'C')\}^2$$
$$\equiv (Aa)(AB)(aC)(BC)(B'C')$$
$$= \tfrac{1}{2}(B'C')[(AB)^2(aC)^2 + (Aa)^2(BC)^2 - (AC)^2(Ba)^2]$$
$$= (AB)^2(aC)^2(B'C') = \{(AB)^2,\ (aC)^2\}^{0.1} + kQ_2 U$$
$$= kQ_2 U.$$

Similarly $$(J_3 T_1,\ J_1)^4 = kQ_1 . U.$$

(iv) Let $J_3 = \phi_x^2 \phi_z''^2$. Then $T_1 = (\phi a)$, and $(\phi a)^2 = 0$.

Hence

$$(J_2 T_1,\ J_1)^4 = \{T_1,\ (J_2,\ J_1)^2\}^2 = \{(\phi a),\ (BC)(B'C')\}^2$$
$$= (\phi a)(BC)(\phi B)(aC))B'C')$$
$$= \tfrac{1}{2}(B'C')[(\phi a)^2(BC)^2 + (\phi B)^2(aC)^2 - (\phi C)^2(aB)^2]$$
$$= (\phi B)^2(aC)^2(B'C') = \{(\phi B)^2,\ (aC)^2\}^{0.1} + k(\phi B)^2 . U$$
$$= k . (\phi B)^2 . U \equiv 0.$$

(v) $P_6 = (J_1, Q_4)^1 = \{J_1, (A'B')^2\}^1 = (\xi A)(A'B')^2$, where $J_1 = \xi_x^4$.

Hence

$$\begin{aligned}(fJ_2^2, P_6)^6 &= (A\xi)^2(BD)^2(a\xi)(aC)(\xi C)(C'D')^2\\ &\equiv \{(A\xi)^2, (BD)^2(aC)(C'D')^2\}^2\\ &= \{(AE)(A'E'), (aC)(BD)^2(C'D')^2\}^2\\ &= (AE)(aC)(Aa)(EC)(BD)^2(A'E')(C'D')^2\\ &= \tfrac{1}{2}(BD)^2(A'E')(C'D')^2[(AE)^2(aC)^2+(Aa)^2(EC)^2-(Ea)^2(AC)^2]\\ &= (Aa)^2(A'E')(BD)^2(EC)^2(C'D')^2\\ &= \{(Aa)^2, (BD)^2(EC)^2(C'D')^2\}^{0.1}+kU.(BD)^2(EC)^2(C'D')^2\\ &= kU.(BD)^2(EC)^2(C'D')^2 \equiv 0,\end{aligned}$$

since $(Aa)^2 = 0.$

Similarly $(fJ_3^2, P_6)^6 \equiv 0.$

The same method will reduce $(f^2J_2, P_6)^6$ and $(f^2J_3, P_6)^6$.

8. The complete system of the binary (2, 1, 1) form is thus found to consist of sixty-five concomitants, which may be classified in the following manner :

The symbol $(\lambda.\mu.\nu)$ denotes that the concomitant is of orders λ, μ, ν in x, y, z respectively.

Degree 1. $f(2.1.1).$

Degree 2.

$J_1(4.0.0),\quad P_2(0.2.2),\quad \Delta_2(0.0.0).$

$J_2(2.2.0),\quad J_3(2.0.2).$

Degree 3.

$(fJ_1)^1(4.1.1),\quad (fJ_1)^2(2.1.1),\quad U(0.1.1).$

$T_1(2.1.3),\quad T_2(2.3.1).$

Degree 4.

$Q_4(4.0.0),\quad (f^2, J_1)^3(2.2.2),\quad (f^2, J_1)^4(0.2.2),\quad \Delta_4(0.0.0).$

$(J_2, J_1)^1(4.2.0),\quad (J_3, J_1)^1(4.0.2).$

$(J_2, J_1)^2(2.2.0),\quad (J_3, J_1)^2(2.0.2).$

$Q_1(0.0.4),\quad Q_2(0.4.0).$

Degree 5.

$$(f,\ Q_4)^1(4\,.\,1\,.\,1), \qquad (f,\ Q_4)^2(2\,.\,1\,.\,1).$$

$$(T_1,\ J_1)^2(2\,.\,1\,.\,3), \qquad (T_2,\ J_1)^2(2\,.\,3\,.\,1).$$

$$(fJ_2,\ J_1)^4(0\,.\,3\,.\,1), \qquad (fJ_3,\ J_1)^4(0\,.\,1\,.\,3).$$

Degree 6.

$P_6(6\,.\,0\,.\,0)$, $(f^2,\ Q_4)^3(2\,.\,2\,.\,2)$, $(f^2,\ Q_4)^4(0\,.\,2\,.\,2)$, $\Delta_6(0\,.\,0\,.\,0)$.

$$(J_2,\ Q_4)^1(4\,.\,2\,.\,0), \qquad (J_3,\ Q_4)^1(4\,.\,0\,.\,2).$$

$$(J_2^2,\ J_1)^3(2\,.\,4\,.\,0), \qquad (J_3^2,\ J_1)^3(2\,.\,0\,.\,4).$$

$$(J_2,\ Q_4)^2(2\,.\,2\,.\,0), \qquad (J_3,\ Q_4)^2(2\,.\,0\,.\,2).$$

$$(fT_1,\ J_1)^4(0\,.\,2\,.\,4), \qquad (fT_2,\ J_1)^4(0\,.\,4\,.\,2).$$

Degree 7.

$$(f,\ P_6)^2(4\,.\,1\,.\,1).$$

$$(fJ_2,\ Q_4)^3(2\,.\,3\,.\,1), \qquad (fJ_3,\ Q_4)^3(2\,.\,1\,.\,3).$$

$$(T_2,\ Q_4)^2(2\,.\,3\,.\,1), \qquad (T_1,\ Q_4)^2(2\,.\,1\,.\,3).$$

$$(fJ_2,\ Q_4)^4(0\,.\,3\,.\,1), \qquad (fJ_3,\ Q_4)^4(0\,.\,1\,.\,3).$$

Degree 8.

$$(f^2,\ P_6)^4(2\,.\,2\,.\,2).$$

$$(J_2,\ P_6)^2(4\,.\,2\,.\,0), \qquad (J_3,\ P_6)^2(4\,.\,0\,.\,2).$$

$$(J_2^2,\ Q_4)^3(2\,.\,4\,.\,0), \qquad (J_3^2,\ Q_4)^3(2\,.\,0\,.\,4).$$

$$(fT_1,\ Q_4)^4(0\,.\,2\,.\,4), \qquad (fT_2,\ Q_4)^4(0\,.\,4\,.\,2).$$

$$(J_2^2,\ Q_4)^4(0\,.\,4\,.\,0), \qquad (J_3^2,\ Q_4)^4(0\,.\,0\,.\,4).$$

Degree 9.

$$(fJ_2,\ P_6)^4(2\,.\,3\,.\,1), \qquad (fJ_3,\ P_6)^4(2\,.\,1\,.\,3).$$

$$(J_2T_2,\ Q_4)^4(0\,.\,5\,.\,1), \qquad (J_3T_1,\ Q_4)^4(0\,.\,1\,.\,5).$$

$$(J_2T_1,\ Q_4)^4(0\,.\,3\,.\,3), \qquad (f^3,\ P_6)^6(0\,.\,3\,.\,3).$$

Degree 10.

$$(J_2^2,\ P_6)^4(2\,.\,4\,.\,0), \qquad (J_3^2,\ P_6)^4(2\,.\,0\,.\,4).$$

Degree 12.

$$(J_2^3,\ P_6)^6(0\,.\,6\,.\,0), \qquad (J_3^3,\ P_6)^6(0\,.\,0\,.\,6).$$

9. I am indebted to Professor W. P. Milne for the following geometrical interpretation of the (2, 1, 1) form.

Let $y_1z_1 = x$, $y_1z_2 = y$, $y_2z_1 = z$, $y_2z_2 = t$, so that $\Theta \equiv xt - yz = 0$.

Put $$\frac{x_1}{x_2} = \lambda.$$

Then $f = x_2^2(\lambda^2 L_1 + 2\lambda L_2 + L_3)$, where L_1, L_2, L_3 represent planes, if x, y, z, t are taken as homogeneous three-dimensional coordinates.

$f = 0$ will then represent the tangent plane to the cone $L_2^2 - L_1L_3 = 0$, which is the same as $P_2 = 0$. Hence the (2 . 1 . 1) form may be regarded as representing the curve of intersection of any tangent plane to the cone with the fundamental quadric Θ. It is proposed to discuss this representation in a later communication. (The transformation is, of course, the well known correlation of quaternary and double binary forms used by Kasner*.)

I am indebted to a referee for the suggestion that I should mention the paper on the (1 . 1 . 1 . 1) binary form by Segre†, which is mainly geometrical, but includes the interpretation of certain of the concomitants. The complete system for this form has not yet been obtained, but would clearly be of considerable complexity, since the (2, 1 . 1), (2 . 2) and binary quartic are all particular cases of it.

References.

1. C. W. Gilham, *Journal London Math. Soc.*, 4 (1929), 166–171.
2. J. H. Grace and A. Young, *Algebra of Invariants*, 167.
3. W. Kasner, *Trans. American Math. Soc.*, 1 (1900), 430–498.
4. G. Peano, *Giorn. matematico*, 20 (1882), 79–100.
5. W. Saddler, *Proc. Camb. Phil. Soc.*, 22 (1925), 688–693.
6. E. Schwartz, *Math. Zeitschrift*, 12 (1922), 18–35.
7. C. Segre, *Annali di Mat.* (3), 29 (1920), 105–140.
8. H. W. Turnbull, *Proc. Royal Soc. Edinburgh*, 43 (1923), 43–50.

* Kasner, 3.

† Segre, 7.

ON THE k-ANALOGUES OF SOME THEOREMS IN THE THEORY OF THE RIEMANN ζ-FUNCTION

By R. E. A. C. PALEY.

[Received and read 12 December, 1929.]

1. In this paper I prove some results about the function $L(s, \chi)$ defined by the equation

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where the series on the right-hand side is convergent, and elsewhere by means of analytic continuation. The numbers $\chi(n)$ satisfy the relations

(*a*) $\chi(nn') = \chi(n)\chi(n')$;

(*b*) $\chi(1) = 1$;

(*c*) $\chi(n) = \chi(n')$ if $n \equiv n' \pmod{k}$;

(*d*) $\chi(n) = 0$, if n is not prime to k.

k is some arbitrarily chosen positive integer. A set of numbers chosen so as to satisfy the relations (*a*), (*b*), (*c*), (*d*) defines a character corresponding to the particular number k. If $\chi(n)$ is a character corresponding to k, then $\bar{\chi}(n)$, the conjugate of $\chi(n)$, is another such character. It may be shown† that, corresponding to a given positive integer k, there are exactly h different characters, where h is Euler's function, the number of numbers less than k and prime to it. It is known that

$$h = k \prod \left(1 - \frac{1}{p}\right),$$

† Landau, **1**, Part 5.

where the product is over all the prime factors of k ($h = k-1$ if k is prime). For every positive number ϵ, $k/h = O(\log_2 k) = O(\log^\epsilon k)$. For each χ corresponding to k, $|\chi(n)| = 1$ if n is prime to k, $\chi(n) = 0$ otherwise. One of the characters corresponding to k, which we call the first or principal character χ_1, is that defined by the relations

$$\chi_1(n) = 1 \quad (n \text{ prime to } k),$$

$$\chi_1(n) = 0 \quad (n \text{ not prime to } k).$$

All the investigations in this paper are about χ's different from χ_1. We assume, except when the contrary is explicitly stated, that we are considering a χ different from χ_1.

Let χ be a character corresponding to a given number k. χ is said to be imprimitive if there exists a divisor K of k, such that $\chi(n) = \chi(n')$ wherever

$$n \equiv n' \pmod{K}; \quad (n, k) = 1; \quad (n', k) = 1.$$

Clearly χ_1 is always imprimitive. If χ is not an imprimitive character it is said to be primitive. If k is prime then all the characters corresponding to k, except the principal character, are primitive.

The function $\chi(n)$ has a period k, and we may clearly define $\chi(n)$ for negative integral values of n, by means of the relation (c) above. We observe that

$$\{\chi(-1)\}^2 = \{\chi(k-1)\}^2 = \chi(k-1)^2 = \chi(1) = 1.$$

Hence $\chi(-1) = \pm 1$. We denote by Case 1 the case in which $\chi(-1) = 1$, by Case 2 the case in which $\chi(-1) = -1$. We use in the paper the following two important properties of characters.

(1)† If $\chi(n)$ is a primitive character, then

$$\sum_{n=1}^{k} \chi(n) \exp\left(\frac{2\pi mni}{k}\right) = \epsilon(\chi)\sqrt{k}\,\bar{\chi}(m) \quad \text{(Case 1)},$$

$$\sum_{n=1}^{k} \chi(n) \exp\left(\frac{2\pi mni}{k}\right) = i\epsilon(\chi)\sqrt{k}\,\bar{\chi}(m) \quad \text{(Case 2)},$$

where $\epsilon(\chi)$ is a constant of modulus unity, depending on χ. Also $\epsilon(\chi)\epsilon(\bar{\chi}) = 1$.

† See Landau, **1**, 483–494.

(2)† If $\chi(n)$ is a primitive character, then

$$\left| \sum_{m=1}^{n} \chi(m) \right| < \sqrt{k} \log k.$$

If χ is a primitive character, the functional equation‡ for $L(s, \chi)$ connects the value of $L(s, \chi)$ with that of $L(1-s, \bar{\chi})$, corresponding to the conjugate character $\bar{\chi}$. If $\chi(-1) = 1$, we have

$$L(s, \chi) = 2\epsilon(\chi) \frac{\Gamma(1-s)}{(2\pi)^{1-s}} k^{\frac{1}{2}-s} \sin \tfrac{1}{2}\pi s \, L(1-s, \bar{\chi}).$$

If $\chi(-1) = -1$, then

$$L(s, \chi) = 2\epsilon(\chi) \frac{\Gamma(1-s)}{(2\pi)^{1-s}} k^{\frac{1}{2}-s} \cos \tfrac{1}{2}\pi s \, L(1-s, \bar{\chi}).$$

Hardy and Littlewood§ have proved the following result for the Riemann ζ-function.

THEOREM A. *Suppose that H and K are positive constants, and*

$$s = \sigma + it, \quad -H \leqslant \sigma \leqslant H,$$

$$x > K, \quad y > K, \quad 2\pi xy = |t|.$$

Then

$$\zeta(s) = \sum_{n<x} n^{-s} + X(s) \sum_{n<y} n^{s-1} + R,$$

where

$$X(s) = 2(2\pi)^{s-1} \sin \tfrac{1}{2}\pi s \, \Gamma(1-s),$$

$$R = O(x^{-\sigma}) + O(y^{\sigma-1} |t|^{\frac{1}{2}-\sigma}).$$

This is the "approximate functional equation". Let $\zeta^{(\mu)}$, $\zeta^{(\nu)}$ represent the μ-th and ν-th derivatives of ζ respectively. Ingham‖ has obtained an asymptotic formula for the integral

$$\int_1^T \zeta^{(\mu)}(a+ti)\, \zeta^{(\nu)}(b-ti)\, dt,$$

† Landau, **2**, 178.
‡ Landau, **1**, 494–498.
§ Hardy and Littlewood, **1**.
‖ Ingham, **1**, Theorem A.

where a, b denote any complex numbers. The following are special cases of his result.

THEOREM B.

$$\int_1^T |\zeta(\sigma+it)|^2\,dt = T\zeta(2\sigma)+(2\pi)^{2\sigma-1}\zeta(2-2\sigma)\frac{T^{2-2\sigma}}{2-2\sigma}+O(T^{1-\sigma+\epsilon}) \quad (\sigma>\tfrac{1}{2}),$$

$$\int_1^T |\zeta(\tfrac{1}{2}+it)|^2\,dt = T\log T-(1+\log 2\pi-2C)\,T+O(T^{\frac{1}{2}+\epsilon}).$$

Hardy and Littlewood† employ Theorem A to establish the following theorem on the mean value of $|\zeta(\sigma+it)|^4$.

THEOREM C.

$$\frac{1}{T}\int_1^T |\zeta(\sigma+it)|^4\,dt \sim \sum_{n=1}^{\infty}\frac{[d(n)]^2}{n^{2\sigma}} = \frac{[\zeta(2\sigma)]^4}{\zeta(4\sigma)} \quad (\sigma>\tfrac{1}{2}),$$

$$\frac{1}{T}\int_1^T |\zeta(\tfrac{1}{2}+it)|^4\,dt = O(T\log^4 T), \tag{1.1}$$

where $d(n)$ denotes the number of divisors of n.

Analogues of ζ-function theorems may be proved for the functions $L(s,\chi)$ for any fixed value of k. For the L-analogues of Theorems A, B, C, we see that σ, k are fixed and t varies. It has been shown by Landau‡ and Littlewood§ that, corresponding to certain theorems of this type, analogous theorems may be proved for which σ, t are fixed and k varies. These are the k-analogues of the original theorems. I have not been able to obtain a k-analogue of Theorem A. In Theorem I, however, I prove an "Abelian" form of the functional equation for $L(s,\chi)$; this is the analogue of a result of my own‖, which gives the corresponding t-theorem. From the Abelian theorem it is possible to obtain the k-analogue of Theorem C. The results are embodied in Theorems III and IV of this paper¶. In

† Hardy and Littlewood, **1**.

‡ Landau, **3**.

§ Littlewood, **1**.

‖ Not yet published.

¶ I have succeeded in obtaining the exact k-analogue of (1.1) only in the case where k is prime.

Theorem II I obtain the k-analogue of Theorem B. This proves that, if $\sigma > \frac{1}{2}$, then

$$\sum_{\chi} |L(s, \chi)|^2 = hL(2\sigma, \chi_1) + A(s)\, h^2 k^{-2\sigma} + O(k^{1-\sigma+\epsilon}),$$

and, if $\sigma = \frac{1}{2}$, then

$$\sum_{\chi} |L(s, \chi)|^2 = \frac{h^2}{k} \log k + \frac{h^2}{k} \sum_{p|k} \frac{\log p}{p-1} + A(s) \frac{h^2}{k} + O(k^{\frac{1}{2}+\epsilon}),$$

where the summation is over all the χ's except the first, and the numbers A are constants depending only on s. It is curious that the proof of this theorem is considerably easier than Ingham's proof of the corresponding t-result. Use of the approximate or Abelian functional equation is not necessary, and the proof is entirely elementary. The method of Theorem II can be extended to obtain an expression for sums of the form

$$\sum_{\chi} L(s, \chi)\, L(s', \bar{\chi}),$$

where s, s' are any two points to the right of the line $\sigma = 0$.

This paper also contains some discussion on the order of the error

$$L(s) - \sum_{n=1}^{\lambda} \chi(n)\, n^{-s}.$$

2. Let $F(x, s)$ denote the function

$$\sum_{n=1}^{\infty} e^{-nx} n^{-s} \quad (\Re x > 0);$$

when $\Re x \leqslant 0$ we denote by $F(x, s)$ the function obtained by analytic continuation over the whole x-plane. Lindelöf† has obtained the following equation, valid for $|x| < 2\pi$, provided that s is not a positive integer,

$$F(x, s) = \Gamma(1-s)\, x^{s-1} + \sum_{\nu=0}^{\infty} \zeta(s-\nu) \frac{(-x)^\nu}{\nu!}. \tag{2.1}$$

If χ is any character and F' is obtained from the corresponding L-function in the same way as F is obtained from ζ, then the equation analogous to

† Lindelöf, **1**, 139.

(2.1) is

$$F'(x, s) = \sum_{\nu=0}^{\infty} L(s-\nu)\frac{(-x)^\nu}{\nu!}, \tag{2.2}$$

valid for all $|x| < 2\pi/k$. The following theorem proves a result which reduces to (2.2) in the circle $|x| < 2\pi/k$ by expansion in powers of x, but which is valid for all real x provided that $\sigma < 1$.

THEOREM I.

Suppose that $0 \leqslant x < \infty$, $\sigma < 1$. *Let* χ *be a primitive character. Then*

$$\sum_{n=1}^{\infty} \chi(n) e^{-nx} n^{-s} = \epsilon(\chi)\frac{\Gamma(1-s)}{(2\pi)^{1-s}} k^{\frac{1}{2}-s} \times \sum_{n=1}^{k} \bar{\chi}(n) \left\{ \frac{\exp\{\frac{1}{2}\pi i(1-s)\}}{\{n+(ikx/2\pi)\}^{1-s}} + \frac{\exp\{-\frac{1}{2}\pi i(1-s)\}}{\{n-(ikx/2\pi)\}^{1-s}} \right\} \quad \text{(Case 1).} \tag{2.3}$$

$$i\sum_{n=1}^{\infty} \chi(n) e^{-nx} n^{-s} = \epsilon(\chi)\frac{\Gamma(1-s)}{(2\pi)^{1-s}} k^{\frac{1}{2}-s} \times \sum_{n=1}^{k} \bar{\chi}(n) \left\{ \frac{\exp\{\frac{1}{2}\pi i(1-s)\}}{\{n+(ikx/2\pi)\}^{1-s}} - \frac{\exp\{-\frac{1}{2}\pi i(1-s)\}}{\{n-(ikx/2\pi)\}^{1-s}} \right\} \quad \text{(Case 2).} \tag{2.4}$$

The equations (2.3), (2.4) are the "Abelian" form of the functional equation for the L-function.

Let $\mathfrak{R}u$ denote the real part of u. Then, if $\mathfrak{R}u > 0$, we write

$$L(s; u) = \sum_{n=1}^{\infty} \chi(n) e^{-nu} n^{-s}. \tag{2.5}$$

The series on the right-hand side of (2.5) is convergent provided that $\mathfrak{R}u > 0$. We define $L(s; u)$, for $\mathfrak{R}u \leqslant 0$, to be the function obtained by analytic continuation of

$$\sum_{n=1}^{\infty} \chi(n) e^{-nu} n^{-s}$$

over the whole u-plane. For $\mathfrak{R}u > 0$,

$$L(0; u) = \frac{\chi(1)e^{-u} + \chi(2)e^{-2u} + \ldots + \chi(k-1)e^{-(k-1)u}}{1-e^{-ku}}. \tag{2.6}$$

Since the sum on the right-hand side of (2.6) represents a function analytic over the whole u-plane, it follows that the equation (2.6) is valid

for all u. We see that, as $\mathfrak{R}u \to \infty$,

$$L(0\,;\,u) = O\{\exp(-\mathfrak{R}u)\},$$

and that, as $\mathfrak{R}u \to -\infty$,

$$L(0\,;\,u) = O\{\exp(\mathfrak{R}u)\}.$$

Let C denote the contour which consists of a straight line from $+\infty$ to π/k just above the real axis, a circle of radius π/k round the origin described counter-clockwise, and a straight line returning to $+\infty$ just below the real axis. Then, if $\sigma > 0$, we have

$$(2.7) \qquad \int_C (-u)^{s-1} e^{-n(u+x)}\,du = -2i \sin(1-s)\pi\,\Gamma(s)\,e^{-nx}\,n^{-s}.$$

If x is real and greater than π/k, we multiply both sides of the equation (2.7) by $\chi(n)$, and sum from $n=1$ to ∞. This is legitimate since the series $\Sigma\chi(n)\,e^{-nx}n^{-s}$ is convergent, and

$$\left| \lim_{N\to\infty} \int_C (-u)^{s-1} \sum_{n=N}^{\infty} e^{-n(u+x)}\,du \right|$$

$$\leqslant \frac{1}{1-e^{-(x-\pi/k)}} \lim_{N\to\infty} \int_C |(-u)^{s-1} e^{-N(u+x)}|\,du = 0.$$

Thus we obtain the equation

$$(2.8) \qquad -2i \sin(1-s)\pi\,\Gamma(s)\,L(s\,;\,x) = \int_C (-u)^{s-1} L(0\,;\,u+x)\,du \qquad (x > \pi/k).$$

Using analytic continuation it is not difficult to see that the equation (2.8) remains valid for all $x > 0$.

Now let C' denote the contour which consists of a straight line from $+\infty$ to $(2M+1)\pi/k$ just above the real axis, a circle of radius $(2M+1)\pi/k$ about the origin, and a straight line returning to ∞ just below the real axis. Then the integral on the right-hand side of (2.8) may be written

$$\int_{C'} (-u)^{s-1} L(0\,;\,u+x)\,du - 2\pi i S,$$

where S denotes the sum of the residues of the function

$$(-u)^{s-1} L(0\,;\,u+x)$$

at the poles inside the circle $|u| \leqslant (\pi/k)(2M+1)$. Writing r for the radius of this circle, we see that

$$\int_{|u|=r} |L(0; u+x)\,du| = O(r)\int_0^{2\pi} \exp(-|r\cos\theta+x|)\,d\theta$$

$$= O(re^{r})\int_0^{2\pi} \exp(-r|\cos\theta|)\,d\theta = O(1).$$

Thus the integral of $(-u)^{s-1}L(0; u+x)$ round the circular part of C' is

$$O(r^{\sigma-1}) = o(1),$$

if $\sigma < 1$. It is easy to see that the integral $(-u)^{s-1}L(0; u+x)$ over the straight portions of C' is $o(1)$. Hence, making $M \to \infty$ in the above result, we obtain

$$2i\sin(1-s)\pi\,\Gamma(s)\,L(s; x) = 2\pi i S', \tag{2.9}$$

where S' now denotes the sum of the residues at all the poles of the function $(-u)^{s-1}L(0; u+x)$. This function has poles at the points

$$\frac{2\pi mi}{k} - x \quad (-\infty < m < \infty).$$

Using the equation

$$\sum_{n=1}^{k-1} \chi(n)\exp\left(\frac{2\pi nmi}{k}\right) = \tau(\chi)\bar{\chi}(n),$$

where $\tau(\chi) = \epsilon(\chi)\sqrt{k}$ in Case 1, $\tau(\chi) = i\epsilon(\chi)\sqrt{k}$ in Case 2, we see that the residue of

$$(-u)^{s-1}\,\frac{\chi(1)\exp\{-(u+x)\}+\chi(2)\exp\{-2(u+x)\}+\ldots+\chi(k-1)\exp\{-(k-1)(u+x)\}}{1-\exp\{-k(u+x)\}}$$

at the point $(2\pi mi/k)-x$ is

$$\frac{(\tau\chi)}{k}\,\bar{\chi}(-m)\left(-\frac{2\pi mi}{k}+x\right)^{s-1}.$$

Since $$\Gamma(s)\,\Gamma(1-s)\sin(1-s)\pi = \pi,$$

we obtain, after a little algebra, the equations (2.3), (2.4).

Let $$\xi(0;u) = \sum_{n=1}^{\infty} e^{-nu} = \frac{e^{-u}}{1-e^{-u}}.$$

We obtain the equation for the ξ-function analogous to (2.1) by considering the integral

$$\int_C (-u)^{s-1}\{\xi(0;u+x)-\xi(0;u)\}\,du,$$

taken round the contour C. This equation is

$$\sum_{n=1}^{\infty} e^{-nx}n^{-s} = \xi(s)+\Gamma(1-s)\,x^{s-1}+\frac{\Gamma(1-s)}{(2\pi)^{1-s}} \times \sum_{1}^{\infty}\left\{\frac{\exp\{\frac{1}{2}\pi i(1-s)\}}{\{n+(ix/2\pi)\}^{1-s}}+\frac{\exp\{-\frac{1}{2}\pi i(1-s)\}}{\{n-(ix/2\pi)^{1-s}}-\frac{2\cos\frac{1}{2}\pi(1-s)}{n^{1-s}}\right\},$$

valid for $\sigma < 1$. From this we have clearly, for $\sigma < 0$,

$$\sum_{n=1}^{\infty} e^{-nx}n^{-s} = \Gamma(1-s)\,x^{s-1}+\frac{\Gamma(1-s)}{2\pi^{1-s}} \times \sum_{1}^{\infty}\left\{\frac{\exp\{\frac{1}{2}\pi i(1-s)\}}{\{n+(ix/2\pi)\}^{1-s}}+\frac{\exp\{-\frac{1}{2}\pi i(1-s)\}}{\{n-(ix/2\pi)\}^{1-s}}\right\}.$$

3. Make $s = 0$ in the equations (2.3), (2.4) and we obtain:

(a) In Case 1

$$L(0;x) = i\epsilon(\chi)\frac{\sqrt{k}}{2\pi}\sum_{n=1}^{\infty}\bar{\chi}(n)\left\{\frac{1}{n+(ikx/2\pi)}-\frac{1}{n-(ikx/2\pi)}\right\}.$$

Writing $N = kx/(2\pi)$, this is

$$O\left\{\sqrt{k}\left|\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\,N}{n^2+N^2}\right|\right\} = O\left\{\sqrt{k}\sum_{n=1}^{\infty}\left(\frac{N}{n^2+N^2}\right)\right\} = O(\sqrt{k}),$$

since $$\sum_{n=1}^{\infty}\left(\frac{N}{n^2+N^2}\right)$$

is bounded.

(*b*) In Case 2

$$L(0;x)-L(0) = \epsilon(\chi)\frac{\sqrt{k}}{2\pi}\sum_{n=1}^{\infty}\bar{\chi}(n)\left\{\frac{1}{n+(ikx/2\pi)}+\frac{1}{n-(ikx/2\pi)}-\frac{2}{n}\right\}$$

$$= -\frac{\epsilon(\chi)\sqrt{k}}{\pi}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)N^2}{n(n^2+N^2)}.$$

Now
$$\left|\sum_{n\geq N}\left\{\frac{\bar{\chi}(n)N^2}{n(n^2+N^2)}\right\}\right| \leqslant \sum_{n\geq N}\frac{N^2}{n^3} = O(1).$$

$$\left|\sum_{1<n<N}\left\{\frac{\bar{\chi}(n)N^2}{n(n^2+N^2)}-\frac{\bar{\chi}(n)}{n}\right\}\right| \leqslant \sum_{1\leqslant n<N}\frac{n}{N^2} = O(1).$$

Hence

$$L(0;x)-L(0) = -\frac{\epsilon(\chi)\sqrt{k}}{\pi}\sum_{1\leqslant n<kx/2\pi}\frac{\bar{\chi}(n)}{n}+O(\sqrt{k}).$$

We see that, for all primitive χ, we have

$$L(0;x)-L(0) = O(\sqrt{k})+O\{\sqrt{k}\,|\log(kx)|\}.$$

4. Consider now, for primitive χ, the expression

$$L(s)-\sum_{1}^{k-1}\chi(n)n^{-s}.$$

We suppose that s is fixed, with $\sigma>0$. Then

(4.1) $$L(s)-\sum_{1}^{k-1}\chi(n)n^{-s}$$

$$= \frac{1}{\Gamma(s)}\int_0^{\infty}u^{s-1}\{\chi(1)e^{-(k+1)u}+\chi(2)e^{-(k+2)u}+\ldots\}\,du$$

$$= \frac{1}{\Gamma(s)}\int_0^{\infty}u^{s-1}e^{-ku}L(0;u)\,du.$$

In Case 1, $L(0;u)=O(\sqrt{k})$. It follows that the expression (4.1) is

$$O(\sqrt{k})\int_0^{\infty}|u^{s-1}e^{-ku}|\,du = O(k^{\frac{1}{2}-\sigma}).$$

Thus

(4.2) $$L(s) = \sum_{1}^{k-1}\chi(n)n^{-s}+O\left(\frac{\sqrt{k}}{k^{\sigma}}\right).$$

In Case 2,

$$L(0;u) = L(0) + O\{\sqrt{k}\log(ku)\} + O(\sqrt{k}).$$

It follows that

$$(4.21)\quad L(s) = \sum_1^{k-1} \chi(n)n^{-s} + \frac{L(0)}{k^s} + O(\sqrt{k})\int_0^\infty |u^{s-1}e^{-ku}|\,du + O(\sqrt{k})\int_0^\infty |u^{s-1}e^{-ku}\log ku|\,du$$

$$= \sum_1^{k-1} \chi(n)n^{-s} + \frac{L(0)}{k^s} + O\left(\frac{\sqrt{k}}{k^\sigma}\right).$$

Since $L(0) = 0$ when $\chi(-1) = 1$, the equations (4.2) and (4.21) show that, for all χ, we have

$$L(s) = \sum_1^{k-1} \chi(n)n^{-s} + \frac{L(0)}{k^s} + O\left(\frac{\sqrt{k}}{k^\sigma}\right) \quad (\sigma > 0).$$

We may extend this argument to show that, if k is a divisor of λ, then

$$(4.3)\qquad L(s) = \sum_1^{\lambda} \chi(n)n^{-s} + \frac{L(0)}{\lambda^s} + O\left(\frac{\sqrt{k}}{\lambda^\sigma}\right) \quad (\sigma > 0).$$

5. We now proceed to obtain the result analogous to (4.3) in the case where λ is not necessarily a multiple of k. It is assumed throughout this paragraph and the next that the character with which we are dealing is primitive. We consider the function $L_m(s)$, defined for $\sigma > 0$ by the equation

$$L_m(s) = \sum_{n=1}^{\infty} \chi(n+m)n^{-s},$$

and defined by means of analytic continuation over the whole of the s-plane. If C denote the contour defined above, we have, for $\sigma > 0$,

$$(5.1)\quad -2i\sin\pi(1-s)\,\Gamma(s)\,L_m(s) = \int_C (-u)^{s-1}\left\{\frac{\chi(m+1)e^{-u} + \chi(m+2)e^{-2u} + \ldots + \chi(m+k)e^{-ku}}{1-e^{-ku}}\right\} du.$$

The integral on the right-hand side represents a function regular in the whole of the s-plane. It follows that the equation (5.1) holds for all

values of s. In considering the functional equation for $L_m(s)$, we distinguish two cases.

$$(a) \quad \chi(m) = 0.$$

We assume that $\sigma < 1$. Then, as $\mathfrak{R}u \to \infty$,

$$(5.2) \qquad \frac{\chi(m+1)e^{-u}+\ldots+\chi(m+k)e^{-ku}}{1-e^{-ku}} = O\{\exp(-\mathfrak{R}u)\},$$

and as $\mathfrak{R}u \to -\infty$, the expression (5.2) is $O\{\exp(\mathfrak{R}u)\}$. If C' denote the contour defined above, we see that

$$\int_{C'} (-u)^{s-1} \left\{\frac{\chi(m+1)e^{-u}+\ldots+\chi(m+k-1)e^{-(k-1)u}}{1-e^{-ku}}\right\} du$$

tends to zero as $M \to \infty$. Hence, calculating the residues in the usual way, we prove that

$$(5.3) \quad L_m(s) = \epsilon(\chi) \frac{2k^{\frac{1}{2}-s}}{(2\pi)^{1-s}} \Gamma(1-s) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \cos\{\frac{1}{2}(1-s)+(2mn/k)\}\pi}{n^{1-s}}$$

(Case 1);

$$(5.4) \quad L_m(s) = \epsilon(\chi) \frac{2k^{\frac{1}{2}-s}}{(2\pi)^{1-s}} \Gamma(1-s) \sum_{n=1}^{\infty} \frac{\bar{\chi}(n) \sin\{\frac{1}{2}(1-s)+(2mn/k)\}\pi}{n^{1-s}}$$

(Case 2).

$$(b) \quad \chi(m) \neq 0.$$

Suppose that $\sigma < 0$. Then, as $\mathfrak{R}u \to \infty$, the expression (5.2) is $O\{\exp(-\mathfrak{R}u)\}$. As $\mathfrak{R}u \to -\infty$, the expression is

$$-\chi(m)+O\{\exp(\mathfrak{R}u)\}.$$

But, since this time $\sigma < 0$, it is still true that

$$\int_{C'} (-u)^{s-1} \left\{\frac{\chi(m+1)e^{-u}+\ldots+\chi(m+k)e^{-ku}}{1-e^{-ku}}\right\} du$$

tends to zero as $M \to \infty$. Thus the equations (5.3), (5.4) still hold.

Suppose now that $s = 0$. Then

$$\int_{C'} (-u)^{-1} \left\{\frac{\chi(m+1)e^{-u}+\ldots+\chi(m+k)e^{-ku}}{1-e^{-ku}}\right\} du$$

$$= -\chi(m) \int_S (-u)^{-1} du + o(1),$$

where S denotes the semicircle $r = \pi(2M+1)/k$, $\frac{1}{2}\pi \leqslant \theta \leqslant 3\pi/2$. Since

$$\int_S (-u)^{-1}\, du = -\pi i,$$

making $M \to \infty$ as before, we obtain the following results:

$$\frac{1}{2}\chi(m)+L_m(0) = -\frac{\sqrt{k}}{\pi}\,\epsilon(\chi)\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)}{n}\sin\left(\frac{2mn\pi}{k}\right) \quad \text{(Case 1)}; \tag{5.5}$$

$$\frac{1}{2}\chi(m)+L_m(0) = \frac{\sqrt{k}}{\pi}\,\epsilon(\chi)\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)}{n}\cos\left(\frac{2mn\pi}{k}\right) \quad \text{(Case 2)}. \tag{5.6}$$

Clearly, arguing in the same way as in Theorem I, we may produce Abelian forms of the equations (5.3), (5.4), (5.5), (5.6). I shall only need these in the case $s = 0$. In this case, by the argument of Theorem I, we obtain, for real and positive x,

$$\begin{aligned}\frac{1}{2}\chi(m)+\chi(m+1)\,e^{-x}+\chi(m+2)\,e^{-2x}+\ldots \\ = -\frac{i\sqrt{k}}{2\pi}\,\epsilon(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\left\{\frac{\exp(-2mn\pi i/k)}{n-(ikx/2\pi)}-\frac{\exp(2mn\pi i/k)}{n+(ikx/2\pi)}\right\} \quad \text{(Case 1)};\end{aligned} \tag{5.7}$$

$$\begin{aligned}\frac{1}{2}\chi(m)+\chi(m+1)\,e^{-x}+\chi(m+2)\,e^{-2x}+\ldots \\ = \frac{\sqrt{k}}{2\pi}\,\epsilon(\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\left\{\frac{\exp(-2mn\pi i/k)}{n-(ikx/2\pi)}+\frac{\exp(2mn\pi i/k)}{n+(ikx/2\pi)}\right\} \quad \text{(Case 2)}.\end{aligned} \tag{5.8}$$

6. We write

$$L_m(0\,;\,x) = \chi(m+1)e^{-x}+\chi(m+2)e^{-2x}+\ldots+\chi(m+\nu)e^{-\nu x}+\ldots.$$

Since, for fixed k,

$$\frac{1}{n\pm(ikx/2\pi)} = \frac{1}{n}+O\left(\frac{x}{n^2}\right),$$

we see at once from the equations (5.7), (5.8), that

$$\frac{1}{2}\chi(m)+L_m(0\,;\,x) = \frac{1}{2}\chi(m)+L_m(0)+O\left(x\sum_{n=1}^{\infty}\frac{1}{n^2}\right).$$

It follows that

$$\begin{aligned}L_m(0) &= \lim_{x\to 0}L_m(0\,;\,x) = \lim_{x\to 0}\{\chi(m+1)\,e^{-x}+\chi(m+2)\,e^{-2x}+\ldots\} \\ &= \lim_{x\to 0}\{\chi(m+1)\,e^{-(m+1)x}+\chi(m+2)\,e^{-(m+2)x}+\ldots\} \\ &= \lim_{x\to 0}\{L(0\,;\,x)-\sum_{n=1}^{m}\chi(n)\,e^{-nx}\} = L(0)-\sum_{n=1}^{m}\chi(n).\end{aligned}$$

Write $N = kx/(2\pi)$. Substituting in the equation (5.7) we obtain, in Case 1,

$$L_m(0;x)-L_m(0)$$

$$= -\frac{i\sqrt{k}}{2\pi}\epsilon(\chi)\sum_{1\le n<N}\bar{\chi}(n)\left\{\frac{\exp(-2mn\pi i/k)}{n-(ikx/2\pi)}-\frac{\exp(2mn\pi i/k)}{n+(ikx/2\pi)}\right.$$

$$\left.-i\frac{\sin(2mn\pi/k)}{n}\right\}+O(\sqrt{k})\sum_{n\ge N}\frac{N}{n^2}$$

$$= O(\sqrt{k}\,|\log kx\,|)+O(\sqrt{k}).$$

Similarly, in Case 2, $L_m(0;x)-L_m(0) = O(\sqrt{k}\,|\log kx\,|)+O(\sqrt{k})$.

Now, if $\sigma > 0$,

$$L(s)-\sum_1^\lambda \chi(n)\,n^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}\left\{\chi(\lambda+1)\,e^{-(\lambda+1)u}+\chi(\lambda+2)\,e^{-(\lambda+2)u}+\ldots\right\}du$$

$$= \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-\lambda u}\,L_\lambda(0;u)\,du$$

$$= \frac{1}{\Gamma(s)}\int_0^\infty u^{s-1}e^{-\lambda u}\,L_\lambda(0)\,du+O(\sqrt{k})\int_0^\infty |\,u^{s-1}e^{-\lambda u}\,|\,du$$

$$+O(\sqrt{k})\int_0^\infty |\,u^{s-1}e^{-\lambda u}\log ku\,|\,du.$$

If $\lambda > \epsilon k$, where ϵ is some positive constant, we obtain, as before,

$$L(s) = \sum_1^\lambda \chi(n)\,n^{-s}+\frac{L_\lambda(0)}{\lambda^s}+O\left(\frac{\sqrt{k}}{\lambda^\sigma}\right).$$

7. I shall now prove an "Ω-result" for the order of the approximation of the finite sum

$$\sum_1^\lambda \chi(n)\,n^{-s}$$

to the function $L(s,\chi)$. I show that, if k is a prime, s is a positive real number, and $\lambda > \epsilon k$ for some positive number ϵ, then

$$\sum_\chi \left|\,L(s,\chi)-\sum_1^\lambda \chi(n)\,n^{-s}\,\right|^2 > Ck^2\lambda^{-2s}.$$

The summation is over all the χ's except the principal character, and C is some positive constant. We see from this result that no better result is

possible than

$$\left| L(s) - \sum_{1}^{\lambda} \chi(n) n^{-s} \right| = O\left(\frac{\sqrt{k}}{\lambda^{\sigma}}\right).$$

We write $\phi(n, m)$ for the function defined by the following rules:

$$\phi(n, m) = -1 \quad \text{if } (k, n) = (k, m) = (k, n-m) = 1;$$

$$\phi(n, m) = k-2 \quad \text{if } (k, n) = (k, m) = 1, \ (k, n-m) = k;$$

$$\phi(n, m) = 0 \quad \text{if } (k, n) = k \text{ or } (k, m) = k.$$

Here (p, q) denotes the highest common factor of p, q. We observe that

$$\sum_{\chi} \chi(n) \bar{\chi}(m) = \phi(n, m),$$

where the summation is over all the χ's except the principal character.

Now let $s = \sigma + it$ and $s' = \sigma' + it'$ be two complex numbers with $\sigma > 1$, $\sigma' > 1$. Since the series

$$\sum_{\lambda+1}^{\infty} \chi(n) n^{-s}, \quad \sum_{\lambda+1}^{\infty} \bar{\chi}(n) n^{-s'}$$

are absolutely convergent we may multiply them together formally. Thus we obtain

$$(7.1) \quad \sum_{\chi} \left\{ L(s, \chi) - \sum_{1}^{\lambda} \chi(n) n^{-s} \right\} \left\{ L(s', \chi) - \sum_{1}^{\lambda} \bar{\chi}(n) n^{-s'} \right\}$$

$$= \sum_{\chi} \left\{ \sum_{\lambda+1}^{\infty} \chi(n) n^{-s} \right\} \left\{ \sum_{\lambda+1}^{\infty} \bar{\chi}(m) m^{-s'} \right\}$$

$$= \sum_{n=\lambda+1}^{\infty} \sum_{m=\lambda+1}^{\infty} n^{-s} m^{-s'} \left\{ \sum_{\chi} \chi(n) \bar{\chi}(m) \right\}$$

$$= \sum_{n=\lambda+1}^{\infty} \sum_{m=\lambda+1}^{\infty} n^{-s} m^{-s'} \phi(n, m).$$

We write $$\Sigma\{\phi(\nu, \mu)\} = \sum_{n=\lambda+1}^{\nu} \sum_{m=\lambda+1}^{\mu} \phi(n, m),$$

$$\Delta(n^{-s} m^{-s'}) = n^{-s} m^{-s'} - (n+1)^{-s} m^{-s'} - n^{-s} (m+1)^{-s'} + (n+1)^{-s} (m+1)^{-s'}.$$

Using Abel's transformation, we see that the expression (7.1) may be written

$$(7.2) \quad \sum_{n=\lambda+1}^{\infty} \sum_{m=\lambda+1}^{\infty} [\Sigma\{\phi(n, m)\} \Delta(n^{-s} m^{-s'})].$$

Now $\Sigma\{\phi(n, m)\}$ is always positive or zero, and it is periodic in both variables with period k. In the square

$$\lambda+1 \leqslant n \leqslant \lambda+k;\ \lambda+1 \leqslant m \leqslant \lambda+k,$$

we have

$$\Sigma\{\phi(n, m)\} = k \text{ Min } \{n-\lambda, m-\lambda\}-(n-\lambda)(m-\lambda)+O(k).$$

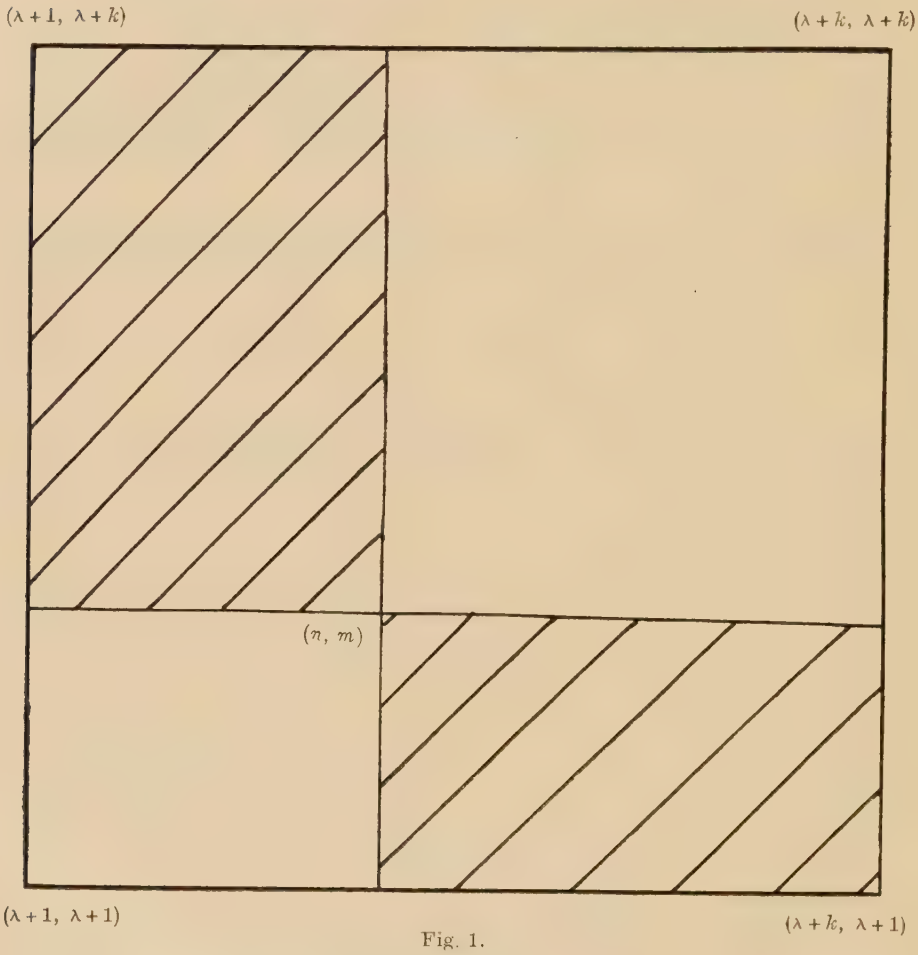


Fig. 1.

If A denote the area of the smaller of the shaded rectangles in Fig. 1, then

$$c_1 A \leqslant \Sigma\{\phi(n, m)\} \leqslant c_2 A, \tag{7.3}$$

where the numbers c_1, c_2 are positive constants. Now since

$$\Delta(n^{-s}m^{-s'}) \sim ss' n^{-s-1}m^{-s'-1},$$

and $\Sigma\{\phi(n, m)\}$ is bounded, we see that the series (7.2) is absolutely convergent for all s, s' with $\sigma > 0$, $\sigma' > 0$. Thus by analytic continuation we see that, for all such s, s', we have

$$(7.4)\quad \sum_{\chi}\left\{L(s,\chi)-\sum_{1}^{\lambda}\chi(n)n^{-s}\right\}\left\{L(s',\bar{\chi})-\sum_{1}^{\lambda}\bar{\chi}(n)n^{-s}\right\} = \sum_{n=\lambda+1}^{\infty}\sum_{m=\lambda+1}^{\infty}[\Sigma\{\phi(n,m)\}\Delta(n^{-s}m^{-s'})].$$

Writing $\sigma = \sigma' > 0$, $t = t' = 0$ in (7.4) and using the inequalities (7.3), it is not difficult to see that

$$c_3 k^2\lambda^{-2s} \leqslant \sum_{\chi}|L(s,\chi)-\sum_{1}^{\lambda}\chi(n)n^{-s}|^2 \leqslant c_4 k^2\lambda^{-2s},$$

where the numbers c_3, c_4 are positive constants.

8. THEOREM II†. *If $\sigma > \frac{1}{2}$, $\epsilon > 0$, $s \neq 1$, then*

$$(8.1)\quad \sum_{\chi}|L(s,\chi)|^2 = hL(2\sigma,\chi_1)+A(s)h^2k^{-2\sigma}+O(k^{1-\sigma+\epsilon}).$$

If $\sigma = \frac{1}{2}$, $\epsilon > 0$, then

$$(8.2)\quad \sum_{\chi}|L(s,\chi)|^2 = \frac{h^2}{k}\log k+\frac{h^2}{k}\sum_{p\,|\,k}\frac{\log p}{p-1}+A(s)\frac{h^2}{k}+O(k^{\frac{1}{2}+\epsilon}).$$

The summation on the left-hand side is in each case over all the χ's corresponding to a given value of k, except the principal character. The summation on the right-hand side of (8.2) *is over all the prime divisors of k. The numbers A depend only on s.*

We write $$\phi(n, m) = \sum_{\chi}\chi(n)\bar{\chi}(m).$$

Then
$$\phi(n, m) = -1 \quad \text{if} \quad (k, n) = (k, m) = 1,\ k \nmid (n-m);$$
$$\phi(n, m) = h-1 \quad \text{if} \quad (k, n) = (k, m) = 1,\ k\,|\,(n-m);$$
$$\phi(n, m) = 0 \quad \text{if} \quad (k, n) > 1 \text{ or } (k, m) > 1.$$

Here (p, q) denotes the highest common factor of p and q; $k\,|\,(n-m)$

† The restriction $s \neq 1$ is inessential, but is included to avoid complexity of detail.

denotes that k is a divisor of $n-m$; $k \nmid (n-m)$ denotes the negation of $k\ (n-m)$.

The series $$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{-s}m^{-\bar{s}}\phi(n, m)$$

is conditionally convergent for $\sigma > 0$. Using Abel's transformation we see that its sum is the same as that of the series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Delta(n^{-s}m^{-\bar{s}})\,\Phi(n, m), \tag{8.21}$$

where $\Phi(n, m)$ denotes

$$\sum_{\nu=1}^{n}\sum_{\mu=1}^{m} \phi(\nu, \mu).$$

Since $\Phi(n, m)$ is bounded we see that the series (8.21) is absolutely convergent for $\sigma > 0$, and using the argument of the last paragraph, we see that it converges to the sum

$$\sum_{\chi} |L(s, \chi)|^2.$$

It follows that

$$\sum_{\chi} |L(s, \chi)|^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{-s}m^{-\bar{s}}\phi(n, m).$$

We split the sum up into four parts

$$\sum_{n=1}^{k}\sum_{m=1}^{k} + \sum_{n=1}^{\infty}\sum_{m=k+1}^{\infty} + \sum_{n=k+1}^{\infty}\sum_{m=1}^{\infty} - \sum_{n=k+1}^{\infty}\sum_{m=k+1}^{\infty} = \Sigma_1+\Sigma_2+\Sigma_3-\Sigma_4.$$

$\Phi(n, m)$ is periodic in both variables with period k. We write

$$n' = n-\left[\frac{n}{p_1}\right]-\left[\frac{n}{p_2}\right]-\ldots+\left[\frac{n}{p_1p_2}\right]+\ldots-\left[\frac{n}{p_1p_2p_3}\right]-\ldots, \tag{8.3}$$

where $p_1, p_2, p_3, \ldots$ are the prime factors of k, and the square brackets denote that only the integral part of the function in brackets is considered. If $n \leqslant k$, $m \leqslant k$, then in the summation

$$\sum_{\nu=1}^{n}\sum_{\mu=1}^{m} \phi(\nu, \mu)$$

the number of non-zero elements is $n'm'$. The number of elements whose

value is $h-1$ is Min (n', m'). Thus we see that, in the square $1 \leqslant n \leqslant k$, $1 \leqslant m \leqslant k$,

$$\Phi(n, m) = \text{Min}\{n'(h-m'),\ m'(h-n')\}.$$

Let d be the number of prime factors of k. Then $k \geqslant d!$. The number of terms in the sum (8.3) is $2^d = O(k^\epsilon)$. It follows that

$$n' = n - \frac{n}{p_1} - \frac{n}{p_2} - \ldots + \frac{n}{p_1 p_2} + \ldots + O(k^\epsilon) = \frac{hn}{k} + O(k^\epsilon);$$

$$m' = \frac{hm}{k} + O(k^\epsilon).$$

Hence, if $1 \leqslant n \leqslant k$, $1 \leqslant m \leqslant k$,

$$\Phi(n, m) = \frac{h^2}{k^2} \text{Min}\{n(k-m),\ m(k-n)\} + O(k^{1+\epsilon}). \tag{8.4}$$

We write $\Phi'(x, y) = (h^2/k^2)\,\text{Min}\{x(k-y),\ y(k-x)\}$

$$(0 \leqslant x \leqslant k,\ \ 0 \leqslant y \leqslant k),$$

$$\Delta(n^{-s} m^{-\bar{s}}) = n^{-s} m^{-\bar{s}} - (n+1)^{-s} m^{-\bar{s}} - n^{-s} (m+1)^{-\bar{s}} + (n+1)^{-s} (m+1)^{-\bar{s}}.$$

Elsewhere we define $\Phi'(x, y)$ to be periodic in both variables with period k.

We obtain now, by Abel's transformation,

$$\Sigma_2 = \sum_{n=1}^{\infty} \sum_{m=k+1}^{\infty} \Phi(n, m)\, \Delta(n^{-s} m^{-\bar{s}}).$$

Since $\Delta(n^{-s} m^{-\bar{s}}) = O(n^{-1-\sigma} m^{-1-\sigma})$, we obtain from (8.4)

$$\Sigma_2 = \sum_{n=1}^{\infty} \sum_{m=k+1}^{\infty} \Phi'(n, m)\, \Delta(n^{-s} m^{-\bar{s}}) + O(k^{1+\epsilon}) \sum_{n=1}^{\infty} \sum_{m=k+1}^{\infty} (nm)^{-1-\sigma}$$

$$= \sum_{n=1}^{\infty} \sum_{m=k+1}^{\infty} \Phi'(n, m)\, \Delta(n^{-s} m^{-\bar{s}}) + O(k^{1-\sigma+\epsilon})$$

$$= \sum_{n=1}^{\infty} \sum_{m=k}^{\infty} \Phi'(n, m)\, \Delta(n^{-s} m^{-\bar{s}}) + O(k^{1-\sigma+\epsilon}).$$

Now $$\Delta(n^{-s} m^{-\bar{s}}) = |s|^2 \int_{x=n}^{n+1} \int_{y=m}^{m+1} x^{-s-1} y^{-\bar{s}-1}\, dx\, dy.$$

Also if $n \leqslant x \leqslant n+1$, $m \leqslant y \leqslant m+1$, then

$$|\Phi'(x, y) - \Phi'(n, m)| \leqslant \text{Max}\left|\frac{\partial}{\partial x} \Phi'(x, y)\right| + \text{Max}\left|\frac{\partial}{\partial y} \Phi'(x, y)\right| < 2k.$$

It follows that

$$\Phi'(n,m)\int_{x=n}^{n+1}\int_{y=m}^{m+1}x^{-s-1}y^{-\bar{s}-1}\,dx\,dy$$

$$=\int_{x=n}^{n+1}\int_{y=m}^{m+1}\Phi'(x,y)\,x^{-s-1}y^{-\bar{s}-1}\,dx\,dy+O(k)\{(n+1)(m+1)\}^{-1-\sigma}.$$

Hence

$$\Sigma_2=|s|^2\int_{x=1}^{\infty}\int_{y=k}^{\infty}\Phi'(x,y)\,x^{-s-1}y^{-\bar{s}-1}\,dx\,dy+O(k^{1-\sigma+\epsilon})$$

$$+O(k)\sum_{n=1}^{\infty}\sum_{m=k+1}^{\infty}(nm)^{-1-\sigma}$$

$$=|s|^2\int_{x=0}^{\infty}\int_{y=k}^{\infty}\Phi'(x,y)\,x^{-s-1}y^{-\bar{s}-1}\,dx\,dy+O(k^{1-\sigma+\epsilon}).$$

Hence $$\Sigma_2=A_1(s)\,h^2k^{-2\sigma}+O(k^{1-\sigma+\epsilon}),$$

where A_1 is a number depending only on s. Similar analysis applies to Σ_3, Σ_4, and we show that

$$\Sigma_2+\Sigma_3-\Sigma_4=A_2(s)\,h^2k^{-2\sigma}+O(k^{1-\sigma+\epsilon}), \tag{8.5}$$

where A_2 depends only upon s.

9. We now write $k_{\alpha\beta\gamma\ldots}$ for $k/(p_\alpha p_\beta p_\gamma\ldots)$. We see that

$$\Sigma_1=h\sum_{n=1}^{k-1}\chi_1(n)\,n^{-2\sigma}-\left|\sum_{n=1}^{k}\chi_1(n)\,n^{-s}\right|^2. \tag{9.1}$$

Now

$$\sum_1^{k-1}\chi_1(n)\,n^{-s}=\sum_1^{k-1}n^{-s}-p_1^{-s}\sum_1^{k_1}n^{-s}-p_2^{-s}\sum_1^{k_2}n^{-s}-\ldots+(p_1p_2)^{-s}\sum_1^{k_{12}}n^{-s}+\ldots.$$

Since, for $s\neq 1$, $$(n_1)^{-s}\sum_1^{k/n_1}n^{-s}=A_3k^{1-s}/n_1+O(1),$$

where A_3 is a number depending only upon s, we see that

$$\sum_1^{k}\chi_1(n)\,n^{-s}=A_3\frac{h}{k}\,k^{1-s}+O(k^{\epsilon}).$$

Hence

$$\left|\sum_{n=1}^{k}\chi_1(n)\,n^{-s}\right|^2=A_3^2h^2k^{-2\sigma}+O(k^{1-\sigma+\epsilon}). \tag{9.2}$$

Suppose that $\sigma > \frac{1}{2}$; then

$$\sum_{n=1}^{k} \chi_1(n)\, n^{-2\sigma} = L(2\sigma, \chi_1) - \sum_{k+1}^{\infty} \chi_1(n)\, n^{-2\sigma}. \tag{9.3}$$

Also $$\sum_{k+1}^{\infty} \chi_1(n)\, n^{-2\sigma} = \sum_{k+1}^{\infty} n^{-2\sigma} - (p_1)^{-2\sigma} \sum_{k_1+1}^{\infty} n^{-2\sigma}$$

$$- (p_2)^{-2\sigma} \sum_{k_2+1}^{\infty} n^{-2\sigma} - \ldots + (p_1 p_2)^{-2\sigma} \sum_{k_{12}+1}^{\infty} n^{-2\sigma} + \ldots.$$

Since $$(n_1)^{-2\sigma} \sum_{k \mid n_1+1}^{\infty} n^{-2\sigma} = A_4 k^{1-2\sigma}/n_1 + O(k^{-2\sigma}),$$

where A_4 is dependent only upon s, we obtain

$$\sum_{k+1}^{\infty} \chi_1(n)\, n^{-2\sigma} = A_4 h k^{-2\sigma} + O(k^{\epsilon-2\sigma}).$$

Combining this with (9.1), (9.2), (9.3), we get

$$\Sigma_1 = hL(2\sigma, \chi_1) - (A_3^2 + A_4)\, h^2 k^{-2\sigma} + O(k^{1-\sigma+\epsilon}).$$

This, with (8.5), proves the theorem for $\sigma > \frac{1}{2}$.

If $\sigma = \frac{1}{2}$, then we consider

$$\sum_{n=1}^{k-1} \chi_1(n)\, n^{-1} = \sum_{1}^{k} n^{-1} - p_1^{-1} \sum_{1}^{k_1} n^{-1} - p_2^{-1} \sum_{1}^{k_2} n^{-1} - \ldots + (p_1 p_2)^{-1} \sum_{1}^{k_{12}} n^{-1} + \ldots.$$

Now

$$\frac{1}{n_1} \sum_{1}^{k \mid n_1} n^{-1} = \frac{1}{n_1} \left\{ \log\left(\frac{k}{n_1}\right) + A_5 \right\} + O(k^{-1}), \tag{9.31}$$

where A_5 is an absolute constant. We see, with a little algebra, that if we expand

$$\sum_{1}^{k} \chi_1(n)\, n^{-1}$$

by means of the equation (9.31), then the coefficient of $\log p_1$ is

$$\frac{1}{p_1}\left\{1 - \frac{1}{p_2} - \frac{1}{p_3} - \ldots + \frac{1}{p_2 p_3} + \ldots\right\} = \frac{1}{p_1} \frac{p_1}{p_1 - 1} \frac{h}{k} = \frac{1}{p_1 - 1} \frac{h}{k}.$$

It follows that

$$\sum_{n-1}^{k-1} \chi_1(n)\, n^{-1} = \frac{h}{k} \log k + \frac{h}{k} \sum_{p \mid k} \frac{\log p}{p-1} + A_5 \frac{h}{k} + O(k^{\epsilon-1}).$$

Thus we see that

$$\Sigma_1 = \frac{h^2}{k}\log k + \frac{h^2}{k}\sum_{p\mid k}\frac{\log p}{p-1} + A(s)\frac{h^2}{k} + O(k^{\frac{1}{2}+\epsilon}).$$

This, with (8.5), proves the theorem for $\sigma = \frac{1}{2}$.

THEOREM II′. *Let* $s = \sigma + it$, $s' = \sigma' + it'$, $0 < \sigma < 1$, $0 < \sigma' < 1$. *Then, if* $s + s' \neq 1$,

$$\sum_\chi L(s,\chi)\,L(s',\bar\chi) = hL(s+s',\chi_1) + A(s,s')\,h^2k^{-s-s'} + O(k^{1-\sigma+\epsilon} + k^{1-\sigma'+\epsilon}).$$

If $s + s' = 1$, *then*

$$\sum_\chi L(s,\chi)\,L(s',\bar\chi) = \frac{h^2}{k}\log k + \frac{h^2}{k}\sum_{p\mid k}\frac{\log p}{p-1} + A(s,s')\frac{h^2}{k} + O(k^{1-\sigma+\epsilon} + k^{1-\sigma'+\epsilon}).$$

The summation on the left-hand side is in each case over all the χ*'s corresponding to a given value of* k, *except the principal character. The summation on the right-hand side of the second equation is over all the prime divisors of* k. *The constants* A *are dependent only on* s, s'.

As in Theorem II we have

$$\begin{aligned}\sum_\chi L(s,\chi)\,L(s',\bar\chi) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n^{-s}m^{-s}\,\phi(n,m) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Phi(n,m)\,\Delta(n^{-s}m^{-s'})\\ &= \sum_{n=1}^{k}\sum_{m=1}^{k} + \sum_{n=1}^{\infty}\sum_{m=k+1}^{\infty} + \sum_{n=k+1}^{\infty}\sum_{m=1}^{\infty} - \sum_{n=k+1}^{\infty}\sum_{m=k+1}^{\infty}\\ &= \Sigma_1 + \Sigma_2 + \Sigma_3 - \Sigma_4.\end{aligned}$$

We may show, as in the last theorem, that

$$\Sigma_2 = A_1(s,s')\,h^2k^{-s-s} + O(k^{1-\sigma+\epsilon} + k^{1-\sigma'+\epsilon}),$$

with similar results for Σ_3, Σ_4. Now

$$\Sigma_1 = h\sum_{n=1}^{k-1}\chi_1(n)\,n^{-s-s'} - \sum_{n=1}^{k-1}\chi_1(n)\,n^{-s}\sum_{n=1}^{k-1}\chi_1(n)\,n^{-s'}. \tag{9.4}$$

As in the last theorem, we may show that, since

$$\sum_{n=1}^{k-1}\chi_1(n)\,n^{-s} = A(s)\frac{h}{k}\,k^{1-s} + O(k^{\epsilon}),$$

the second term of (9.4) is

$$A_2(s,s')\,h^2k^{-s-s'} + O(k^{1-\sigma+\epsilon} + k^{1-\sigma'+\epsilon}).$$

It remains to consider the term

$$h \sum_{n=1}^{k-1} \chi_1(n)\, n^{-s-s'}.$$

We have already shown that, if $s+s'=1$, then this term is

$$\frac{h^2}{k}\log k + \frac{h^2}{k}\sum_{p\mid k}\frac{\log p}{p-1} + A_3(s, s')\frac{h^2}{k} + O(k^{\epsilon-1}),$$

and from this we may prove the second half of the theorem.

We now need the following lemma.

LEMMA. *Let* $z = x+iy$, $x > 0$, $z \neq 1$. *Then*

$$\sum_{n=1}^{m} n^{-z} = \frac{m^{1-z}}{1-z} + \zeta(z) + O(m^{-x}).$$

We have, since $z \neq 1$,

$$\sum_{n=1}^{m} n^{-z} - \frac{m^{1-z}}{1-z} = 1 - \frac{1}{1-z} + \int_1^m \left\{ \{[\xi]+1\}^{-z} - \xi^{-z} \right\} d\xi,$$

where the square brackets denote that we take the integral part of ξ. Since the integral is convergent at infinity,

$$\lim_{m\to\infty}\left\{ \sum_{n=1}^{m} n^{-z} - \frac{m^{1-z}}{1-z} \right\} = 1 - \frac{1}{1-z} + \int_1^{\infty} \left\{ \{[\xi]+1\}^{-z} - \xi^{-z} \right\} d\xi.$$

The expression on the right-hand side represents a function regular everywhere to the right of $x = 0$, except possibly at $z = 1$. If $x > 1$, it is easy to see that the right-hand is equal to $\zeta(z)$. It follows that, if $x > 0$, $z \neq 1$,

$$\lim_{m\to\infty}\left\{ \sum_{n=1}^{m} n^{-z} - \frac{m^{1-z}}{1-z} \right\} = \zeta(z).$$

Hence
$$\sum_{n=1}^{m} n^{-z} - \frac{m^{1-z}}{1-z} - \zeta(z) = \int_m^{\infty} \left\{ \{[\xi]-1\}^{-z} - \xi^{-z} \right\} d\xi.$$

Since the integrand is $O(\xi^{-x-1})$, we obtain the result of the lemma.

Now
$$\sum_{n=1}^{k-1} \chi_1(n)\, n^{-s-s'} = \sum_{n=1}^{k} n^{-s-s'} - \frac{1}{p_1^{s+s'}}\sum_{1}^{k_1} n^{-s-s'} - \frac{1}{p_2^{s+s'}}\sum_{1}^{k_2} n^{-s-s'}$$

$$- \ldots + \frac{1}{(p_1 p_2)^{s+s'}}\sum_{1}^{k_{12}} n^{-s-s'} + \ldots$$

where $p_1, p_2, \ldots,$ are the prime divisors of k. Thus, using the lemma, we obtain

$$\sum_{n=1}^{k-1} \chi_1(n)\, n^{-s-s'} = \frac{k^{1-s-s'}}{1-s-s'} + \zeta(s+s') + O(k^{-\sigma-\sigma'})$$

$$- \frac{1}{(p_1)^{s+s'}} \left\{ \frac{k_1^{1-s-s'}}{1-s-s'} + \zeta(s+s') + O(k_1^{-\sigma-\sigma'}) \right\}$$

$$- \ldots + \frac{1}{(p_1 p_2)^{s+s'}} \left\{ \frac{k_{12}^{1-s-s'}}{1-s-s'} + \zeta(s+s') + O(k_{12}^{-\sigma-\sigma'}) \right\} + \ldots$$

$$= \frac{k^{1-s-s'}}{1-s-s'} \left\{ 1 - \frac{1}{p_1} - \frac{1}{p_2} - \ldots + \frac{1}{p_1 p_2} + \ldots \right\}$$

$$+ \zeta(s+s') \left\{ 1 - \frac{1}{p_1^{s+s'}} - \frac{1}{p_2^{s+s'}} - \ldots + \frac{1}{(p_1 p_2)^{s+s'}} + \ldots \right\} + O(k^{\epsilon-\sigma-\sigma'}).$$

Now clearly, for $\sigma+\sigma' > 1$,

$$\zeta(s+s') \prod_{p \mid k} \left(1 - \frac{1}{p^{s+s'}}\right) = L(s+s', \chi_1), \tag{9.5}$$

and hence

$$\frac{\zeta(s+s')}{L(s+s', \chi_1)} = \left\{ \prod_{p \mid k} \left(1 - \frac{1}{p^{s+s'}}\right) \right\}^{-1}.$$

Both sides are functions regular all over the plane, and it follows that (9.5) is true everywhere. Hence

$$\sum_{n=1}^{k-1} \chi_1(n) n^{-s-s'} = L(s+s', \chi_1) + A_4(s, s')\, h k^{-s-s'} + O(k^{\epsilon-\sigma-\sigma'}),$$

from which we obtain the required theorem.

By considering the product of the series

$$\Sigma \chi(n) \log^{\mu} n\, n^{-s}, \quad \Sigma \bar{\chi}(n) \log^{\nu} n\, n^{-s'},$$

we may produce similar approximations for the sum

$$\sum_{\chi} L^{(\mu)}(s, \chi)\, L^{(\nu)}(s', \bar{\chi}).$$

The arguments are on the same lines as those of the last two theorems.

10. THEOREM III. *If* $\sigma > \frac{1}{2}$,

$$\frac{1}{h-1}\sum_{\chi}|L(s,\chi)|^4 \sim \sum_{n=1}^{\infty}\chi_1(n)[d(n)]^2 n^{-2\sigma}, \tag{10.1}$$

where the summation on the left-hand side of (10.1) *is over all the* χ*'s corresponding to a given value of* k *except the principal character, and* $d(n)$ *represents the number of divisors of* n.

We first prove the theorem in the case where k is a prime. Since every character except χ_1 is primitive, we may employ Theorem I. Writing $x = 1/N$ in the equations (2.3), (2.4), we get, for $\sigma < 1$,

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)\,e^{-n/N}n^{-s} + X(s,\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\,a_n n^{s-1}, \tag{10.2}$$

where a_n has the same value for all the primitive characters for which $\chi(-1) = +1$, and has the same value for all the primitive characters for which $\chi(-1) = -1$. In either case we have

$$a_n = 1 + O\left(\frac{nN}{k}\right) \quad \left(n \leqslant \frac{k}{N}\right),$$

$$a_n = O\left(\frac{k}{nN}\right) \quad \left(n \geqslant \frac{k}{N}\right).$$

We change s into $1-s$, $\chi(n)$ into $\bar{\chi}(n)$, and employ the functional equation for $L(s,\chi)$. We thus obtain, from (10.2),

$$L(s,\chi) = \sum_{n=1}^{\infty}\chi(n)\,a_n n^{-s} + X(s,\chi)\sum_{n=1}^{\infty}\bar{\chi}(n)\,e^{-n/N}n^{s-1}, \tag{10.3}$$

valid for $\sigma > 0$. We write $N = k/N = \sqrt{k}$, and consider the first term on the right-hand side of (10.3), namely

$$\sum_{n=1}^{\infty}\chi(n)\,a_n n^{-s} = \sum_{1}^{[N]}\chi(n)\,n^{-s} + \sum_{1}^{[N]}\chi(n)\,n^{-s}(a_n - 1) + \sum_{[N]+1}^{\infty}\chi(n)\,n^{-s}a_n = \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{10.4}$$

We suppose that $\sigma > \frac{1}{2}$, and we consider separately the sets of characters for which $\chi(-1) = \pm 1$. If by $\phi_+(n,m)$, $\phi_-(n,m)$ we denote

$$\sum_{\chi_+}\chi(n)\,\bar{\chi}(m), \quad \sum_{\chi_-}\chi(n)\,\bar{\chi}(m),$$

the sums being over the primitive characters mod (k) for which

$$\chi(-1) = +1 \quad \text{and} \quad \chi(-1) = -1$$

respectively, then

$\phi_+(n, m) = -1$ $\quad (k, n) = 1,\ (k, m) = 1,\ k \nmid (n \sim m),\ k \nmid (n+m)$;

$\phi_+(n, m) = \frac{1}{2}(k-3)$ $\quad (k, n) = 1,\ (k, m) = 1,\ k \mid (n \sim m)$ or $k \mid (n+m)$;

$\phi_+(n, m) = 0$ $\quad (k, n) > 1$ or $(k, m) > 1$;

$\phi_-(n, m) = \frac{1}{2}(k-1)$ $\quad (k, n) = 1,\ (k, m) = 1,\ k \mid (n \sim m)$;

$\phi_-(n, m) = -\frac{1}{2}(k-1)$ $\quad (k, n) = 1,\ (k, m) = 1,\ k \mid (n+m)$;

$\phi_-(n, m) = 0$ $\quad$ otherwise†.

Now
$$\Sigma_2^2 = \sum_1^{[N]^2} \chi(n)\, n^{-s} d'(n)\, \beta(n),$$

where $d'(n)$ denotes the number of ways of expressing n as the product of two factors, both of them less than $[N]$, and $\beta(n)$ is the same for all primitive characters for which $\chi(-1) = +1$, and for all primitive characters for which $\chi(-1) = -1$, while in either case $\beta(n) = O(n/N^2)$. Thus, considering the sum over the primitive characters for which $\chi(-1) = +1$,

$$(10.5) \quad \sum_{\chi_+} \Big| \sum_{n=1}^{[N]} \chi(n)\, n^{-s}(a_n - 1) \Big|^4 = \sum_{n=1}^{[N]^2} \sum_{m=1}^{[N]^2} \{ n^{-s} m^{-\bar{s}}\, d'(n)\, d'(m)\, \beta(n)\, \bar{\beta}(m)\, \phi_+(n, m) \}.$$

Since $d(n) = O(n^\delta)$, the expression (10.5) is

$$(10.6) \quad O(k^{-2}) \sum_{n=1}^{k} \sum_{m=1}^{k} (nm)^{1+\delta-\sigma} |\phi_+(n, m)|,$$

for every positive value of δ. The expression (10.6) is of the form

$$O(k^{-2}) \sum_{n=1}^{k} \sum_{m=1}^{k} (nm)^{1+\delta-\sigma} + O(k^{-1}) \sum_{n=1}^{k} n^{2+2\delta-2\sigma} + O(k^{-1}) \sum_{n=1}^{k-1} n^{1+\delta-\sigma}(k-n)^{1+\delta-\sigma}$$

$$= O(k^{2+2\delta-2\sigma}).$$

† $n \sim m$ means $|n-m|$.

We may prove a similar result for the sum over the characters for which $\chi(-1) = -1$. Combining these two results, we obtain

$$\sum_{\chi} \left| \sum_{n=1}^{[N]} \chi(n) n^{-s} (a_n - 1) \right|^4 = O(k^{2+2\delta-2\sigma}). \tag{10.61}$$

Next $$\Sigma_3^2 = \sum_{([N]+1)^2}^{\infty} \chi(n)\, n^{-s}\, d''(n)\, \gamma(n),$$

where $d''(n)$ denotes the number of ways of expressing n as the product of two factors both greater than $[N]$, and $\gamma_n = O(N^2/n)$ [as before, γ_n is the same for all primitive characters for which $\chi(-1) = +1$ or $\chi(-1) = -1$]. Then

$$\begin{aligned}
&\sum_{\chi+} \left| \sum_{n=[N]+1}^{\infty} \chi(n) n^{-s} a_n \right|^4 \\
&\quad = \sum_{n=([N]+1)^2}^{\infty} \sum_{m=([N]+1)^2}^{\infty} \{ n^{-s} m^{-\bar{s}}\, d''(n)\, d''(m)\, \gamma(n)\, \bar{\gamma}(m)\, \phi_+(n, m) \} \\
&\quad = O(k^2) \sum_{n=k+1}^{\infty} \sum_{m=k+1}^{\infty} (nm)^{-1+\delta-\sigma} |\phi_+(n, m)| \\
&\quad = O(k^2) \sum_{n=k+1}^{\infty} \sum_{m=k+1}^{\infty} (nm)^{-1+\delta-\sigma} + O(k^3) \sum_{\substack{n=k+1 \\ k \mid (n \sim m)}}^{\infty} \sum_{m=k+1}^{\infty} (nm)^{-1+\delta-\sigma} \\
&\qquad\qquad + O(k^3) \sum_{\substack{n=k+1 \\ k \mid (n+m)}}^{\infty} \sum_{m=k+1}^{\infty} (nm)^{-1+\delta-\sigma}.
\end{aligned} \tag{10.7}$$

The first term of (10.7) is clearly $O(k^{2+2\delta-2\sigma})$. For the second term we observe that

$$\sum_{\substack{n=Lk+1 \\ k \mid (n \sim m)}}^{(L+1)k} \sum_{m=Mk+1}^{(M+1)k} \{ (nm)^{-1+\delta-\sigma} \} = O(k) (Lk)^{-1+\delta-\sigma} (Mk)^{-1+\delta-\sigma}.$$

Hence the second term of (10.7) is

$$O(k^{2+2\delta-2\sigma}) \sum_{L=1}^{\infty} \sum_{M=1}^{\infty} \{ (LM)^{-1+\delta-\sigma} \} = O(k^{2+2\delta-2\sigma}).$$

Similarly the third term of (10.7) is of the form $O(k^{2+2\delta-2\sigma})$. We may obtain a similar result for the sum over the primitive characters for which $\chi(-1) = -1$. Combining the two results, we obtain

$$\sum_{\chi}{}' \left| \sum_{n=[N]+1}^{\infty} \chi(n)\, n^{-s} a_n \right|^4 = O(k^{2+2\delta-2\sigma}). \tag{10.8}$$

$$\text{Now} \quad \sum_{\chi}\left|\sum_{n=1}^{[N]}\chi(n)\,n^{-s}\right|^4 = \sum_{n=1}^{[N]^2}\sum_{m=1}^{[N]^2}\{n^{-s}m^{-\bar{s}}\,d'(n)\,d'(m)\,\phi(n,m)\}$$

$$= h\sum_{n=1}^{[N]^2}\chi_1(n)\{d'(n)\}^2 n^{-2\sigma} - \left|\sum_{n=1}^{[N]}\chi_1(n)\,d'(n)\,n^{-s}\right|^2, \tag{10.9}$$

where $\phi(n, m)$ denotes the function defined in Theorem II. The second term of (10.9) is $O(k^{2+\delta-2\sigma})$. We see that

$$\sum_{n=[N]}^{[N]^2}\chi_1(n)\Big(d'(n)\Big)^2 n^{-2\sigma} = O(N^{1+\delta-2\sigma}) = o(1);$$

$$\sum_{n=[N]}^{\infty}\chi_1(n)\Big(d(n)\Big)^2 n^{-2\sigma} = O(N^{1+\delta-2\sigma}) = o(1).$$

Since, for $n < N$, $d'(n) = d(n)$, we see that the first term of (10.9) is

$$\sum_{n=1}^{\infty}\chi_1(n)\Big(d(n)\Big)^2 n^{-2\sigma} + o(1).$$

Hence

$$\sum_{\chi}\left|\sum_{n=1}^{[N]}\chi(n)\,n^{-s}\right|^4 = h\sum_{n=1}^{\infty}\chi_1(n)\Big(d(n)\Big)^2 n^{-2\sigma} + o(h). \tag{10.91}$$

11. We now consider the second term of the expression (10.3). We observe that

$$X(s, \chi) = O(k^{\frac{1}{2}-\sigma}).$$

We write $$\sum_{n=1}^{\infty}\bar{\chi}(n)\,e^{-n/N}\,n^{s-1} = \sum_{n=1}^{[N]} + \sum_{n=[N]+1}^{\infty} = \Sigma_4 + \Sigma_5.$$

In the first place $$\Sigma_4^2 = \sum_{n=1}^{[N]^2}\bar{\chi}(n)\,n^{s-1}\,\eta(n)\,d'(n),$$

where $\eta(n) = O(1)$, and so

$$\sum_{\chi}\left|\sum_{n=1}^{[N]}\bar{\chi}(n)\,e^{-n/N}\,n^{s-1}\right|^4$$

$$= \sum_{n=1}^{[N]^2}\sum_{m=1}^{[N]^2}\{n^{s-1}m^{\bar{s}-1}\,\eta(n)\,\bar{\eta}(m)\,d'(n)\,d'(m)\,\phi(n,m)\}$$

$$= O(k)\sum_{n=1}^{[N]^2} n^{2\sigma-2+\delta} + O(1)\left|\sum_{n=1}^{[N]^2} n^{\sigma-1+\delta}\right|^2 = O(k^{2\sigma+2\delta}). \tag{11.1}$$

Also $$\Sigma_5^2 = \sum_{n=([N]+1)^2}^{\infty}\bar{\chi}(n)\,n^{s-1}\,\theta(n)\,d''(n),$$

where $\theta(n) = O(e^{-\sqrt{n}/N}) = O(N^{2\sigma+2}/n^{\sigma+1})$, and so

$$\sum_{\chi}\left|\sum_{n=[N]+1}^{\infty}\bar{\chi}(n)\,e^{-n/N}\,n^{s-1}\right|^4$$

$$= \sum_{n=([N]+1)^2,\ m=([N]+1)^2}^{\infty}\sum^{\infty}\{n^{s-1}\,m^{\bar{s}-1}\,\theta(n)\,\bar{\theta}(m)\,d''(n)\,d''(m)\,\phi(n,m)\}$$

$$= O(k^{2\sigma+2})\sum_{n=k}^{\infty}\sum_{m=k}^{\infty}\{(nm)^{-2}\,d''(n)\,d''(m)|\,\phi(n,m)\,|\}.$$

This may be shown by the arguments of the last paragraph to be $O(k^{2\sigma+2\delta})$.

We write

$$S_4 = X(s,\chi)\,\Sigma_4; \quad S_5 = X(s,\chi)\,\Sigma_5.$$

Then

$$\sum_{\chi}|S_4|^4 = O(k^{2-4\sigma})\sum_{\chi}|\Sigma_4|^4 = O(k^{2+2\delta-2\sigma}) = o(h); \tag{11.2}$$

$$\sum_{\chi}|S_5|^4 = O(k^{2+2\delta-2\sigma}) = o(h). \tag{11.3}$$

Finally $$L(s,\chi) = \Sigma_1+\Sigma_2+\Sigma_3+S_4+S_5,$$

Writing for simplicity $S_1 = \Sigma_1$, $S_2 = \Sigma_2$, $S_3 = \Sigma_3$, we see from (10.61), (10.8), (10.91), (11.2), and (11.3), that

$$\sum_{\chi}|S_1|^4 = h\sum_{1}^{\infty}\chi_1(n)\Big(d(n)\Big)^2 n^{-2\sigma}+o(h);$$

$$\sum_{\chi}|S_\nu|^4 = o(h) \quad (\nu = 2,\ 3,\ 4,\ 5).$$

Then, by applying Hölder's inequality twice, we may show that

$$\sum_{\chi}(S_\lambda\bar{S}_\mu S_\nu\bar{S}_\rho) \leqslant \{\sum_{\chi}|S_\lambda|^4\}^{\frac{1}{4}}\{\sum_{\chi}|S_\mu|^4\}^{\frac{1}{4}}\{\sum_{\chi}|S_\nu|^4\}^{\frac{1}{4}}\{\sum_{\chi}|S_\rho|^4\}^{\frac{1}{4}} = o(h),$$

unless $\lambda = \mu = \nu = \rho = 1$. Hence $\Sigma|L(s,\chi)|^4$ can be expressed as a finite sum of terms, each of which except the term $\Sigma|S_1|^4$ is $o(h)$. Hence

$$\sum_{\chi}|L(s,\chi)|^4 = \sum_{\chi}|S_1|^4+o(h) = h\sum_{n=1}^{\infty}\chi_1(n)\Big(d(n)\Big)^2 n^{-2\sigma}+o(h),$$

which proves the theorem in the case where k is prime.

12. We now proceed to the proof of the theorem for general values of k. The argument of the last paragraph shows that, if $\chi(n)$ is a primitive character corresponding to k, then, for $\sigma > \frac{1}{2}$,

$$L(s,\chi) = \sum_{n=1}^{[\sqrt{k}]}\chi(n)\,n^{-s}+R,$$

where R can be expressed as a function of χ which, when summed over *all* χ's, gives

$$\sum_{\chi} |R|^4 = O(k^{2+2\delta-2\sigma}). \tag{12.1}$$

A fortiori, if the sum (12.1) is restricted to the primitive characters mod k, we obtain the same result.

Now let μ be any number satisfying the inequalities

$$k^{\frac{1}{\sigma}} \leqslant \mu \leqslant k^{\frac{1}{2}}.$$

Then

$$\sum_{\chi} \left| \sum_{\mu+1}^{[\sqrt{k}]} \chi(n) n^{-s} \right|^4$$

may easily be seen to be

$$O(k^{2+2\delta-2\sigma}) + O(k) \sum_{(\mu+1)^2}^{k} \Big(d(n)\Big)^2 n^{-2\sigma} = O(k\mu^{2+2\delta-4\sigma}).$$

It follows that, if χ is primitive, we may write

$$L(s, \chi) = \sum_{n=1}^{\mu} \chi(n) n^{-s} + \rho',$$

where

$$\Sigma |\rho'|^4 = O\{k^{1+\frac{1}{4}(1+\delta-2\sigma)}\},$$

the last sum being taken over all the primitive characters corresponding to k.

Let $\chi(n)$ be an imprimitive character corresponding to k. Then there exists a divisor K of k, such that

$$L(s, \chi) = L(s, \chi^*) \Pi \{1 - \chi^*(p) p^{-s}\}, \tag{12.2}$$

where $\chi^*(n)$ is a primitive character corresponding to K, and the product on the right-hand side is taken over all the prime divisors of k/K. First let K_ν be some fixed divisor of k, for which $k^{\frac{3}{4}} \leqslant K < k$. We write $\mu = k^{\frac{1}{\sigma}}$, so that

$$K_\nu^{\frac{1}{\sigma}} \leqslant \mu \leqslant K_\nu^{\frac{1}{2}}.$$

We denote by S_ν the set of imprimitive characters χ, which can be expressed in the form (12.2), where χ^* is a character corresponding to K_ν. Then we see that

$$L(s, \chi^*) = \sum_{n=1}^{\mu} \chi^*(n) n^{-s} + R_0, \tag{12.3}$$

where, by (12 . 1),

$$\sum_{S_\nu} |R_0|^4 = O\{K_\nu^{1+\frac{1}{4}(1+\delta-2\sigma)}\}.$$

Let $p_1, p_2, \ldots, p_\lambda$ be the prime factors of k/K_ν. Then

$$\left\{\sum_{n=1}^{\mu} \chi^*(n)\, n^{-s}\right\} \{1-\chi^*(p_1)\, p_1^{-s}\} = \sum_{n=1}^{\mu} \chi^{(1)}(n)\, n^{-s} - \sum_{\substack{\mu+1 \\ p_1 \mid n}}^{p_1\mu} \chi^*(n)\, n^{-s},$$

where $\chi^{(1)}(n)$ is a character corresponding to $K_\nu p_1$, and is defined by the equations

$$\chi^{(1)}(n) = \chi^*(n) \quad (n, p_1) = 1;$$

$$\chi^{(1)}(n) = 0 \qquad (p_1 \mid n).$$

But, since $p_1\mu < \sqrt{k}$, we may show by the argument used several times already that, if R_1 denotes

$$-\sum_{\substack{\mu+1 \\ p_1 \mid n}}^{p_1\mu} \chi^*(n)\, n^{-s},$$

then

$$\sum_{S_\nu} |R_1|^4 = O\{K_\nu^{1+\frac{1}{4}(1+\delta-2\sigma)}\}.$$

We may carry out this process λ times, and obtain

$$\sum_{n=1}^{\mu} \chi^*(n)\, n^{-s} \prod \{1-\chi^*(p)\, p^{-s}\} = \sum_{n=1}^{\mu} \chi(n)\, n^{-s} + \rho,$$

where

$$\rho = \sum_{m=1}^{\lambda} R_m \prod_{r=m+1}^{\lambda} \{1-\chi^*(p_r)\, p_r^{-s}\}.$$

Combining this with (12 . 2) and (12 . 3), we obtain

$$L(s, \chi) = \sum_{n=1}^{\mu} \chi(n)\, n^{-s} + \rho',$$

where

$$\rho' = \sum_{m=0}^{\lambda} R_m \sum_{r=m+1}^{\lambda} \{1-\chi^*(p_r)\, p_r^{-s}\}. \tag{12 . 4}$$

Since $|1-\chi^*(p_r)\, p_r^{-s}| \leqslant 2$, for every r, (12 . 4) gives

$$|\rho'| \leqslant \sum_{m=0}^{\lambda} |R_m| 2^\lambda = O(k^\delta) \sum_{m=0}^{\lambda} |R_m|.$$

It follows that

$$(12.5) \qquad \sum_{S_\nu} |\rho'|^4 = O(k^{4\delta}\lambda^4) \sum_{m=0}^{\lambda} \left\{ \sum_{S_\nu} |R_m|^4 \right\} = O\left\{ k^{5\delta+1+\frac{1}{4}(1+\delta-2\sigma)} \right\}.$$

Now denote by S the set of imprimitive characters for which the corresponding value of K_ν is greater than or equal to $k^{\frac{3}{4}}$. Since the number of divisors of k is $O(k^\delta)$, it follows from (12.5) that

$$(12.6) \qquad \sum_{S} |\rho'|^4 = O\left\{ k^{6\delta+1+\frac{1}{4}(1+\delta-2\sigma)} \right\},$$

and if δ is suitably chosen the expression (12.6) is $o(h)$.

We now consider the characters for which the corresponding value of K is less than $k^{\frac{3}{4}}$. Let K'_ν ($< k^{\frac{3}{4}}$) be a fixed factor of k, and let T_ν denote the set of characters for which χ^* is a character corresponding to K'_ν. Then

$$\sum_{T_\nu} |L(s, \chi^*)|^4 = O(K'_\nu) = O(k^{\frac{3}{4}}).$$

It follows that

$$\sum_{T_\nu} |L(s, \chi)|^4 = O(2^{4d} k^{\frac{3}{4}}),$$

where $d = o(\log k)$ denotes the number of prime divisors of k. We denote by T the set of imprimitive characters for which the corresponding value of K is less than $k^{\frac{3}{4}}$. Then we see that

$$(12.7) \qquad \sum_{T} |L(s, \chi)|^4 = O(k^\delta)(2^{4d} k^{\frac{3}{4}}) = O(k^{\frac{3}{4}+2\delta}).$$

Also, for any character whatever,

$$\sum_{n=1}^{\mu} \chi(n) n^{-s} = O(\mu^{1-\sigma}) = O\left\{ k^{\frac{1}{4}(1-\sigma)} \right\}.$$

Since the number of characters belonging to T is $O(k^{\frac{3}{4}+\delta})$, the last result gives

$$\sum_{T} \left| \sum_{n=1}^{\mu} \chi(n) n^{-s} \right|^4 = O\left\{ k^{\frac{3}{4}+\delta+\frac{1}{2}(1-\sigma)} \right\}.$$

Combining this result with (12.7), we see that if, as previously, ρ' is defined by the equation

$$(12.8) \qquad L(s, \chi) = \sum_{n=1}^{\mu} \chi(n) n^{-s} + \rho',$$

then

$$(12.9) \qquad \sum_{T} |\rho'|^4 = O\left\{ k^{\frac{3}{4}+2\delta+\frac{1}{2}(1-\sigma)} \right\} = o(h),$$

if δ is sufficiently small.

Let W denote the set of primitive characters corresponding to k. We have already seen that, if ρ' is defined by the equation (12.8), then

$$\sum_W |\rho'|^4 = o(h).$$

Combining this with the results (12.6) and (12.9), and continuing the argument as in the proof of the theorem in the case where k is prime, we may prove Theorem III for general values of k.

13. THEOREM IV. *If* $\sigma = \frac{1}{2}$, $\epsilon > 0$, *then*

$$\sum_\chi |L(s, \chi)|^4 = O(k \log^4 k) \quad (k \text{ prime}),$$

$$\sum_\chi |L(s, \chi)|^4 = O(k^{1+\epsilon}) \quad (\text{all } k).$$

The summation on the left-hand side is over all the χ*'s corresponding to* k *except the principal character.*

We consider first the case where k is prime. Again, writing

$$x = \frac{1}{N} = \frac{N}{k} = \frac{1}{\sqrt{k}}$$

in the equations (2.3), (2.4), we obtain

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} a_n + X(s, \chi) \left\{ \sum_{n=1}^{\infty} \bar{\chi}(n) n^{s-1} a'_n \right\}, \tag{13.1}$$

where, as in Theorem III, a_n, a'_n have the same values for the sets of primitive characters for which $\chi(-1) = +1$ and $\chi(-1) = -1$ respectively. Also we have, in both cases,

$$a_n = e^{-n/N} = 1 + O\left(\frac{n}{N}\right) \quad (n \leqslant N),$$

$$a_n = O\left(\frac{N}{n}\right) \quad (n > N),$$

$$a'_n = 1 + O\left(\frac{n}{N}\right) \quad (n \leqslant N),$$

$$a'_n = O\left(\frac{N}{n}\right) \quad (n > N),$$

and $|X(s, \chi)| = 1$. We first need some lemmas.

LEMMA I. $$d(1)+d(2)+\ldots+d(n) \sim n \log n,$$

$$\{d(1)\}^2+\{d(2)\}^2+\ldots+\{d(n)\}^2 \sim \frac{1}{\pi^2} n \log^3 n.$$

The first result is well known. The second is due to Ramanujan†.

LEMMA II.

$$\sum_{m=1}^{n} d(m)\, m^{-\frac{1}{2}} = O(\sqrt{n} \log n), \tag{13.2}$$

$$\sum_{m=n}^{\infty} d(m)\, m^{-\frac{3}{2}} = O(n^{-\frac{1}{2}} \log n), \tag{13.3}$$

$$\sum_{m=1}^{n} \frac{\left(d(m)\right)^2}{m} = O(\log^4 n). \tag{13.4}$$

These results follow from those of Lemma I by partial summation.

LEMMA III.

$$\sum_{m=1}^{n-1} \frac{d(m)\, d(n-m)}{\sqrt{m}\,\sqrt{(n-m)}} = O(\log^4 n). \tag{13.5}$$

If λ *is a positive integer and* $3 \leqslant \lambda \leqslant 2k$, *then*

$$\sum_{n=k+1}^{\lambda k-k-1} \frac{d(n)\, d(\lambda k-n)}{n^{\frac{3}{2}}\,(\lambda k-n)^{\frac{3}{2}}} = O\left(\frac{k^{-2}\log^3 k}{\lambda}\right). \tag{13.6}$$

Using Hölder's inequality we have

$$\sum_{m=1}^{n-1} \frac{d(m)\, d(n-m)}{\sqrt{m}\,\sqrt{(n-m)}} \leqslant \left\{\sum_{m=1}^{n-1} \frac{\left(d(m)\right)^2}{m}\right\}^{\frac{1}{2}} \left\{\sum_{m=1}^{n-1} \frac{\left(d(n-m)\right)^2}{n-m}\right\}^{\frac{1}{2}}$$

$$= \sum_{m=1}^{n-1} \frac{\left(d(m)\right)^2}{m} = O(\log^4 n),$$

by (13.4). This proves (13.5).

Similarly, Hölder's inequality gives

$$\sum_{n=k+1}^{\lambda k-k-1} \frac{d(n)\, d(\lambda k-n)}{n^{\frac{3}{2}}\,(\lambda k-n)^{\frac{3}{2}}} \leqslant \left\{\sum_{n=k+1}^{\lambda k-k-1} \frac{\left(d(n)\right)^2}{n^2(\lambda k-n)}\right\}^{\frac{1}{2}} \left\{\sum_{n=k+1}^{\lambda k-k-1} \frac{\left(d(\lambda k-n)\right)^2}{n(\lambda k-n)^2}\right\}^{\frac{1}{2}}$$

$$= \sum_{n=k+1}^{\lambda k-k-1} \frac{\left(d(n)\right)^2}{n^2(\lambda k-n)}.$$

† Ramanujan, 1.

Using Abel's transformation and Lemma I, this may be seen to be of the form

$$O(\log^3 k)\left\{\sum_{n=k+1}^{\lambda k-k-1}\left(\frac{1}{n(\lambda k-n)^2}+\frac{1}{n^2(\lambda k-n)}\right)+\frac{1}{\lambda k^2}\right\}$$

$$= O(k^{-2}\log^3 k)\left\{\frac{1}{\lambda}+\frac{\log\lambda}{\lambda^2}\right\} = O\left\{\frac{k^{-2}\log^3 k}{\lambda}\right\}.$$

LEMMA IV.

$$\sum_{n=k+1}^{k^2}\frac{\left(d(n)\right)^2}{n^3} = O(k^{-2}\log^3 k). \tag{13.7}$$

If λ *is an integer and* $1 \leqslant \lambda \leqslant k$, *then*

$$\sum_{n=k+1}^{k^2}\frac{d(n)\,d(n+\lambda k)}{n^{\frac{3}{2}}\,(n+\lambda k)^{\frac{3}{2}}} = O\left(\frac{k^{-2}\log^3 k}{\lambda}\right). \tag{13.8}$$

The result (13.7) follows by partial summation from Lemma I. For (13.8) we observe that, by Hölder's inequality, if $M \leqslant k^2$,

$$\sum_{n=k+1}^{M} d(n)\,d(n+\lambda k) \leqslant \left\{\sum_{n=k+1}^{M}\left(d(n)\right)^2\right\}^{\frac{1}{2}}\left\{\sum_{n=k+1}^{M}\left(d(n+\lambda k)\right)^2\right\}^{\frac{1}{2}}$$

$$= O\{\sqrt{M}\sqrt{(M+\lambda k)}\}\log^3 k.$$

The expression on the left-hand side of (13.8) is

$$O(\lambda k)^{-\frac{3}{2}}\sum_{n=k+1}^{k^2}\frac{d(n)\,d(n+\lambda k)}{n^{\frac{3}{2}}}.$$

Using Abel's transformation, this is

$$O(\lambda k)^{-\frac{3}{2}}\left\{\sum_{n=k+1}^{k^2}\left(\frac{\sqrt{n}\sqrt{(n+\lambda k)}\log^3 k}{n^2}\right)+\frac{k^2\log^3 k}{k^3}\right\}$$

$$= O(\lambda k)^{-\frac{3}{2}}\log^3 k\left\{\sum_{n=k+1}^{\infty}\left(\frac{\sqrt{(n+\lambda k)}}{n^2}\right)+k^{-1}\right\}$$

$$= O(\lambda k)^{-\frac{3}{2}}\log^3 k\left\{\frac{\sqrt{(\lambda k)}}{k}+k^{-1}\right\} = O\left(\frac{k^{-2}\log^3 k}{\lambda}\right).$$

14. We now consider the first summation on the right-hand side of (13.1). This may be written

$$(14.1)\qquad \sum_{n=1}^{[N]} \chi(n)\, n^{-s}\, a_n + \sum_{n=[N]+1}^{k} \chi(n)\, n^{-s}\, a_n + \sum_{n=k+1}^{k^2} \chi(n)\, n^{-s}\, a_n$$

$$= \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Since a_n is decreasing and

$$\sum_{n=k+1}^{M} \chi(n) = O(\sqrt{k}\log k),$$

we see by partial summation that $\Sigma_3 = O(\log k)$. We have

$$\Sigma_1^2 = \sum_{1}^{[N]^2} \chi(n)\, n^{-s}\, d'(n)\, \beta(n),$$

where, as in the last theorem, $\beta(n)$ has the same value for the two sets of primitive characters defined by $\chi(-1) = +1$, $\chi(-1) = -1$, and $\beta(n) = O(1)$. Thus we see that, with the notation of Theorem III,

$$\sum_{\chi_+} |\Sigma_1|^4 = \sum_{n=1}^{[N]^2} \sum_{m=1}^{[N]^2} n^{-s} m^{-\bar{s}}\, d'(n)\, d'(m)\, \beta(n)\, \bar{\beta}(m)\, \phi_+(n, m)$$

$$= O(k) \sum_{n=1}^{k-1} \frac{\big(d(n)\big)^2}{n} + O(k) \sum_{n=1}^{k-1} \frac{d(n)\, d(k-n)}{\sqrt{n}\,\sqrt{(k-n)}} + O(1) \left[\sum_{n=1}^{k-1} \frac{d(n)}{\sqrt{n}}\right]^2$$

$$= O(k \log^4 k),$$

by (13.4), (13.5), and (13.2). It follows similarly that

$$\sum_{\chi_-} |\Sigma_1|^4 = O(k \log^4 k); \quad \sum_{\chi} |\Sigma_1|^4 = O(k \log^4 k).$$

$$\Sigma_2^2 = \sum_{n=([N]+1)^2}^{k^2} \chi(n)\, n^{-s}\, d'''(n)\, \gamma(n),$$

where $\gamma(n)$ has the same value for the two sets of primitive characters defined by $\chi(-1) = +1$, $\chi(-1) = -1$, and $\gamma(n) = O(k/n)$, while $d'''(n)$ denotes the number of ways in which n can be expressed as the product of two numbers n_1, n_2 for which $[N]+1 \leqslant n_1 \leqslant k$, $[N]+1 \leqslant n_2 \leqslant k$.

It follows by the now familiar argument that

$$\sum_{\chi_+} |\Sigma_2|^4 = \sum_{n=([N]+1)^2}^{k^2} \sum_{m=([N]+1)^2}^{k^2} \left(n^{-s} m^{-\bar{s}} d'''(n)\, d'''(m)\, \gamma(n)\, \bar{\gamma}(m)\, \phi_+(n, m) \right)$$

$$(14.2) \qquad = O(1) \left[\sum_{n=([N]+1)^2}^{k^2} \{ k n^{-\frac{3}{2}} d(n) \} \right]^2$$

$$+ O(k) \sum_{\substack{n=([N]+1)^2 \\ k \nmid (n \sim m)}}^{k^2} \sum_{m=([N]+1)^2}^{k^2} \{ k^2 (nm)^{-\frac{3}{2}} d(n)\, d(m) \}$$

$$+ O(k) \sum_{\substack{n=([N]+1)^2 \\ k \nmid (n+m)}}^{k^2} \sum_{m=([N]+1)^2}^{k^2} \{ k^2 (nm)^{-\frac{3}{2}} d(n)\, d(m) \}.$$

The first term of (14.2) is $O(k \log^2 k)$ in virtue of (13.3). The second term is

$$O(k^3) \sum_{\lambda=0}^{k-1} \left\{ \sum_{n=k}^{k^2} \{ n(n+\lambda k) \}^{-\frac{3}{2}} d(n)\, d(n+\lambda k) \right\}.$$

In virtue of (13.7) and (13.8), this is

$$O(k \log^3 k) \sum_{\lambda=1}^{k-1} \left(\frac{1}{\lambda} \right) = O(k \log^4 k).$$

The third term of (14.2) is

$$O(k^3) \sum_{\lambda=3}^{2k-1} \left\{ \sum_{n=k+1}^{\lambda k - k + 1} \{ n(\lambda k - n) \}^{-\frac{3}{2}} d(n)\, d(\lambda k - n) \right\}.$$

In virtue of (13.6) this is

$$O(k \log^3 k) \sum_{\lambda=3}^{2k-1} \left(\frac{1}{\lambda} \right) = O(k \log^4 k).$$

Dealing in the same way with the sum over the characters for which $\chi(-1) = -1$, we obtain

$$\sum_{\chi} |\Sigma_2|^4 = O(k \log^4 k).$$

Thus we have shown that

$$\sum_{\chi} |\Sigma_\nu|^4 = O(k \log^4 k) \quad (\nu = 1, 2, 3).$$

Similar analysis is applicable to the second term on the right-hand side

of (13.1). A slight modification is needed in dealing with the term Σ_3. It is no longer true that a'_n is real and decreasing. We need, however, only observe that, since

$$|a'_n - a'_{n+1}| = O\left(\frac{N}{n^2}\right),$$

the variation of a_n in the range (k, ∞) is bounded. Since a'_n is also itself bounded, an application of Abel's transformation and the equation (14.1) shows that $\Sigma_3 = O(\log k)$. Finally, arguing in the same way as in the last theorem, we may prove that

$$\sum_{\chi} |L(s, \chi)|^4 = O(k \log^4 k).$$

This proves the theorem in the case where k is prime.

15. Now consider the case where k is not necessarily a prime. We denote by $\Lambda(s, \chi)$ the function

$$\sum_{n=1}^{\infty} \chi(n) n^{-s} a_n + X(s, \chi) \left\{ \sum_{n=1}^{\infty} \bar{\chi}(n) n^{s-1} a'_n \right\},$$

where a_n, a'_n depend on k and are the same as in equation (13.1). If χ is primitive, then $L(s, \chi) = \Lambda(s, \chi)$. Let K_ν be any factor of k. As in the last theorem we denote by S_ν the set of imprimitive characters χ corresponding to k, for which $L(s, \chi)$ can be expressed in the form (12.2), where χ^* is a primitive character corresponding to K_ν. Then from (12.2) it follows that

$$\sum_{S_\nu} |L(s, \chi)|^4 \leqslant 2^{4d} \sum_{\chi^*} |\Lambda(s, \chi^*)|^4, \tag{15.1}$$

where d denotes the number of prime divisors of k. Since $d = o(\log k)$, the expression (15.1) is $O(k^\epsilon) K_\nu \log^4 K_\nu = O(k^{1+2\epsilon})$, for every positive ϵ. Since the number of divisors of k is $O(k^\epsilon)$, we see that

$$\sum_{S} |L(s, \chi)|^4 = O(k^{1+3\epsilon}),$$

S denoting the set of imprimitive characters corresponding to k. Since the sum of $|L(s, \chi)|^4$, taken over the primitive characters corresponding to k, is of the form $O(k \log^4 k)$, the required result follows immediately.

References.

Hardy and Littlewood, **1**. "The approximate functional equation in the theory of the zeta-function, with applications to the divisor-problems of Dirichlet and Piltz", *Proc. London Math. Soc.* (2), 21 (1921), 39–74.

Ingham, **1**. "Mean-value theorems in the theory of the Riemann zeta-function", *Proc. London Math. Soc.* (2), 27 (1928), 273–300.

Landau, **1**. *Handbuch der Lehre von der Verteilung der Primzahlen*, 1 (1909).

——— **2**. *Vorlesungen über Zahlentheorie*, 2 (1927).

——— **3**. "Uber das Nichtverschwinden der Dirichletschen Reihen, welche komplexen Charakteren entsprechen", *Math. Annalen*, 70 (1911), 69–78.

Lindelöf, **1**. *Le calcul des résidus et ses applications à la théorie des fonctions* (1905).

Littlewood, **1**. "On the class number of the corpus $P(\sqrt{-k})$", *Proc. London Math. Soc.* (2), 27 (1928), 358–372.

Ramanujan, **1**. "Some formulae in the analytic theory of numbers", *Messenger of Math.*, 45 (1916), 81–84.

IDENTICAL RELATIONS SATISFIED IN AN ALGEBRA

By D. E. LITTLEWOOD.

[Received 28 February, 1930.—Read 13 March, 1930.]

1. In all algebras identical relations exist, that is, polynomial equations satisfied by every number in the algebra. The determination of all such expressions is a necessary preliminary to the theory of equations in that algebra*.

Every irreducible algebra with a finite base is the sum of a nilpotent algebra, and the direct product of a matrix algebra and a division algebra†. The only division algebras over the reals are those of the reals, the complex numbers, and quaternions. Avoiding nilpotent algebras, then, which have not yet been classified, we are left with matrices, quaternions, and quaternion matrices.

The identical quadratic relations for quaternions are known*. In this paper we first obtain the identical quadratic relations for matrices. An artifice gives us an alternative method of obtaining the corresponding relations for quaternions. The general bilinear relations are then obtained for these algebras, and the quadratic and bilinear relations in quaternion matrices.

Finally we show that all identical polynomial and polylinear relations can be deduced from the quadratic and bilinear relations, and a method for obtaining the number of such relations for any given degree is given.

2. *Matrix algebras.*

Consider a matrix algebra of order r^2. Denote by e_{ij} the matrix with unity in the i-th row, j-th column, and zero elsewhere.

* A. R. Richardson, "Equations over a division algebra", *Messenger of Math.*, 57 (1928), 1–6.

† G. Frobenius, *Journal für Math.*, 84 (1878), 59.

Let X be the variable matrix $\Sigma x_{ij} e_{ij}$, the variable coefficients x_{ij} being scalar.

The general quadratic in the variable X is

$$\Sigma \lambda_{fgijmn} e_{fg} X e_{ij} X e_{mn}.$$

The term $e_{fg} X e_{ij} X e_{mn}$ represents a matrix with $x_{gi} x_{jm}$ in the f-th row, n-th column, and zero elsewhere. Clearly, since the x's are scalar and commutative,

$$e_{fg} X e_{ij} X e_{mn} = e_{fj} X e_{mg} X e_{in}. \tag{1}$$

There will be $\frac{1}{2} r^4 (r^2 - 1)$ such equations. Apart from these identities, the terms in the general quadratic will either represent different products $x_{gi} x_{jm}$, or these will be in a different position in the matrix. Thus they will be linearly independent over the reals.

The equations (1), then, will include all quadratic identities.

By using the representation of quaternions as two rowed complex matrices, we can obtain from this the quaternion quadratic identities.

Using ϵ for the commutative square root of -1, and i, j, k having their usual significance, there is a matrix representation of quaternions as follows:

$$1 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad i = \begin{vmatrix} \epsilon & 0 \\ 0 & -\epsilon \end{vmatrix}, \quad j = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad k = \begin{vmatrix} 0 & \epsilon \\ \epsilon & 0 \end{vmatrix}.$$

From this we obtain

$$\left.\begin{aligned} 2e_{11} &= 1 - \epsilon i, & 2e_{12} &= j - \epsilon k, \\ 2e_{21} &= -j - \epsilon k, & 2e_{22} &= 1 + \epsilon i \end{aligned}\right\}. \tag{2}$$

Equation (1) gives

$$e_{11} X e_{12} X e_{11} = e_{12} X e_{11} X e_{11},$$

and 23 other similar relations.

Substituting from (2), we obtain

$$(1 - \epsilon i) X (j - \epsilon k) X (1 - \epsilon i) = (j - \epsilon k) X (1 - \epsilon i) X (1 - \epsilon i).$$

The conjugate equation

$$(1 + \epsilon i) X (j + \epsilon k) X (1 + \epsilon i) = (j + \epsilon k) X (1 + \epsilon i) X (1 + \epsilon i)$$

is also true, for it corresponds to the equation

$$-e_{22} X e_{21} X e_{22} = -e_{21} X e_{22} X e_{22}.$$

We can therefore ake the real part, and obtain

$$XjX-iXkX-iXjXi-XkXi = jX^2-kXiX-kX^2i-jXiXi. \qquad (3)$$

This is a quadratic identity. Together with the 23 others obtained similarly, it gives the 24 quadratic identities that are linearly independent over the reals. If we allow an arbitrary quaternion multiplier on the left, there will be six linearly independent ones, all deducible from the form

$$XiXi+XjXj+XkXk = iXiX+jXjX+kXkX^*.$$

Bilinear identities.

If, in addition to $X = \Sigma x_{ij} e_{ij}$, we have a second matrix variable $Y = \Sigma y_{ij} e_{ij}$, then the general bilinear form in the two variables is

$$\Sigma \lambda_{fgijmn} e_{fg} Xe_{ij} Ye_{mn} + \Sigma \mu_{fgijmn} e_{fg} Ye_{ij} Xe_{mn}.$$

$e_{fg} Xe_{ij} Ye_{mn}$ is a matrix with $x_{gi} y_{jm}$ in the f-th row, n-th column, and zero elsewhere.

Clearly, then, the r^6 equations

$$e_{fg} Xe_{ij} Ye_{mn} = e_{fj} Ye_{mg} Xe_{in} \qquad (4)$$

represent all the bilinear relations linearly independent over the reals.

The quaternion bilinear identities can be deduced from this. Using the substitution (2) in the equation

$$e_{11} Xe_{12} Ye_{11} = e_{12} Ye_{11} Xe_{11},$$

we obtain

$$XjY-iXkY-iXjYi-XkYi = jYX-kYiX-kYXi-jYiXi.$$

If we put $X = Y$, we obtain the quadratic identity.

A more comprehensive form is obtained in the following way:

$$\begin{aligned} XY &= (e_{11}+e_{22})\, X(e_{11}+e_{22})\, Y(e_{11}+e_{22}) \\ &= e_{11} Xe_{11} Ye_{11} + e_{11} Xe_{11} Ye_{22} + e_{11} Xe_{22} Ye_{11} \\ &\qquad + e_{22} Xe_{11} Ye_{11} + e_{11} Xe_{22} Ye_{22} + e_{22} Xe_{11} Ye_{22} \\ &\qquad + e_{22} Xe_{22} Ye_{11} + e_{22} Xe_{22} Ye_{22} \\ &= e_{11} Ye_{11} Xe_{11} + e_{11} Ye_{21} Xe_{12} + e_{12} Ye_{11} Xe_{21} \\ &\qquad + e_{21} Ye_{12} Xe_{11} + e_{12} Ye_{21} Xe_{22} + e_{21} Ye_{22} Xe_{12} \\ &\qquad + e_{22} Ye_{12} Xe_{21} + e_{22} Ye_{22} Xe_{22}\,; \end{aligned}$$

* A. R. Richardson, *loc. cit.*

whence

$$\begin{aligned} 4XY = {} & YX - iYXi - jYXj - kYXk - iYiX - YiXi - jYjX \\ & - YjXj - kYkX - YkXk + iYjXk \\ & + jYkXi + kYiXj - iYkXj - kYjXi - jYiXk. \end{aligned} \tag{5}$$

For future reference we write this shortly in the form

$$4XY = \sum_{s=1}^{16} \lambda_s Y \mu_s X \nu_s,$$

where λ_s, μ_s, and ν_s are the appropriate quaternion units, λ_s carrying a minus sign, if necessary, for the s-th term in (5).

4. *Quaternion matrix algebras.*

We shall obtain immediately the bilinear identities. The quadratic identities may be obtained by putting $Y = X$,

$$Y = \Sigma y_{fg} e_{fg}, \qquad X = \Sigma x_{fg} e_{fg}.$$

y_{fg} and x_{fg} are quaternions, and i, j, and k are commutative with e_{fg}.

Since y_{fg} and x_{fg} are not in general commutative, equation (4), namely

$$e_{fg} X e_{pq} Y e_{mn} = e_{fq} Y e_{mg} X e_{pm},$$

is not true. Instead, we must make use of equation (5), and thus obtain

$$4 e_{fg} X e_{pq} Y e_{mn} = \sum_{s=1}^{16} \lambda_s e_{fq} Y \mu_s e_{mg} X \nu_s e_{pn}. \tag{6}$$

λ_s, μ_s, and ν_s have the values stated in Section 3.

This general form cannot be simplified for the quadratic identities except for the special case when $g = q$ and $p = m$. In this case we can use the quaternion quadratic identities rather than the quaternion bilinear identities. For example

$$\begin{aligned} & e_{fg} Xie_{mg} Xie_{mn} - e_{fg} iXe_{mg} iXe_{mn} \\ & + e_{fg} Xje_{mg} Xje_{mn} - e_{fg} jXe_{mg} jXe_{mn} \\ & + e_{fg} Xke_{mg} Xke_{mn} - e_{fg} kXe_{mg} kXe_{mn} = 0 \end{aligned}$$

is the identity corresponding to

$$(Xi)^2-(iX)^2+(Xj)^2-(jX)^2+(Xk)^2-(kX)^2=0$$

in quaternions.

5. *Polynomial and polylinear identities.*

Finally we show that all polynomial and polylinear identities can be deduced from the quadratic and bilinear identities.

For matrix algebras, consider the homogeneous polynomial in X, of degree n,

$$\Sigma\lambda_{\substack{a_0\dots a_n\\ b_0\dots b_n}} e_{a_0b_0}Xe_{a_1b_1}Xe_{a_2b_2}\dots Xe_{a_nb_n}.$$

The term given without the coefficient is equivalent to a matrix with $\Pi x_{b_{s-1}a_s}$ in the a_0-th row, b_n-th column, and zero elsewhere.

The only polynomial identities are obtained by permuting the terms in the product, namely the terms $x_{b_{r-1}a_r}$. Any permutation can be obtained by a series of interchanges of adjacent terms, and the corresponding identities will be left- and right-hand multiples of quadratic identities. Hence the polynomial identities may be obtained from the quadratic identities.

The case of the polylinear identities is similar.

Since complex quaternions form a matrix algebra, the theorem must also be true for complex quaternions and for complex quaternion matrices. The identities in real and complex quaternions will be the same, and our proof is complete.

In the theory of equations, it is useful to know the number of identities of a given degree linearly independent over the reals. We give a method of estimating this.

Consider first matrices of order r^2. Let $\prod_{s=1}^{n} x_{b_{s-1}a_s}$ be any product of n similar or dissimilar constituents of X. If the x's may be permuted in P different ways, to this product will correspond $P-1$ identities. $\Sigma(P-1)$ for all such products will give the number of identities, linearly independent over the algebra. We multiply by r^2 to obtain the number linearly independent over the reals.

Take any partition of n,

$$n=a_1+a_2\dots+a_s.$$

Of the numbers $a_1, a_2, \dots, a_s$, suppose that e_1 have one value, that e_2

have another value, etc., so that

$$s = e_1 + e_2 \ldots + e_t.$$

Then the number of polynomial identities of degree n, linearly independent over the reals, is obtained by summing

$$\frac{r^2 P_s}{(e_1!)(e_2!)\ldots(e_t!)} \left\{ \frac{n!}{(a_1!)(a_2!)\ldots(a_s!)} - 1 \right\} r^2$$

for all partitions of n.

For quaternions take $r = 2$. For quaternion matrices take r as twice the number of rows or columns.

As an example, we may find the number of quaternion quadratic and cubic identities. The partitions of 2 are

$$2 = 2 \quad \text{and} \quad 2 = 1+1.$$

Since $(2!/2!)-1 = 0$, we need consider only the second. For quaternions $r^2 = 4$.

We have $$a_1 = a_2 = 1,$$

$$e_1 = 2.$$

Hence the number of identities is

$$\frac{{}^4P_2}{2!}(2!-1)4 = 6 \times 4 = 24.$$

There are 24 quadratic identities linearly independent over the reals. This is in conformity with the result already obtained.

The partitions of 3 are $3 = 3$, which can be neglected, and

$$3 = 2+1 \quad \text{and} \quad 3 = 1+1+1.$$

Hence the number of identities is

$${}^4P_2 \left(\frac{3!}{2!} - 1 \right) 4 + \frac{{}^4P_3}{3!}(3!-1)4 = 96+80 = 176.$$

There are 176 cubic identities linearly independent over the reals.

There are sufficient polylinear identities to enable one to put the variables in all terms in any assigned order.

6. Concerning nilpotent algebras, or algebras with properly nilpotent elements, no exhaustive account can be given, for these have not been

classified. If the nilpotent elements of the variable can be removed by a linear transformation, then clearly we can obtain quadratic and bilinear identities corresponding to those of the algebra bereft of its nilpotent elements.

For example, if to the algebra of complex matrices of order four be added a nilpotent element θ, commutative with e_{11}, e_{12}, e_{21}, and e_{22}, and satisfying

$$\theta^2 = 0 \quad \text{and} \quad i\theta = -\theta i,$$

then the general variable x in this algebra will not satisfy

$$e_{11}xe_{12}xe_{11} = e_{12}xe_{11}xe_{11}.$$

If x is $\theta e_{11}+ie_{21}$, the left-hand side becomes $e_{11}\theta i$, and the right-hand side $e_{11}i\theta$. But the transformation

$$2X = x-ixi$$

enables us to express that part of x which does not involve θ as a linear function of x.

Thus $$e_{11}Xe_{12}Xe_{11} = e_{12}Xe_{11}Xe_{11},$$

or $$e_{11}(x-ixi)\,e_{12}(x-ixi)\,e_{11} = e_{12}(x-ixi)\,e_{11}(x-ixi)\,e_{11}.$$

There will be corresponding bilinear identities.

Apart from these relations, there will be certain identities that will follow from the nature of the nilpotent elements themselves. To illustrate these, a list of the identities is given here for the irreducible algebras with three and four elements tabulated in the *Encyklopädie der Math. Wiss.** Algebras that can be made equivalent by a complex transformation are not distinguished. Of the algebras with four elements, number VI is omitted, since it is the algebra of quaternions, which we have already discussed. All the others involve nilpotent elements.

The algebras with three elements have multiplication tables as follows:

I.	e_0	e_1	e_2	II.	e_0	e_1	e_2	III.	e_0	e_1	e_2
	e_1	e_2	0		e_1	e_0	e_2		e_1	0	0
	e_2	0	0		e_2	$-e_2$	0		e_2	0	0.

Numbers I and III are commutative, and there can be no identities.

* *Encyklopädie der Math. Wiss.*, I_1 (1904), 167.

In the second algebra there is one linear identity:

$$(e_0-e_1)\,x(e_0+e_1)=0,$$

upon which all identities are dependent.

Of the algebras with four elements the following are commutative and can have no identities:

I.				IV.				V.				X.			
e_0	e_1	e_2	e_3	e_0	e_1	e_3	e_3	e_0	e_1	e_2	e_3	e_0	e_1	e_2	e_3
e_1	e_2	e_3	0	e_1	e_3	0	0	e_1	e_2	0	0	e_1	0	0	0
e_2	e_3	0	0	e_2	0	e_3	0	e_2	0	0	0	e_2	0	0	0
e_3	0	0	0	e_3	0	0	0	e_3	0	0	0	e_3	0	0	0.

The second algebra has the same identities as the second one with three elements, as given above:

II.				
e_0	e_1	e_2	e_3	
e_1	e_0	e_2	0	$(e_0-e_1)\,x(e_0+e_1)=0.$
e_2	$-e_2$	0	0	
e_3	0	0	0	

The third algebra has five linear identities and two independent quadratic identities. There are two corresponding bilinear identities:

III.			
e_0	e_1	e_2	e_3
e_1	e_3	e_3	0
e_2	$-e_3$	e_3	0
e_3	0	0	0

$$xe_3=e_3x=e_1xe_1=e_1xe_2=e_2xe_2=-e_2xe_1$$

$$e_1x^2-2xe_1x+x^2e_1=0$$

$$e_2x^2-2xe_2x+x^2e_2=0$$

$$e_1xy-xe_1y-ye_1x+yxe_1=0$$

$$e_2xy-xe_2y-ye_2x+yxe_2=0.$$

The seventh has two linear identities:

VII.				
e_0	e_1	e_2	e_3	
e_1	e_0	e_3	e_2	$e_1xe_2-xe_3=0$
e_2	$-e_3$	0	0	$e_2xe_1+e_3x=0.$
e_3	$-e_2$	0	0	

The eighth has three linear and two quadratic or two bilinear identities:

VIII.

	e_0	e_1	e_2	e_3
e_1		0	e_3	0
e_2		$-e_3$	0	0
e_3		0	0	0

$$xe_3-e_3x = 0$$
$$e_1xe_2-xe_3 = 0$$
$$e_2xe_1+e_3x = 0$$
$$e_1x^2-2xe_1x+x^2e_1 = 0$$
$$e_2x^2-2xe_2x+x^2e_2 = 0$$
$$e_1xy-xe_1y-ye_1x+yxe_1 = 0$$
$$e_2xy-xe_2y-ye_2x+yxe_2 = 0.$$

The ninth has one independent linear identity:

IX.

	e_0	e_1	e_2	e_3
e_1		e_0	e_2	e_3
e_2		$-e_2$	0	0
e_3		$-e_3$	0	0

$$(e_0-e_1)\,x\,(e_0+e_1) = 0.$$

It will be seen that all these identities are aspects of the nilpotent nature of the elements. They can scarcely be considered as important as the identities in algebras containing no properly nilpotent element.

I wish to thank Prof. Richardson for introducing the subject to me, and for his advice.

DEFORMATIONS OF A CURVE, IN A RIEMANNIAN n-SPACE, WHICH DISPLACE CERTAIN VECTORS PARALLELLY AT EACH POINT

By H. A. HAYDEN.

[Received 28 January, 1930.—Read 6 February, 1930.]

1. It is well known, for a Euclidean 3-space, that if there is a 1-1-correspondence between the points of two curves, and if the tangents at corresponding points are parallel, then the principal normals are also parallel and hence, of course, the binormals also. Two such curves are said to be deducible from each other by a Combescure transformation.*

In the first part of this paper we examine whether this result can be extended to the case of a Riemannian n-space, using the word "parallel" in the accepted sense for a Riemannian space.

Since a parallel displacement of a vector is dependent on the path of displacement, if the path is of finite length, we restrict ourselves to curves which are deducible from each other by an infinitesimal displacement. We shall, in fact, regard a curve as given a *continuous* deformation, and examine whether every deformation which displaces the tangent parallelly at each point also displaces the 1-normal, 2-normal, ..., $(n-1)$-normal parallelly at each point. A deformation which effects this we shall call a "generalized Combescure transformation", or, briefly, a "G.C. transformation"; while, if a curve is such that every deformation which displaces its tangents parallelly is a G.C. transformation, we shall say that the curve "possesses the G.C. property". Further, if a space is such that every curve in it possesses the G.C. property, we shall say that the space possesses the G.C. property.

In § 2 we show that the only Riemannian spaces which possess the G.C. property are flat spaces.

In §§ 3–4 we examine certain special cases of deformations in a general Riemannian space which displace the tangents parallelly.

In § 5 we obtain some results connected with G.C. transformations in a flat n-space.

---

* Combescure, *Ann. Sci. de l'Ecole Normale*, 4 (1867), 102–122.

2. 2.1. Let C be a curve whose equations are

$$x^i = x^i(s),$$

and let each point be given an infinitesimal displacement $\delta x^i \equiv \epsilon\lambda^i$, where ϵ is an infinitesimal constant and λ^i is a vector-field defined along C, the vector λ^i being, in general, of variable finite length, λ say. [$\lambda \equiv \sqrt{(g_{ij}\lambda^i\lambda^j)}$, where $ds^2 = g_{ij}\,dx^i\,dx^j$ is the metric of the space*.] Then the equations of the deformed curve, which we shall denote by $\bar{C}$, are

$$\bar{x}^i(\bar{s}) = x^i + \epsilon\lambda^i. \tag{2.11}$$

Let $\xi^i_{(1)}$, $\xi^i_{(2)}$, ..., $\xi^i_{(n)}$ be the unit tangent vector, unit 1-normal, ..., unit $(n-1)$-normal, respectively at any point P of the curve C, and let $\bar{\xi}^i_{(1)}$, $\bar{\xi}^i_{(2)}$, ..., $\bar{\xi}^i_{(n)}$ be the unit tangent, etc., at the corresponding point $\bar{P}$ of $\bar{C}$. Then

$$\xi^i_{(1)} = \frac{dx^i}{ds},$$

$$\bar{\xi}^i_{(1)} = \frac{d\bar{x}^i}{d\bar{s}} = \left(\xi^i_{(1)} + \epsilon\frac{d\lambda^i}{ds}\right)\frac{ds}{d\bar{s}}.$$

Let $\bar{\bar{\xi}}^i_{(1)}$ be the vector at $\bar{P}$ parallel to the vector $\xi^i_{(1)}$ at P; so that, to the first order,

$$\begin{aligned}\bar{\bar{\xi}}^i_{(1)} &= \xi^i_{(1)} - \Gamma^i_{jk}\,\xi^j_{(1)}(\bar{x}^k - x^k),\\ &= \xi^i_{(1)} - \epsilon\Gamma^i_{jk}\,\xi^j_{(1)}\lambda^k,\end{aligned}$$

where Γ^i_{jk} are the Christoffel symbols of the second kind, viz.

$$\Gamma^i_{jk} \equiv \tfrac{1}{2}g^{i\alpha}\left(\frac{\partial g_{j\alpha}}{\partial x^k} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha}\right).$$

Then $\bar{\xi}^i_{(1)} - \bar{\bar{\xi}}^i_{(1)}$ is a vector (infinitesimal) which we shall denote by $\delta\xi^i_{(1)}$, and $\lim_{\delta s\to 0}\frac{\delta\xi^i_{(1)}}{\delta s}$ is a vector (finite) which we shall denote by $\Delta\xi^i_{(1)}$, where

$$\delta s \equiv \sqrt{(g_{ij}\,\delta x^i\,\delta x^j)} = \epsilon\lambda\dagger.$$

---

* We consider, for simplicity, the case of a positive definite metric; it will not be difficult to make the necessary alterations for the case of an indefinite metric

$$ds^2 = e g_{ij}\,dx^i\,dx^j \quad (e = \pm 1).$$

† Infinitesimal deformations of a curve in a Riemannian space have been considered by Levi-Civita, "Sur l'écart géodésique", *Math. Annalen*, 97 (1927), 291–320, and by Hlavaty, "Sur la déformation infinitésimale d'une courbe dans la variété métrique avec torsion", *Bulletin de la Soc. Math. de France*, 56 (1923), 18–25. The latter considers, not the vector $\delta\xi^i_{(1)}$, but the vector $\bar{\xi}^i_{(1)} - {}^*\xi^i_{(1)}$, where ${}^*\xi^i_{(1)} \equiv \xi^i_{(1)} + (\bar{x}^k - x^k)\,\partial\xi^i_{(1)}/\partial x^k$ ["le champ $\xi^i_{(1)}$ déplacé de x^i en $\bar{x}^i$"]; this, however, appears to have no meaning, since the vector-field $\xi^i_{(1)}$ is defined only along the curve C (so that $\partial\xi^i_{(1)}/\partial x^k$ is meaningless).

The condition for the tangents to be displaced parallelly during deformation —briefly we can say that λ^i is a "parallel-tangent deformation"—is that $\Delta\xi^i_{(1)} = 0$ at every point of C.

Now $$\delta\xi^i_{(1)} = \xi^i_{(1)}\left(\frac{ds}{d\bar{s}} - 1\right) + \epsilon\left(\frac{d\lambda^i}{ds} \cdot \frac{ds}{d\bar{s}} + \Gamma^i_{jk}\lambda^j\xi^k_{(1)}\right),$$

to the first order. We can find $ds/d\bar{s}$ from the fact that $\bar{\xi}^i_{(1)}$ is a unit vector at $\bar{P}$; thus $\bar{g}_{ij}\bar{\xi}^i_{(1)}\bar{\xi}^j_{(1)} = 1$, where $\bar{g}_{ij}$ is the value of g_{ij} at $\bar{P}$, *i.e.*

$$\bar{g}_{ij} = g_{ij} + \frac{\partial g_{ij}}{\partial x^k}(\bar{x}^k - x^k) = g_{ij} + \epsilon\lambda^k\frac{\partial g_{ij}}{\partial x^k}$$

$$= g_{ij} + \epsilon\lambda^k(g_{i\alpha}\Gamma^\alpha_{jk} + g_{j\alpha}\Gamma^\alpha_{ik}).$$

Hence

$$\left(\frac{d\bar{s}}{ds}\right)^2 = [g_{ij} + \epsilon\lambda^k(g_{i\alpha}\Gamma^\alpha_{jk} + g_{j\alpha}\Gamma^\alpha_{ik})]\left[\xi^i_{(1)} + \epsilon\frac{d\lambda^i}{ds}\right]\left[\xi^j_{(1)} + \epsilon\frac{d\lambda^j}{ds}\right]$$

$$= g_{ij}\xi^i_{(1)}\xi^j_{(1)} + 2\epsilon g_{ij}\xi^i_{(1)}\left(\frac{d\lambda^j}{ds} + \Gamma^j_{\alpha\beta}\lambda^\alpha\xi^\beta_{(1)}\right), \text{ to the first order,}$$

$$= 1 + 2\epsilon\xi_{(1)i}D\lambda^i,$$

where $D\lambda^i$ is the vector derivative* (or "derived vector") of λ^i with regard to the curve C. If we put

$$\xi_{(1)i}D\lambda^i = \phi, \tag{2.12}$$

so that ϕ is the component of the vector $D\lambda^i$ along the tangent, then, to the first order,

$$\frac{d\bar{s}}{ds} = \sqrt{(1 + 2\epsilon\phi)} = 1 + \epsilon\phi, \tag{2.13}$$

and $$\frac{ds}{d\bar{s}} = 1 - \epsilon\phi;$$

whence $$\delta\xi^i_{(1)} = \epsilon(D\lambda^i - \phi\xi^i_{(1)}),$$

to the first order, and

$$\Delta\xi^i_{(1)} = \frac{1}{\lambda}(D\lambda^i - \phi\xi^i_{(1)}), \tag{2.14}$$

exactly.

* Bianchi, "Sul parallelismo vincolato di Levi-Civita nella metrica degli spazi curvi", *Rend. della R. Acc. di Napoli*, 28 (1922), 150–171; Levi-Civita, *The absolute differential calculus* (1927), 140.

Thus the condition for the deformation λ^i to be a parallel-tangent deformation is that

$$D\lambda^i = \phi\xi^i_{(1)}, \tag{2.15}$$

i.e.

$$D\lambda^i \cdot \xi_{(\alpha)i} = 0 \quad (\alpha = 2, 3, \ldots, n). \tag{2.16}$$

2.2. Suppose that u^i is any vector defined along the curve C, and that $\bar{u}^i$ is the corresponding vector defined analogously along $\bar{C}$; for example, if $u^i = \xi^i_{(\alpha)}$, then

$$\bar{u}^i = \bar{\xi}^i_{(\alpha)}.$$

Then, by definition of δu^i,

$$\begin{aligned}\bar{u}^i &= u^i - \epsilon\Gamma^i_{jk}u^j\lambda^k + \delta u^i \\ &= u^i + \epsilon(\lambda\Delta u^i - \Gamma^i_{jk}u^j\lambda^k),\end{aligned}$$

to the first order.

If $\bar{D}$ denote vector differentiation along $\bar{C}$,

$$\begin{aligned}\bar{D}\bar{u}^i &= \frac{d\bar{u}^i}{d\bar{s}} + \bar{\Gamma}^i_{jk}\bar{u}^j\bar{\xi}^k_{(1)} \\ &= (1-\epsilon\phi)\left[\frac{du^i}{ds} + \epsilon\frac{d}{ds}(\lambda\Delta u^i - \Gamma^i_{jk}u^j\lambda^k)\right] \\ &\quad + \left(\Gamma^i_{jk} + \epsilon\frac{\partial\Gamma^i_{jk}}{\partial x^l}\lambda^l\right)(u^j + \epsilon\lambda\Delta u^j - \epsilon\Gamma^j_{\alpha\beta}u^\alpha\lambda^\beta)(\xi^k_{(1)} + \epsilon\lambda\Delta\xi^k_{(1)} - \epsilon\Gamma^k_{\alpha\beta}\xi^\alpha_{(1)}\lambda^\beta) \\ &= Du^i + \epsilon\Big[D(\lambda\Delta u^i) - \lambda\Gamma^i_{jk}\Delta u^j\xi^k_{(1)} - \Gamma^i_{jk}u^j(D\lambda^k - \Gamma^k_{\alpha\beta}\lambda^\alpha\xi^\beta_{(1)}) \\ &\qquad - \Gamma^i_{jk}\lambda^k(Du^j - \Gamma^j_{\alpha\beta}u^\alpha\xi^\beta_{(1)}) - \frac{\partial\Gamma^i_{jk}}{\partial x^l}\xi^l_{(1)}u^j\lambda^k \\ &\qquad - \phi(Du^i - \Gamma^i_{jk}u^j\xi^k_{(1)}) + \frac{\partial\Gamma^i_{jk}}{\partial x^l}\lambda^l u^j\xi^k_{(1)} \\ &\qquad - \Gamma^i_{jk}\Gamma^j_{\alpha\beta}u^\alpha\lambda^\beta\xi^k_{(1)} - \Gamma^i_{jk}\Gamma^k_{\alpha\beta}u^j\lambda^\beta\xi^\alpha_{(1)} \\ &\qquad + \lambda\Gamma^i_{jk}\xi^k_{(1)}\Delta u^j + \lambda\Gamma^i_{jk}u^j\Delta\xi^k_{(1)}\Big],\end{aligned}$$

to the first order.

Now $$\delta(Du^i) = \bar{D}\bar{u}^i - (Du^i - \epsilon\Gamma^i_{jk}Du^j\lambda^k),$$

and, by a little manipulation of the dummy suffixes in the expression for $\bar{D}\bar{u}^i$ with the help of (2.14), this reduces to the form

$$\delta(Du^i) = \epsilon\left[\lambda D(\Delta u^i) + \frac{d\lambda}{ds}\Delta u^i - \phi Du^i - R^i_{.jkl}u^j\xi^k_{(1)}\lambda^l\right],$$

to the first order, where $R^i_{.jkl}$ is the curvature tensor of the space, viz.

$$R^i_{.jkl} \equiv \frac{\partial\Gamma^i_{jl}}{\partial x^k} - \frac{\partial\Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{ak}\Gamma^a_{jl} - \Gamma^i_{al}\Gamma^a_{jk}.$$

Thus, finally, we have

$$\Delta(Du^i) - D(\Delta u^i) = \frac{1}{\lambda}\left[\frac{d\lambda}{ds}\Delta u^i - \phi Du^i - R^i_{.jkl}u^j\xi^k_{(1)}\lambda^l\right]. \quad (2.21)$$

2.3. If we put $u^i = \xi^i_{(a)}$, then we have, as necessary conditions that λ^i should be a G.C. transformation, $\Delta\xi^i_{(a)} = 0$ $(a = 1, 2, \ldots, n)$, and hence, on substituting in equation (2.21),

$$\Delta(D\xi^i_{(a)}) = -\frac{1}{\lambda}[\phi D\xi^i_{(a)} + R^i_{.jkl}\xi^j_{(a)}\xi^k_{(1)}\lambda^l] \quad (a = 1, \ldots, n). \quad (2.31)$$

The Frenet formulae for a curve in a Riemannian n-space are

$$D\xi^i_{(a)} = \kappa_a\xi^i_{(a+1)} - \kappa_{a-1}\xi^i_{(a-1)}$$

$$(a = 1, \ldots, n;\ \kappa_0 = 0,\ \kappa_n = 0),$$

where $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$, are the first, second, ..., $(n-1)$-th curvatures of the curve relative to the n-space.*

Hence

$$\Delta(D\xi^i_{(a)}) = \kappa_a\Delta\xi^i_{(a+1)} - \kappa_{a-1}\Delta\xi^i_{(a-1)} + \Delta\kappa_a\xi^i_{(a+1)} - \Delta\kappa_{a-1}\xi^i_{(a-1)},$$

and therefore, for a G.C. transformation,

$$\Delta(D\xi^i_{(a)}) = \Delta\kappa_a\xi^i_{(a+1)} - \Delta\kappa_{a-1}\xi^i_{(a-1)};$$

whence, from equations (2.31),

$$-R^i_{.jkl}\xi^j_{(a)}\xi^k_{(1)}\lambda^l = (\lambda\Delta\kappa_a + \phi\kappa_a)\xi^i_{(a+1)} - (\lambda\Delta\kappa_{a-1} + \phi\kappa_{a-1})\xi^i_{(a-1)}$$

$$(a = 1, 2, \ldots, n). \quad (2.32)$$

* Blaschke, "Frenets Formeln für den Raum von Riemann", *Math. Zeitschrift*, 6 (1920), 94–99; J. A. Schouten, *Der Ricci-Kalkül* (1924), 229.

These necessary conditions for a G.C. transformation are equivalent to

$$R^i_{.jkl}\xi^j_{(\alpha)}\xi^k_{(1)}\xi_{(\beta)i}\lambda^l = 0 \qquad (2.33)$$

$$(\beta = 1, 2, \ldots, \alpha-2, \alpha, \alpha+2, \ldots, n\,;\ \alpha = 1, \ldots, n),$$

since $$\xi^i_{(\alpha)}\xi_{(\beta)i} = \delta^\alpha_\beta,$$

which is 0 if $\alpha \neq \beta$, and 1 if $\alpha = \beta$.

These conditions must be satisfied by any λ^i which satisfies $D\lambda^i = \phi\xi^i_{(1)}$, in order that the deformation may be a G.C. transformation. If these are satisfied, then $\Delta\kappa_\alpha$ ($\alpha = 1, \ldots, n$) are given by the equations

$$\lambda\Delta\kappa_\alpha + \phi\kappa_\alpha = -R^i_{.jkl}\xi^j_{(\alpha)}\xi^k_{(1)}\xi_{(\alpha+1)i}\lambda^l \quad (\kappa_0 = \kappa_n = 0), \qquad (2.34)$$

which are obtained from (2.32) by multiplying throughout by $\xi_{(\alpha+1)i}$ and summing for i*.

Now consider any (arbitrarily) given point P_0 in the space and a given curve C through that point. If we give ϕ any arbitrary value along C, and λ^i ($i = 1, \ldots, n$) any arbitrary values at P_0, then the equation $D\lambda^i = \phi\xi^i_{(1)}$ will define a vector field λ^i all along C, which will displace the tangents parallelly.

Thus, if the curve C is to have the G.C. property, the conditions (2.33) must be satisfied all along C, and in particular at P_0, for arbitrary λ^i at P_0; whence we must have

$$R^i_{.jkl}\xi^j_{(\alpha)}\xi^k_{(1)}\xi_{(\beta)i} = 0$$

$$(\alpha, l = 1, \ldots, n\,;\ \beta = 1, 2, \ldots, \alpha-2, \alpha, \alpha+2, \ldots, n).$$

Further, if the space itself is to have the G.C. property, these conditions must be satisfied for every curve, *i.e.* for arbitrary ξ's at P_0, and this requires

$$R^i_{.jkl} = 0 \quad (i, j, k, l = 1, 2, \ldots, n)$$

at the point P_0. Moreover, P_0 is quite arbitrary, and thus we must have $R^i_{.jkl} = 0$ at every point of the space.

These conditions are also *sufficient* in order that a space should possess the G.C. property, as is evident from (2.33).

Hence *the only spaces which have the G.C. property are flat spaces.*

---

* *Added March 31st*, 1930. A referee has called my attention to a recent paper by McConnell, *Proc. R. Irish Acad.*, 39 A (1929), in which results are obtained which are equivalent to equations (2.32), (2.34), of this paper, for a general displacement.

In the case of a flat space, the equations (2 . 34) for the $\Delta\kappa_a$ reduce to

$$\Delta\kappa_a = -\frac{\phi}{\lambda}\kappa_a \quad (a = 1, \ldots, n-1),$$

whence
$$\bar{\kappa}_a \equiv \kappa_a + \delta\kappa_a = \kappa_a + \epsilon\lambda\Delta\kappa_a = (1-\epsilon\phi)\kappa_a$$
$$= \kappa_a \frac{ds}{d\bar{s}},$$

as is well known for Combescure transformations in a Euclidean 3-space.

3. 3 . 1. Let us express the deformation in terms of its components along the tangent and normals at each point. Thus, let us put

$$\lambda^i = \sum_{a=1}^{n} c_a \xi^i_{(a)}, \tag{3 . 11}$$

the c's being scalar functions of s.

Then

$$D\lambda^i = \sum_{a=1}^{n}\left[c_a D\xi^i_{(a)} + \frac{dc_a}{ds}\xi^i_{(a)}\right]$$
$$= \sum_{a=1}^{n}\left[c_a(\kappa_a\xi^i_{(a+1)} - \kappa_{a-1}\xi^i_{(a-1)}) + \frac{dc_a}{ds}\xi^i_{(a)}\right]$$
$$= \sum_{a=1}^{n}\left[\left(\frac{dc_a}{ds} - \kappa_a c_{a+1} + \kappa_{a-1}c_{a-1}\right)\xi^i_{(a)}\right] \quad (\kappa_0 = \kappa_n = 0), \tag{3 . 12}$$

so that
$$\phi \equiv \xi_{(1)i} D\lambda^i = \frac{dc_1}{ds} - \kappa_1 c_2,$$

and thence
$$\frac{ds}{d\bar{s}} = 1 - \epsilon\left(\frac{dc_1}{ds} - \kappa_1 c_2\right), \tag{3 . 13}$$

to the first order.

This involves only c_1 and c_2, and shows that for a displacement perpendicular to the tangent and 1-normal the first variation of the arc is zero*.

Also

$$\Delta\xi^i_{(1)} \equiv \frac{1}{\lambda}(D\lambda^i - \phi\xi^i_{(1)}) = \frac{1}{\lambda}\sum_{a=2}^{n}\left[\left(\frac{dc_a}{ds} - \kappa_a c_{a+1} + \kappa_{a-1}c_{a-1}\right)\xi^i_{(a)}\right]. \tag{3 . 14}$$

---

* *Cf.* J. L. Synge, "The first and second variations of the length-integral in Riemannian space", *Proc. London Math. Soc.* (2), 25 (1926), 247–264.

In terms of the c's, the conditions for λ^i to be a parallel-tangent deformation are, therefore,

$$\frac{dc_\alpha}{ds} = \kappa_\alpha c_{\alpha+1} - \kappa_{\alpha-1} c_{\alpha-1} \quad (\alpha = 2, 3, \ldots, n). \quad (3.15)$$

These are $n-1$ equations for the n scalar functions $c_1, c_2, \ldots, c_n$, so that we can always take one of the c's to be arbitrarily defined along the curve.

3.2. We cannot, however, in general have a parallel-tangent deformation for which $c_n = 0$; for the equations (3.15), written at length, are

$$\begin{aligned}
\frac{dc_2}{ds} &= \kappa_2 c_3 - \kappa_1 c_1, \\
\frac{dc_3}{ds} &= \kappa_3 c_4 - \kappa_2 c_2, \\
\ldots &\quad \ldots \quad \ldots \\
\ldots &\quad \ldots \quad \ldots \\
\frac{dc_{n-1}}{ds} &= \kappa_{n-1} c_n - \kappa_{n-2} c_{n-2}, \\
\frac{dc_n}{ds} &= -\kappa_{n-1} c_{n-1},
\end{aligned}$$

and $c_n = 0$ would require $c_{n-1} = 0$ (unless $\kappa_{n-1} = 0$), and thence $c_{n-2} = 0, \ldots, c_1 = 0$ (in general), which would give no deformation at all.

3.3. If a curve is such that none of its curvatures vanishes at any point, we shall call it a "non-degenerate curve"*, and we shall, from this point onwards, consider only curves which are non-degenerate.

3.4. First, let us examine whether it is possible to have a parallel-tangent deformation for which all the c's are constants. This would require

$$\begin{aligned}
\kappa_2 c_3 - \kappa_1 c_1 &= 0, \\
\kappa_3 c_4 - \kappa_2 c_2 &= 0, \\
\ldots &\quad \ldots \quad \ldots \\
\ldots &\quad \ldots \quad \ldots \\
\kappa_{n-1} c_n - \kappa_{n-2} c_{n-2} &= 0, \\
\kappa_{n-1} c_{n-1} &= 0,
\end{aligned}$$

* Blaschke, *loc. cit.*, 95, uses the word *allgemein* in a similar sense.

and hence $c_{n-1} = 0$, and further, therefore,

$$c_{n-3} = c_{n-5} = \ldots = c_2 = 0, \text{ if } n \text{ is odd};$$

or

$$c_{n-3} = c_{n-5} = \ldots = c_1 = 0, \text{ if } n \text{ is even.}$$

Moreover, this would require

$$\frac{\kappa_{n-1}}{\kappa_{n-2}} = \frac{c_{n-2}}{c_n} = \text{const.},$$

$$\frac{\kappa_{n-3}}{\kappa_{n-4}} = \frac{c_{n-4}}{c_{n-2}} = \text{const.},$$

and so on, down to $\kappa_2/\kappa_1 = \text{const.}$, if n is odd, or $\kappa_3/\kappa_2 = \text{const.}$, if n is even.

For a Euclidean 3-space this is a helix, and we shall therefore call such a curve in a Riemannian n-space a "generalized helix".

Thus, *a parallel-tangent transformation with constant c's is possible only for a "generalized helix", and then it must be a deformation in the* $(\frac{1}{2}n+\frac{1}{2})$*-space, or* $\frac{1}{2}n$*-space, according as n is odd or even, defined by* $\xi^i_{(n)}, \xi^i_{(n-2)}, \ldots, \xi^i_{(1)}$ (*or* $\xi^i_{(2)}$). *Moreover, the first variation of the arc-length is, or is not, zero, according as n is odd or even.*

This last fact is evident from the expression

$$\frac{d\bar{s}}{ds} = 1 - \epsilon\left(\frac{dc_1}{ds} - \kappa_1 c_2\right) \quad \text{(to the first order)}.$$

If n is odd, $c_2 = 0$ and $c_1 = \text{const.}$, whence $d\bar{s}/ds = 1$, to the first order.

If n is even, $c_1 = 0$ and $c_2 \neq 0$, since if c_2 were zero then we should also have $c_4 = c_6 = \ldots = c_n = 0$, and thus all the c's would vanish, giving no deformation at all. Thus $d\bar{s}/ds$ is definitely not equal to 1, to the first order, except possibly at certain points.

In the case of a flat n-space the deformed curve will also be a generalized helix, since, if the space is flat, the deformation will be a G.C. transformation and

$$\bar{\kappa}_\alpha = \kappa_\alpha \frac{ds}{d\bar{s}} \quad (\alpha = 1, 2, \ldots, n-1),$$

whence

$$\frac{\bar{\kappa}_{n-1}}{\bar{\kappa}_{n-2}} = \frac{\kappa_{n-1}}{\kappa_{n-2}} = \text{const.}, \quad \text{etc.}$$

4. *Two special types of parallel-tangent transformations.*

4.1. *Those for which the first variation of the arc-length is zero.*

This requires $\phi = 0$, and hence $D\lambda^i = 0$, so that λ^i must be a parallel-vector-field along C of constant magnitude. Hence the deformation is one

of constant magnitude along directions which are parallel with regard to the curve.

In the case of a Euclidean space this is a translation, and we shall therefore call such a displacement in a Riemannian space a *translation of the curve C.*

4.2. *Those for which the displacement is of constant magnitude.* (4.1 is, of course, included in this type.)

This requires $$g_{ij}\lambda^i\lambda^j = \text{const.},$$

i.e. $$D(g_{ij}\lambda^i\lambda^j) = 0,$$

i.e. $$g_{ij}\lambda^i D\lambda^j = 0,$$

and hence, either $D\lambda^i = 0$, which is type 4.1 above, or $D\lambda^i$ is perpendicular to λ^i.

But, for a parallel-tangent transformation, $D\lambda^i$ must be parallel to $\xi^i_{(1)}$, unless $\phi = 0$ (which again gives type 4.1). Thus the only parallel-tangent deformations of constant magnitude, apart from translations, are those for which λ^i is perpendicular to the tangent $\xi^i_{(1)}$. This requires $c_1 = 0$, and the equations (3.15) for the c's reduce to

$$\frac{dc_2}{ds} = \kappa_2 c_3,$$

$$\frac{dc_3}{ds} = \kappa_3 c_4 - \kappa_2 c_2,$$

$$\cdots \quad \cdots \quad \cdots$$

$$\cdots \quad \cdots \quad \cdots$$

$$\frac{dc_{n-1}}{ds} = \kappa_{n-1} c_n - \kappa_{n-2} c_{n-2},$$

$$\frac{dc_n}{ds} = -\kappa_{n-1} c_{n-1}.$$

These are $n-1$ ordinary differential equations, in normal form, for the $n-1$ functions $c_2, c_3, \ldots, c_n$, and such a system is always integrable, the solution involving $n-1$ arbitrary constants, corresponding to the $n-1$ degrees of arbitrariness of λ^i at any one point of the curve.

For $n = 2$, we have simply $dc_2/ds = 0$, *i.e.* $c_2 = \text{const.}$ Thus the

only parallel-tangent deformations of constant magnitude, apart from translations, are constant displacements along the normals.

For $n = 3$ our equations are

$$\frac{dc_2}{ds} = \kappa_2 c_3, \quad \frac{dc_3}{ds} = -\kappa_2 c_2.$$

Putting $t = \int \kappa_2 ds$, we have

$$\frac{dc_2}{dt} = c_3, \quad \frac{dc_3}{dt} = -c_2,$$

whence $$c_2 = A \sin(t+a), \quad c_3 = A \cos(t+a),$$

where A, a are arbitrary constants.

Moreover, a can be included in the arbitrary constant in $\int \kappa_2 ds$, and hence the only possible parallel-tangent deformations of constant magnitude, other than translations, are those of the type

$$c_1 = 0, \quad c_2 = A \sin\left(\int \kappa_2 ds\right), \quad c_3 = A \cos\left(\int \kappa_2 ds\right).$$

5. *Some results for a flat n-space.*

In the case of a flat space (*i.e.* one for which $R^i_{.jkl} = 0$, for $i, j, k, l = 1, 2, \ldots, n$) we know* that we can find a coordinate system in which g_{ij} $(i, j = 1, \ldots, n)$ are all constants and hence Γ^i_{jk} $(i, j, k = 1, \ldots, n)$ are all zero at every point of the space.

For such spaces there is a great simplification in choosing such a "Cartesian"† coordinate system, since the equations of parallelism for any vector-field μ^i are, in such a coordinate system, simply $d\mu^i/ds = 0$, *i.e.* $\mu^i = \text{const.}$ Thus the condition for the vector $\bar{u}^i$ to be parallel to the vector u^i is that $\bar{u}^i = u^i$ in a Cartesian coordinate system.

Moreover, in a flat space the variation of a vector during a parallel displacement between two points is independent of the path of displacement, so that we can consider C, $\bar{C}$ as two perfectly distinct curves, at a finite distance apart, with a 1-1 correspondence between their points. Several results may be obtained more easily in this way.

5.1. We can, for instance, prove that every parallel-tangent transformation in a flat space is a G.C. transformation as follows.

* Eisenhart, *Riemannian geometry* (1926), 25.

† Eisenhart, *loc. cit.*, 187.

Choose a Cartesian coordinate system. Then, for a parallel-tangent transformation, we have

$$\bar{\xi}^i_{(1)} = \xi^i_{(1)},$$

in that coordinate system. Hence

$$\bar{D}\bar{\xi}^i_{(1)} = \frac{d\bar{\xi}^i_{(1)}}{d\bar{s}} = \frac{d\xi^i_{(1)}}{ds}\frac{ds}{d\bar{s}} = D\xi^i_{(1)}\frac{ds}{d\bar{s}},$$

in these coordinates, *i.e.*

$$\bar{\kappa}_1\bar{\xi}^i_{(2)} = \kappa_1\xi^i_{(2)}\frac{ds}{d\bar{s}},$$

whence

$$\bar{g}_{ij}\bar{\kappa}_1^2\bar{\xi}^i_{(2)}\bar{\xi}^j_{(2)} = g_{ij}\kappa_1^2\xi^i_{(2)}\xi^j_{(2)}\left(\frac{ds}{d\bar{s}}\right)^2,$$

since $\bar{g}_{ij} = g_{ij}$ in our selected coordinate system.

But $\xi^i_{(2)}$ and $\bar{\xi}^i_{(2)}$ are each unit vectors; whence

$$\bar{\kappa}_1^2 = \kappa_1^2\left(\frac{ds}{d\bar{s}}\right)^2, \quad i.e. \quad \bar{\kappa}_1 = \kappa_1\frac{ds}{d\bar{s}},$$

and hence $\bar{\xi}^i_{(2)} = \xi^i_{(2)}$, in the special coordinate system; so that the vector $\bar{\xi}^i_{(2)}$ is parallel to the vector $\xi^i_{(2)}$.

Differentiating the relation $\bar{\xi}^i_{(2)} = \xi^i_{(2)}$, and using the previous relations, viz. $\bar{\xi}^i_{(1)} = \xi^i_{(1)}$ and $\bar{\kappa}_1 = \kappa_1\, ds/d\bar{s}$, we find that $\bar{\xi}^i_{(3)}$ is parallel to $\xi^i_{(3)}$, and so on; or we can proceed by induction. In this way the result follows.

By starting from the relation $\bar{\xi}^i_{(n)} = \xi^i_{(n)}$, differentiating repeatedly, and working through the Frenet formulae backwards, we can also prove the following:

If the $(n-1)$*-normals at corresponding points of two curves are parallel, then so are the* $(n-2)$*-normals, ..., 1-normals and tangents, and*

$$\frac{\bar{\kappa}_1}{\kappa_1} = \frac{\bar{\kappa}_2}{\kappa_2} = \ldots = \frac{\bar{\kappa}_{n-1}}{\kappa_{n-1}} = \frac{ds}{d\bar{s}}.$$

But we *cannot* start in the middle of the Frenet formulae and work towards both ends, as trial will show. This is well known for a Euclidean 3-space, for if the principal normals to two curves at corresponding points are parallel, it does *not* follow that the tangents are parallel and the binormals are parallel. Bertrand curves are a case in point. The single term on the right-hand side of the first and last of the Frenet formulae enables us to start from either end, and thus we have a duality of theorems

dealing with the tangent and certain normals and the $(n-1)$-normal and certain other normals.

5.2. *If $p < \frac{1}{2}n+1$, then the only deformations which displace the 2-space defined by $\xi^i_{(1)}$ and $\xi^i_{(p)}$ parallelly at each point are G.C. transformations.*

If the 2-space defined by $\bar{\xi}^i_{(1)}$ and $\bar{\xi}^i_{(p)}$ is parallel to that defined by $\xi^i_{(1)}$ and $\xi^i_{(p)}$, then, in a Cartesian coordinate system, $\bar{\xi}^i_{(1)}$ and $\bar{\xi}^i_{(p)}$ must each be of the form $a\xi^i_{(p)}+b\xi^i_{(1)}$, where a, b are scalar functions of s.

We shall assume that $\bar{\xi}^i_{(1)}$ is of this form and find the conditions for $\bar{\xi}^i_{(p)}$ to be also of this form.

If
$$\bar{\xi}^i_{(1)} = a\xi^i_{(p)}+b\xi^i_{(1)},$$

then
$$\bar{D}\bar{\xi}^i_{(1)}\frac{d\bar{s}}{ds} = D(a\xi^i_{(p)}+b\xi^i_{(1)}),$$

i.e.
$$\bar{\kappa}_1\frac{d\bar{s}}{ds}\bar{\xi}^i_{(2)} = a\kappa_p\xi^i_{(p+1)}+a'\xi^i_{(p)}-a\kappa_{p-1}\xi^i_{(p-1)}+b\kappa_1\xi^i_{(2)}+b'\xi^i_{(1)},$$

where dashes denote differentiations with regard to s. Differentiating again, we get

$$\bar{\kappa}_1\bar{\kappa}_2\left(\frac{d\bar{s}}{ds}\right)^2\bar{\xi}^i_{(3)}+\text{terms in } \bar{\xi}^i_{(2)},\ \bar{\xi}^i_{(1)} = a\kappa_p\kappa_{p+1}\xi^i_{(p+2)}+\text{terms in } \xi^i_{(p+1)},\ \xi^i_{(p)},\ \ldots.$$

Differentiating $p-1$ times altogether, we have, finally,

$$\bar{\kappa}_1\bar{\kappa}_2\ldots\bar{\kappa}_{p-1}\left(\frac{d\bar{s}}{ds}\right)^{p-1}\bar{\xi}^i_{(p)}+\text{terms in } \bar{\xi}^i_{(p-1)},\ \bar{\xi}^i_{(p-2)},\ \ldots$$
$$= a\kappa_p\kappa_{p+1}\ldots\kappa_{2p-2}\xi^i_{(2p-1)}+\text{terms in } \xi^i_{(2p-2)},\ \text{etc.}$$

Now we can substitute for $\bar{\xi}^i_{(1)}$, $\bar{\xi}^i_{(2)}$, ..., $\bar{\xi}^i_{(p-1)}$, in the left-hand side, from the previous equations, and they will contain no term in $\xi^i_{(2p-1)}$, since this term enters the right-hand side only at the last stage. Thus, if $\bar{\xi}^i_{(p)}$ is to be of the form $\alpha\xi^i_{(p)}+\beta\xi^i_{(1)}$, the left-hand side will contain altogether no term in $\xi^i_{(2p-1)}$. Hence, multiplying throughout by $\xi_{(2p-1)i}$ and summing for i, and remembering that $\xi^i_{(\alpha)}\xi_{(\beta)i} = 0$ unless $\beta = \alpha$, while $\xi^i_{(\alpha)}\xi_{(\alpha)i} = 1$, we have
$$a\kappa_p\kappa_{p+1}\ldots\kappa_{2p-2} = 0.$$

For a non-degenerate curve, to which we restrict ourselves, the only κ's which are zero are κ_0 and κ_n; these having no geometrical significance, but being introduced merely to enable us to write all the Frenet equations

in a single formula. Thus, if $2p-2 < n$, *i.e.* if $p < \frac{1}{2}n+1$, we must have $a = 0$; whence $\bar{\xi}^i_{(1)}$ must be parallel to $\xi^i_{(1)}$ and the deformation must be a G.C. transformation.

5.3. The dual theorem is:

If $p > \frac{1}{2}n$, the only deformations which displace the 2-space defined by $\xi^i_{(p)}$ and $\xi^i_{(n)}$ parallelly at each point are G.C. transformations.

To prove this, we put $\bar{\xi}^i_{(n)} = a\xi^i_{(p)} + b\xi^i_{(n)}$, using a Cartesian coordinate system as before, differentiate $n-p$ times, keeping in this case the normals of *lowest* order in prominence, and finally multiply throughout by $\xi_{(2p-n)i}$ and sum for i. This gives

$$a\kappa_{p-1}\kappa_{p-2}\dots\kappa_{2p-n} = 0,$$

and thus, if $2p-n > 0$, *i.e.* if $p > \frac{1}{2}n$, we must have $a = 0$, and therefore the vector $\bar{\xi}^i_{(n)}$ must be parallel to the vector $\xi^i_{(n)}$, which (for a flat space) is a G.C. transformation.

5.4. *The only deformations which displace the p-space defined by $\xi^i_{(1)}, \xi^i_{(2)}, \dots, \xi^i_{(p)}$ ($p < n$) parallelly at each point are G.C. transformations.*

Such a deformation requires that, in a Cartesian coordinate system, $\bar{\xi}^i_{(1)}, \bar{\xi}^i_{(2)}, \dots, \bar{\xi}^i_{(p)}$ shall each be of the form $\sum_{a=1}^{p} a_a \xi^i_{(a)}$, the a's being scalar functions of s.

Thus, suppose that $\bar{\xi}^i_{(1)} = \sum_{a=1}^{p} a_a \xi^i_{(a)}$;

then, on differentiating,

$$\bar{\kappa}_1 \frac{d\bar{s}}{ds} \bar{\xi}^i_{(2)} = a_p \kappa_p \xi^i_{(p+1)} + \text{terms in } \xi^i_{(p)}, \xi^i_{(p-1)}, \dots.$$

If $\bar{\xi}^i_{(2)}$ is to be of the form $\sum_{a=1}^{p} b_a \xi^i_{(a)}$ then, on multiplying by $\xi_{(p+1)i}$ and summing for i, we must have

$$a_p \kappa_p = 0,$$

and hence, if $p < n$, $\qquad a_p = 0.$

Thus

$$\bar{\kappa}_1 \frac{d\bar{s}}{ds} \bar{\xi}^i_{(2)} = a_{p-1}\kappa_{p-1}\xi^i_{(p)} + \text{terms in } \xi^i_{(p-1)}, \xi^i_{(p-2)}, \dots,$$

and, differentiating again,

$$\bar{\kappa}_1\bar{\kappa}_2\left(\frac{d\bar{s}}{ds}\right)^2\bar{\xi}^i_{(3)}+\text{terms in }\bar{\xi}^i_{(2)},\ \bar{\xi}^i_{(1)}$$
$$=a_{p-1}\kappa_{p-1}\kappa_p\xi^i_{(p+1)}+\text{terms in }\xi^i_{(p)},\ \xi^i_{(p-1)},\ \ldots.$$

Again, for $\bar{\xi}^i_{(3)}$ to be of the required form, we must have, on multiplying by $\xi_{(p+1)i}$ and summing for i,

$$a_{p-1}\kappa_{p-1}\kappa_p=0,$$

and hence, if $p<n$, $$a_{p-1}=0,$$

for a non-degenerate curve; and thus

$$\bar{\kappa}_1\bar{\kappa}_2\left(\frac{d\bar{s}}{ds}\right)^2\bar{\xi}^i_{(3)}+\text{terms in }\bar{\xi}^i_{(2)},\ \bar{\xi}^i_{(1)}$$
$$=a_{p-2}\kappa_{p-2}\kappa_{p-1}\xi^i_{(p)}+\text{terms in }\xi^i_{(p-1)},\ \xi^i_{(p-2)},\ \ldots.$$

Continuing in this way, we find that

$$a_p=a_{p-1}=a_{p-2}=\ldots=a_2=0,$$

and hence $a_1=1$, since $\bar{\xi}^i_{(1)}$ is a unit vector.

Our result at this stage is

$$\bar{\kappa}_1\bar{\kappa}_2\ldots\bar{\kappa}_{p-1}\left(\frac{d\bar{s}}{ds}\right)^{p-1}\bar{\xi}^i_{(p)}+\text{terms in }\bar{\xi}^i_{(p-1)},\ \bar{\xi}^i_{(p-2)},\ \cdots$$
$$=\kappa_1\kappa_2\ldots\kappa_{p-1}\xi^i_{(p)}+\text{terms in }\xi^i_{(p-1)},\ \xi^i_{(p-2)},\ \ldots,$$

which is quite consistent and gives no new conditions.

(If $p=n$ then, of course, the p-space is the whole space and there is no case to consider.)

The dual theorem is:

The only deformations which displace the p-space defined by $\xi^i_{(n)},\ \xi^i_{(n-1)},\ \ldots,\ \xi^i_{(n-p+1)}$ parallelly at each point are G.C. transformations.

This, of course, follows immediately from the last and is not a separate result, since each of the vectors $\xi^i_{(1)},\ \xi^i_{(2)},\ \ldots,\ \xi^i_{(n-p)}$ is orthogonal to the p-space defined by $\xi^i_{(n)},\ \xi^i_{(n-1)},\ \ldots,\ \xi^i_{(n-p+1)}$, and thus a deformation which displaces this p-space parallelly also displaces the $(n-p)$-space defined by $\xi^i_{(1)},\ \xi^i_{(2)},\ \ldots,\ \xi^i_{(n-p)}$ parallelly, and by the previous result must be a G.C. transformation.

5.5. We saw earlier (§ 3.4) that, for a flat space, a parallel-tangent transformation (and hence a G.C. transformation) with constant c's trans-

forms a generalized helix into another generalized helix. For such a transformation

$$\bar{\kappa}_\alpha = \kappa_\alpha \frac{ds}{d\bar{s}} = \frac{\kappa_\alpha}{1+\epsilon\kappa_1 c_2} \quad (\alpha = 1, \ldots, n).$$

Thus, if the κ's are constant, so are the $\bar{\kappa}$'s; whence a curve of constant curvatures deforms into another curve of constant curvatures by such a deformation.

Also, if n is odd, then $d\bar{s}/ds = 1$, to the first order, and thus $\bar{\kappa}_\alpha = \kappa_\alpha$ $(\alpha = 1, \ldots, n)$; this is, in fact, a translation.

If n is even, $d\bar{s}/ds \neq 1$ (to the first order), (*vide* § 3.4), and thus $\bar{\kappa}_\alpha \neq \kappa_\alpha$; but $c_1 = 0$ in this case, so that the deformation is one of constant magnitude, but not constant in direction with regard to the curve.

For example, for a Euclidean 3-space, the curve is a circular helix and the deformationis merely a translation; for a Euclidean 2-space (a plane) the curve is a circle and the deformation ($c_1 = 0$, $c_2 = \text{const.}$) is one of constant magnitude along the normals, *i.e.* not in a constant direction. Moreover, in this case the deformed curve is still a circle, but the curvatures of the two curves are not equal.

ON A GENERALIZED HELIX IN A RIEMANNIAN n-SPACE

By H. A. Hayden.

[Received 28 January, 1930.—Read 6 February, 1930.]

1.

1.1. In a previous paper* we had occasion to introduce, for a Riemannian n-space, a type of curve which we called a "generalized helix", and which was defined by the properties

$$\frac{\kappa_{n-1}}{\kappa_{n-2}} = \text{const.}, \quad \frac{\kappa_{n-3}}{\kappa_{n-4}} = \text{const.}, \quad \ldots, \quad \frac{\kappa_2}{\kappa_1} = \text{const.}, \quad \text{if } n \text{ is odd,}$$

or

$$\frac{\kappa_{n-1}}{\kappa_{n-2}} = \text{const.}, \quad \frac{\kappa_{n-3}}{\kappa_{n-4}} = \text{const.}, \quad \ldots, \quad \frac{\kappa_3}{\kappa_2} = \text{const.}, \quad \text{if } n \text{ is even;}$$

where $\kappa_1, \kappa_2, \ldots, \kappa_{n-1}$ are the first, second, ..., $(n-1)$-th curvatures of the curve relative to the n-space.

For a Euclidean 3-space this is the ordinary helix, and we know that the tangents to a helix make a constant angle with a fixed direction (in the 3-space); conversely, we know that if a curve is such that its tangents make a constant angle with a fixed direction then it is a helix.

The object of the present paper is to investigate whether there is a corresponding constant angle property for a generalized helix in a Riemannian space.

In this case the phrase "fixed direction" will have no meaning of course, and we therefore try replacing it by "a vector-field parallel with regard to the curve", parallel in the accepted sense in Riemannian geometry.

1.2. If we express that the tangent makes a constant angle with a parallel vector-field along the curve, we get only one relation between the curvatures. For $n = 3$, this relation turns out to be $\kappa_1/\kappa_2 = \text{const.}$, so

* "Deformations of a curve, in a Riemannian n-space, which displace certain vectors parallelly at each point", *Proc. London Math. Soc.* (2), 32 (1931), 321–336.

that the curve is a helix; but for $n > 3$ we shall require more relations if the curve is to be restricted to be a helix.

The number of relations between the curvatures in our definition of a generalized helix, and the fact that they are grouped in pairs from the end, suggest that we should try the vectors $\xi^i_{(n)}$, $\xi^i_{(n-2)}$, ..., $\xi^i_{(3)}$, or $\xi^i_{(4)}$, according as n is odd or even; where $\xi^i_{(1)}$, $\xi^i_{(2)}$, ..., $\xi^i_{(n)}$ are the unit tangent-vector, unit 1-normal, ..., unit $(n-1)$-normal to the curve.

We have, in fact, the following results.

2. *Case of n odd.*

2.1. *If n is odd, and if the vectors* $\xi^i_{(n)}$, $\xi^i_{(n-2)}$, ..., $\xi^i_{(3)}$ *make constant angles with a parallel vector-field along the curve, then the curve is a generalized helix; moreover,* $\xi^i_{(1)}$ *also makes a constant angle with the given vector-field, and* $\xi^i_{(n-1)}$, $\xi^i_{(n-3)}$, ..., $\xi^i_{(2)}$ *are each perpendicular to the given vector-field.*

To prove this, we can suppose, without any loss of generality, that the given vector-field is a unit vector-field, say μ^i.

Suppose, now, that we are given

$$\mu_i \xi^i_{(\alpha)} = c_\alpha \quad (\alpha = n, n-2, \ldots, 3), \qquad (2.11)$$

the c's being constants (and $\mu_i \equiv g_{ij}\mu^j$, where $ds^2 = g_{ij}dx^i dx^j$ is the metric of the space). Then

$$\frac{d}{ds}[\mu_i \xi^i_{(\alpha)}] = 0 \quad (\alpha = n, n-2, \ldots, 3);$$

i.e.

$$D[\mu_i \xi^i_{(\alpha)}] = 0,$$

since $\mu_i \xi^i_{(\alpha)}$ is invariant; where the operator D denotes "absolute" differentiation with regard to the curve; that is,

$$Da^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} \equiv \frac{da^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}}{ds} + \left\{ \sum_{\alpha=1}^{p} \Gamma^{i_\alpha}_{kl} a^{i_1 \ldots i_{\alpha-1} k i_{\alpha+1} \ldots i_p}_{j_1 \ldots j_q} - \sum_{\beta=1}^{q} \Gamma^{k}_{j_\beta l} a^{i_1 \ldots i_p}_{j_1 \ldots j_{\beta-1} k j_{\beta+1} \ldots j_q} \right\} \frac{dx^l}{ds},$$

where $x^\nu = x^\nu(s)$ are the equations of the curve*. Thus

$$\mu_i D\xi^i_{(\alpha)} + \xi^i_{(\alpha)} D\mu_i = 0;$$

and, since μ^i is a parallel vector-field, $D\mu_i = 0$, whence

$$\mu_i D\xi^i_{(\alpha)} = 0 \quad (\alpha = n, n-2, \ldots, 3).$$

* Bianchi, "Sul parallelismo vincolato di Levi-Civita nella metrica degli spazi curvi", *Rend. R. Acc. di Napoli*, 28 (1922), 150-171; Levi-Civita, *The absolute differential calculus* (1927), 140.

Using the Frenet formulae*, viz.

$$D\xi^i_{(\alpha)} = \kappa_\alpha \xi^i_{(\alpha+1)} - \kappa_{\alpha-1}\xi^i_{(\alpha-1)} \quad (\kappa_0 = \kappa_n = 0),$$

we have

$$\kappa_\alpha \mu_i \xi^i_{(\alpha+1)} - \kappa_{\alpha-1}\mu_i \xi^i_{(\alpha-1)} = 0 \quad (\alpha = n,\ n-2,\ \ldots,\ 3). \tag{2.12}$$

Putting $\alpha = n$, we have

$$-\kappa_{n-1}\mu_i \xi^i_{(n-1)} = 0,$$

and hence, for a non-degenerate curve†,

$$\mu_i \xi^i_{(n-1)} = 0; \tag{2.13}$$

so that $\xi^i_{(n-1)}$ is perpendicular to μ^i.

Putting $\alpha = n-2$ in equations (2.12), and using (2.13), we get

$$-\kappa_{n-3}\mu_i \xi^i_{(n-3)} = 0,$$

so that $\xi^i_{(n-3)}$ is also perpendicular to μ^i (for a non-degenerate curve).

Proceeding in this way, using the previous equations, we find that each of the vectors $\xi^i_{(n-1)}$, $\xi^i_{(n-3)}$, ..., $\xi^i_{(2)}$ is perpendicular to μ^i; whence μ^i must be of the form

$$\mu^i = a_n \xi^i_{(n)} + a_{n-2}\xi^i_{(n-2)} + \ldots + a_3 \xi^i_{(3)} + a_1 \xi^i_{(1)},$$

where the a's are scalar functions of s.

Multiplying by $\xi_{(n)i}$, and summing for i, we have, since $\xi^i_{(\alpha)}\xi_{(\beta)i} = \delta^\alpha_\beta$ (*i.e.* $= 0$ if $\alpha \neq \beta$, 1 if $\alpha = \beta$),

$$a_n = \mu^i \xi_{(n)i} = \mu_i \xi^i_{(n)} = c_n.$$

Similarly $a_{n-2} = c_{n-2}$, ..., $a_3 = c_3$.

Also, since μ^i is a unit vector,

$$g_{ij}\mu^i\mu^j = 1,$$

whence

$$a_n^2 + a_{n-2}^2 + \ldots + a_1^2 = 1,$$

i.e.

$$a_1 = \sqrt{\{1 - (c_n^2 + c_{n-2}^2 + \ldots + c_3^2)\}} = \text{const.};$$

this constant will be called c_1.

Thus

$$\mu^i = c_n \xi^i_{(n)} + c_{n-2}\xi^i_{(n-2)} + \ldots + c_3 \xi^i_{(3)} + c_1 \xi^i_{(1)}. \tag{2.14}$$

* Blaschke, "Frenets Formeln für den Raum von Riemann", *Math. Zeitschrift*, 6 (1920), 94-99; J. A. Schouten, *Der Ricci-Kalkül* (1924), 229.

† *Vide* Hayden, "Deformations of a curve, in a Riemannian n-space, which displace certain vectors parallelly at each point", *Proc. London Math. Soc.* (2), 32 (1931), 321-336.

From the relations $$\mu_i \xi^i_{(\alpha)} = 0 \quad (\alpha = n-1, n-3, \ldots, 2)$$

we have $$D[\mu_i \xi^i_{(\alpha)}] = 0,$$

and hence $$\mu_i D\xi^i_{(\alpha)} = 0, \quad \text{since } D\mu_i = 0,$$

i.e. $$\kappa_\alpha \mu_i \xi^i_{(\alpha+1)} - \kappa_{\alpha-1} \mu_i \xi^i_{(\alpha-1)} = 0 \quad (\alpha = n-1, n-3, \ldots, 2).$$

By the relation (2.14) these are equivalent to

$$\frac{\kappa_{n-1}}{\kappa_{n-2}} = \frac{c_{n-2}}{c_n} = \text{const.},$$

$$\frac{\kappa_{n-3}}{\kappa_{n-4}} = \frac{c_{n-4}}{c_{n-2}} = \text{const.},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{\kappa_2}{\kappa_1} = \frac{c_1}{c_3} = \text{const.}$$

Thus the curve is a generalized helix.

Also $\mu_i \xi^i_{(1)} = c_1 = \text{const.}$; so that $\xi^i_{(1)}$ makes a constant angle with the given vector-field μ^i along the curve.

2.2. We can prove, in a similar manner, that,

If n IS ODD, *and if the vectors $\xi^i_{(1)}$, $\xi^i_{(3)}$, ..., $\xi^i_{(n-2)}$ to a curve make constant angles with a parallel vector-field along the curve, then the curve is a generalized helix; moreover, $\xi^i_{(n)}$ makes a constant angle with the given vector-field, and $\xi^i_{(2)}$, $\xi^i_{(4)}$, ..., $\xi^i_{(n-1)}$ are each perpendicular to the given vector-field.*

2.3. *The converse is also true,* viz.,

If n IS ODD, *the vectors $\xi^i_{(n)}$, $\xi^i_{(n-2)}$, ..., $\xi^i_{(1)}$ associated with a generalized helix make constant angles with a certain parallel vector-field along the curve.*

To prove this, let

$$\frac{\kappa_{n-1}}{\kappa_{n-2}} = c_{n-2},$$

$$\frac{\kappa_{n-3}}{\kappa_{n-4}} = \frac{c_{n-4}}{c_{n-2}},$$

$$\frac{\kappa_{n-5}}{\kappa_{n-6}} = \frac{c_{n-6}}{c_{n-4}},$$

$$\ldots \quad \ldots \quad \ldots$$

$$\frac{\kappa_2}{\kappa_1} = \frac{c_1}{c_3}.$$

Since the curve is a generalized helix c_{n-2} is a constant, and thence, also, c_{n-4}, c_{n-6}, ..., c_1 are constants.

If we multiply the equations

$$\begin{aligned}
D\xi^i_{(n)} &= -\kappa_{n-1}\xi^i_{(n-1)},\\
D\xi^i_{(n-2)} &= \kappa_{n-2}\xi^i_{(n-1)} - \kappa_{n-3}\xi^i_{(n-3)},\\
D\xi^i_{(n-4)} &= \kappa_{n-4}\xi^i_{(n-3)} - \kappa_{n-5}\xi^i_{(n-5)},\\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots &\\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots &\\
D\xi^i_{(3)} &= \kappa_3\xi^i_{(4)} - \kappa_2\xi^i_{(2)},\\
D\xi^i_{(1)} &= \kappa_1\xi^i_{(2)},
\end{aligned}$$

by 1, c_{n-2}, c_{n-4}, ..., c_1, respectively, and add, we have, in virtue of our definitions of the c's,

$$D[\xi^i_{(n)} + c_{n-2}\xi^i_{(n-2)} + c_{n-4}\xi^i_{(n-4)} + \ldots + c_3\xi^i_{(3)} + c_1\xi^i_{(1)}] = 0.$$

Hence the vector

$$\xi^i_{(n)} + c_{n-2}\xi^i_{(n-2)} + c_{n-4}\xi^i_{(n-4)} + \ldots + c_1\xi^i_{(1)} \tag{2.31}$$

must be a parallel vector-field along the curve. Let us call it μ^i.

Now

$$g_{ij}\mu^i\xi^j_{(n)} = 1, \quad g_{ij}\mu^i\xi^j_{(n-2)} = c_{n-2}, \quad \ldots, \quad g_{ij}\mu^i\xi^j_{(1)} = c_1,$$

so that the vectors $\xi^i_{(n)}$, $\xi^i_{(n-2)}$, ..., $\xi^i_{(1)}$ make constant angles with the vector-field μ^i defined by (2.31).

3. *Case of n even.*

In §§2.1–2.2 we saw that it is possible to have a non-degenerate curve in a Riemannian n-space, when n is odd, whose vectors $\xi^i_{(1)}$, $\xi^i_{(3)}$, ..., $\xi^i_{(n)}$ make constant angles with a parallel vector-field along it.

When n is even we find that there is an essential difference, expressed in the following results.

3.1. *If n is even, there does not exist a non-degenerate curve whose vectors $\xi^i_{(1)}$, $\xi^i_{(3)}$, ..., $\xi^i_{(n-1)}$ make constant angles with a parallel vector-field along it.*

Also

If n IS EVEN, *there does not exist a non-degenerate curve whose vectors $\xi^i_{(n)}$, $\xi^i_{(n-2)}$, ..., $\xi^i_{(2)}$ make constant angles with a parallel vector-field along it.*

We shall prove here the first of these theorems; the proof of the second is on similar lines.

If such a curve exist we must have

$$\mu_i \xi^i_{(a)} = c_a \quad (a = 1, 3, \ldots, n-1),$$

where μ^i is some parallel vector-field along the curve and the c's are some constants.

There is no loss of generality in taking μ^i to be a unit vector. Hence

$$D[\mu_i \xi^i_{(a)}] = 0,$$

i.e. $$\mu_i D\xi^i_{(a)} = 0, \quad \text{since } D\mu_i = 0;$$

i.e. $$\kappa_a \mu_i \xi^i_{(a+1)} - \kappa_{a-1} \mu_i \xi^i_{(a-1)} = 0$$

$$(a = 1, 3, \ldots, n-1;\ \kappa_0 = \kappa_n = 0). \quad (3.11)$$

Putting $a = 1, 3, \ldots, n-3$ in succession, and using at each stage the results of the previous substitutions (*cf.* § 2.1), we find that the vectors $\xi^i_{(2)}, \xi^i_{(4)}, \ldots, \xi^i_{(n-2)}$ must each be perpendicular to μ^i.

Taking $a = n-1$ in (3.11), we get

$$\kappa_{n-1} \mu_i \xi^i_{(n)} - \kappa_{n-2} \mu_i \xi^i_{(n-2)} = 0,$$

and hence, since $\xi^i_{(n-2)}$ is to be perpendicular to μ^i,

$$\kappa_{n-1} \mu_i \xi^i_{(n)} = 0. \quad (3.12)$$

For a non-degenerate curve, this requires $\xi^i_{(n)}$ to be perpendicular to μ^i, and hence μ^i to be of the form

$$\mu^i = c_1 \xi^i_{(1)} + c_3 \xi^i_{(3)} + \ldots + c_{n-1} \xi^i_{(n-1)}. \quad (3.13)$$

From the fact that $\xi^i_{(2)}, \xi^i_{(4)}, \ldots, \xi^i_{(n)}$ are perpendicular to μ^i, we have

$$D[\mu_i \xi^i_{(a)}] = 0 \quad (a = 2, 4, \ldots, n),$$

whence $$\kappa_a \mu_i \xi^i_{(a+1)} - \kappa_{a-1} \mu_i \xi^i_{(a-1)} = 0$$

$$(a = 2, 4, \ldots, n;\ \kappa_0 = \kappa_n = 0).$$

Taking $\alpha = n$, we must have

$$\kappa_{n-1} c_{n-1} = 0, \tag{3.14}$$

whence, for a non-degenerate curve,

$$c_{n-1} = 0.$$

Taking $\alpha = n-2, \ldots, 2$, we have

$$\begin{aligned}
\kappa_{n-2} c_{n-1} - \kappa_{n-3} c_{n-3} &= 0, \\
\kappa_{n-4} c_{n-3} - \kappa_{n-5} c_{n-5} &= 0, \\
\ldots \quad \ldots \quad \ldots \quad \ldots & \\
\ldots \quad \ldots \quad \ldots \quad \ldots & \\
\kappa_2 c_3 - \kappa_1 c_1 &= 0,
\end{aligned}$$

whence, for a non-degenerate curve, $c_{n-3} = c_{n-5} = \ldots = c_1 = 0$. But this is impossible, since $c_1^2 + c_3^2 + \ldots + c_{n-1}^2 = 1$, because μ^i is a unit vector.

3.2. Suppose, however, that we put $\kappa_{n-1} = 0$ in the proof of § 3.1. Then equation (3.12) is satisfied without $\xi^i_{(n)}$ being perpendicular to μ^i. Thus in place of equation (3.13) we have

$$\mu^i = c_1 \xi^i_{(1)} + c_3 \xi^i_{(3)} + \ldots + c_{n-1} \xi^i_{(n-1)} + a_n \xi^i_{(n)}, \tag{3.21}$$

where $$a_n = \sqrt{\{1 - (c_1^2 + c_3^2 + \ldots + c_{n-1}^2)\}} = \text{const.}$$

Further, (3.14) can be satisfied by $c_{n-1} \neq 0$, and the subsequent equations then give

$$\begin{aligned}
\frac{\kappa_1}{\kappa_2} = \frac{c_3}{c_1} &= \text{const.}, \\
\frac{\kappa_3}{\kappa_4} = \frac{c_5}{c_3} &= \text{const.}, \\
\ldots \quad \ldots & \quad \ldots \\
\frac{\kappa_{n-3}}{\kappa_{n-2}} = \frac{c_{n-1}}{c_{n-3}} &= \text{const.}
\end{aligned}$$

Thus there *can* exist a curve satisfying the conditions of the first theorem of § 3.1, but it must be a degenerate curve. Moreover,

$$\mu^i \xi_{(n)i} = a_n = \text{const.},$$

so that in this case $\xi^i_{(n)}$ also makes a constant angle with the parallel vector-field μ^i.

In fact, if $\kappa_{n-1} = 0$, $\xi^i_{(n)}$ is a parallel vector-field along the curve, as is evident from the last Frenet formula

$$D\xi^i_{(n)} = -\kappa_{n-1}\xi^i_{(n-1)} = 0$$

in this case*.

The curve so obtained is not what we have previously defined as a helix, but since this curve has a constant angle property associated with it (*vide* also § 3.3), whereas there appears to be no analogous property associated with a helix in a space of even dimensions, we shall call a curve of the type

$$\frac{\kappa_1}{\kappa_2} = \text{const.}, \quad \frac{\kappa_3}{\kappa_4} = \text{const.}, \quad \ldots, \quad \kappa_{n-1} = 0 \quad (n \text{ even}),$$

a "helix of the second kind".

We have, therefore, the result:

If n IS EVEN, *and if the vectors $\xi^i_{(1)}$, $\xi^i_{(3)}$, ..., $\xi^i_{(n-1)}$ to a curve make constant angles with a parallel vector-field along the curve, then the curve is a helix of the second kind; and, further, $\xi^i_{(n)}$ is a parallel vector-field along the curve.*

3.3. We can also prove the converse of this (by a method similar to that in §2.3); namely

If n IS EVEN, *the vectors $\xi^i_{(1)}$, $\xi^i_{(3)}$, ..., $\xi^i_{(n-1)}$ associated with a helix of the second kind make constant angles with a certain parallel vector-field along the curve.*

3.4. We can further obtain the following result, which is true for n odd or even.

For a curve of the type

$$\frac{\kappa_1}{\kappa_2} = \text{const.}, \quad \frac{\kappa_3}{\kappa_4} = \text{const.}, \quad \ldots, \quad \frac{\kappa_{2p-3}}{\kappa_{2p-2}} = \text{const.}, \quad \kappa_{2p-1} = \kappa_{2p} = \ldots = \kappa_{n-1} = 0,$$

* A little explanation, with regard to degenerate curves, is necessary. We have called a curve "degenerate" if any of its curvatures vanishes, the reason being that if, say, $\kappa_p = 0$, then the vectors $\xi^i_{(p+1)}$, $\xi^i_{(p+2)}$, ..., $\xi^i_{(n)}$ are indeterminate. The case of $\kappa_{n-1} = 0$, and none of the other curvatures zero, is, however, slightly different, since, $\xi^i_{(1)}$, $\xi^i_{(2)}$, ..., $\xi^i_{(n-1)}$ being determined, a unit vector $\xi^i_{(n)}$ perpendicular to each of those vectors is then uniquely determined whether κ_{n-1} vanishes or not. The last Frenet equation still holds with this definition of $\xi^i_{(n)}$.

where p is any positive integer, there exists a parallel vector-field along the curve which makes constant angles with the vectors $\xi^i_{(1)}$, $\xi^i_{(3)}$, ..., $\xi^i_{(2p-1)}$, and it is perpendicular to each of the vectors $\xi^i_{(2)}$, $\xi^i_{(4)}$, ... $\xi^i_{(2p-2)}$.

The parallel vector-field is, in fact, of the form

$$c_1\xi^i_{(1)}+c_3\xi^i_{(3)}+\ldots+c_{2p-1}\xi^i_{(2p-1)}+a_{2p}\lambda^i_{(2p)}+a_{2p+1}\lambda^i_{(2p+1)}+\ldots+a_n\lambda^i_{(n)};$$

where c_1, c_3, ..., c_{2p-1} are constants, such that

$$\frac{c_3}{c_1}=\frac{\kappa_1}{\kappa_2},\quad \frac{c_5}{c_3}=\frac{\kappa_3}{\kappa_4},\quad \ldots,\quad \frac{c_{2p-1}}{c_{2p-3}}=\frac{\kappa_{2p-3}}{\kappa_{2p-2}},$$

a_{2p}, a_{2p+1}, ..., a_n are scalar functions of s, and $\lambda^i_{(2p)}$, $\lambda^i_{(2p+1)}$, ..., $\lambda^i_{(n)}$ are any $n-2p+1$ mutually orthogonal vectors along the curve, each orthogonal to the $(2p-1)$-space defined by $\xi^i_{(1)}$, $\xi^i_{(2)}$, ..., $\xi^i_{(2p-1)}$.

THE DERIVATES AND APPROXIMATE DERIVATES OF MEASURABLE FUNCTIONS

By J. C. BURKILL *and* U. S. HASLAM-JONES.

[Received 14 May, 1930.—Read 15 May, 1930.]

1. Well known theorems concerning the possible values of the derivates of a function are due to Denjoy and to Mrs. Young. An investigation in some ways more searching than theirs has been carried out by Besicovitch*. We are able to present the main part of Besicovitch's work in a simpler form. Theorem 1 of this paper gives the essence of three theorems of his memoir. Our presentation differs considerably from his. The idea of the approximate derivative is fundamental in his treatment, but he does not work, as we do, with approximate derivates; he states his results in terms of the related functions $\alpha(x)$, $\beta(x)$ of our Lemma 2. Moreover, we replace the continuous functions of his theorems by measurable functions.

It is clear from the lack of references to Besicovitch's work that it has generally escaped notice and it may be inaccessible to many readers. We have therefore made our account complete in itself, and for this purpose we have reproduced Theorem 2 from Besicovitch's paper without any essential change, beyond the extension from a continuous function to a measurable function.

2. *Definition of approximate derivates.* We suppose throughout that $f(x)$ is measurable and finite. We proceed to define the upper right approximate derivate at the point x.

---

* A. S. Besicovitch, "Diskussion der stetigen Funktionen im Zusammenhang mit der Frage über ihre Differentiierbarkeit", *Bull. de l'Academie des Sciences de Russie* (1925), 97–122 and 527–540. A discussion on similar lines was given by S. Saks, "Sur les nombres dérivées des fonctions", *Fundamenta Math.*, 5 (1924), 98–104.

Let $AD^+(f, x, \lambda)$, the upper right λ-derivate, be the lower bound of numbers a such that the set of points ξ for which

$$f(\xi)-f(x) \geqslant a(\xi-x) \quad (\xi > x)$$

(which is clearly measurable) has upper right-hand density* at x less than or equal to λ, where $\lambda < 1$.

As λ decreases, $AD^+(f, x, \lambda)$ increases, and we define the upper right approximate derivate, $AD^+f(x)$, to be

$$\lim_{\lambda \to 0} AD^+(f, x, \lambda).$$

There are corresponding definitions for the lower right approximate derivate AD_+ and the left derivates AD^-, AD_-. Thus we have

$$AD_+(f, x, \lambda) = -AD^+(-f, x, \lambda),$$

$$AD^-(f, x, \lambda) = -AD_+\{f(-x), -x, \lambda\}.$$

When the upper and lower right approximate derivates are equal we may speak of a right-hand approximate derivative, and when all four approximate derivates are equal we have the approximate derivative $ADf(x)$.

3. We prove first the necessary lemmas on measurability, showing finally that the λ-derivates and approximate derivates are measurable.

LEMMA 1. *Let $m(x, x+h)$ be the measure of the set of points ξ in $(x, x+h)$ for which $f(\xi) \geqslant f(x)$. Then, for fixed h, $m(x, x+h)$ is a measurable function of x.*

For, let E be the set in which $m(x, x+h) < \lambda h$. We define $\phi(x, x+h, \lambda)$ to be the upper bound (which is attained) of numbers c such that the measure of the set of x' in $(x, x+h)$ for which $f(x') \geqslant c$ is greater than or equal to λh. Then E is the set in which

$$f(x) > \phi(x, x+h, \lambda).$$

---

* There is no ambiguity here in the use of the word "density" rather than "metric density". Let $e(x_1, x_2)$ be the measure of the part of a measurable set E which lies in the nterval (x_1, x_2). The upper and lower right-hand densities of E at the point x are the upper and lower limits, as h tends to zero, of $e(x, x+h)/h$. Similarly, left-hand densities are defined as the limits of $e(x-h, x)/h$. When all these four limits are equal their common value is the "density" of E at x. Points at which the density of E is 1 are said to be "points of density" of E.

The lemma will be proved if we can show that (for fixed h, λ) $\phi(x, x+h, \lambda)$ is a measurable function of x. It is easy to show that it is, in fact, an upper semi-continuous function*.

For suppose that $\phi(x_0, x_0+h, \lambda) = c_0$; then we wish to prove that

$$\overline{\lim_{x \to x_0}} \phi(x, x+h, \lambda) \leqslant c_0. \tag{3.1}$$

From the definition of ϕ, the set of x' in (x_0, x_0+h) for which $f(x') \geqslant c_0+\epsilon$ has measure less than λh, say $\lambda h-\delta$. Then, for $|x-x_0|<\delta$, the set of x' in $(x, x+h)$ for which $f(x') \geqslant c_0+\epsilon$ has measure less than λh. Hence, if $|x-x_0|<\delta$,

$$\phi(x, x+h, \lambda) \leqslant c_0+\epsilon.$$

Therefore

$$\overline{\lim_{x \to x_0}} \phi(x, x+h, \lambda) \leqslant c_0+\epsilon,$$

and, since ϵ is arbitrary, inequality (3.1) is established.

Define now, as in Besicovitch's paper, $\alpha(x)$ and $\beta(x)$ as the lower and upper right-hand densities at x of the points ξ in the enunciation of Lemma 1. That is, $\alpha(x)$ and $\beta(x)$ are the lower and upper limits, as h tends to 0, of $m(x, x+h)/h$.

LEMMA 2. *The functions $\alpha(x)$, $\beta(x)$ are measurable.*

Let $M(x, x+h)$ be the lower bound of $m(x, x+l)/l$, for $0<l<h$.

Since $m(x, x+l)/l$ is a continuous function of l in this range, the lower bound is the same if only rational values of l are considered. Then $M(x, x+h)$, being the lower bound of an enumerable set of measurable functions, is measurable.

As h decreases, $M(x, x+h)$ increases. Hence $\alpha(x)$, being the limit as h tends to 0 of $M(x, x+h)$, is measurable.

A similar proof holds for $\beta(x)$.

COROLLARY. *If $\alpha_a(x)$, $\beta_a(x)$ are the lower and upper densities at x of points ξ, for which*

$$f(\xi)-f(x) \geqslant a(\xi-x) \quad (\xi > x),$$

then the above argument applied to $f(x)-ax$, in place of $f(x)$, shows that, for each a, $\alpha_a(x)$, $\beta_a(x)$ are measurable.

---

* H. Looman, "Sur deux catégories remarquables de fonctions de variable réelle" *Fundamenta Math.*, 5 (1924), 105–111.

LEMMA 3. *The λ-derivates and approximate derivates of a measurable function are measurable.*

Let E be the set in which

$$AD^+(f, x, \lambda) \leqslant a.$$

Let a_n be a decreasing sequence with limit a, and let E_n be the set of x such that the set of ξ for which

$$f(\xi)-f(x) \geqslant a_n(\xi-x) \quad (\xi > x),$$

has upper right-hand density at x less than or equal to λ.

Then clearly the sets E_n form a decreasing sequence with limit E. But E_n is the set for which $\beta_{a_n}(x) \leqslant \lambda$, and is therefore measurable by the corollary to Lemma 2. Hence E is measurable, and so $AD^+(f, x, \lambda)$ is measurable.

The approximate derivates, being limits of λ-derivates, are measurable.

We need also the following lemma for the proof of Theorem 1.

LEMMA 4. *The set of points x such that*

$$f(x')-f(x) < a(x'-x)$$

for all points x' in $(x, x+l)$, with the possible exception of a set of measure less than $(\lambda+\epsilon)l$, where l may take any value in $0<l\leqslant h$, is measurable.

We may work with $f(x)$ in place of $f(x)-ax$.

As in Lemma 2, we need consider only rational values of l.

The set defined is then the product of the enumerable infinity of measurable sets given by

$$m(x, x+l) < (\lambda+\epsilon)\, l,$$

and so is measurable.

4. THEOREM 1. *Suppose that at points x of a measurable set E, for some λ less than 1,*

$$AD^+(f, x, \lambda) < +\infty.$$

Then $f(x)$ has a finite approximate derivative almost everywhere in E

Given ϵ (which we shall suppose to be less than $1-\lambda$), we can fix a and M (sufficiently large) such that

$$AD^+(f, x, \lambda) < a,$$

$$-M < f(x) < M$$

in a set of measure greater than $mE-\epsilon$.

By choosing h (small enough) we can find a set E_1 of measure greater than $mE-2\epsilon$ such that, for x in E_1,

$$f(x')-f(x) < a(x'-x)$$

for all points x' in $(x, x+l)$, with the possible exception of a set of measure less than $(\lambda+\epsilon)l$, where l may take any value in $0<l\leqslant h$.

The measurability of E_1 is secured by Lemma 4.

Take a subset E_2 of E_1 and numbers γ (small), k (between $\lambda+\epsilon$ and 1) such that, if x is a point of E_2,

(1) $m\{E_1(x-\delta, x)\} > k\delta$ if $\delta \leqslant \gamma$,

(2) f is approximately continuous at x,

(3) $mE_2 > mE-3\epsilon$.

Take any interval (b, c), of length less than $\min(h, \gamma)$, which contains points of E_2. We write

$$f_1(x) = f(x)-ax,$$

and define $\phi(x)$ as the lower bound of $f_1(\xi)$ for points ξ of the set E_2 in the interval $b \leqslant \xi \leqslant x$.

Clearly, $f_1(x) \geqslant \phi(x)$ at points of E_2 in (b, c). We prove that, at all these points, $f_1(x) = \phi(x)$.

Suppose, on the contrary, that, for some x_1 in E_2 and some η greater than 0,

$$f_1(x_1) = \phi(x_1)+\eta.$$

We shall establish the existence of a sequence of points ξ_n of E_1, tending to x_1 as limit from the left, such that

$$f_1(\xi_n) < f_1(x_1)-\tfrac{1}{2}\eta.$$

Then, by definition of E_1, the set of points in (ξ_n, x_1) for which

$$f_1(x) > f_1(x_1)-\tfrac{1}{2}\eta$$

has measure less than $(\lambda+\epsilon)(x_1-\xi_n)$. Since $\lambda+\epsilon<1$, and ξ_n may be taken arbitrarily near to x_1, this inequality contradicts the approximate continuity of $f_1(x)$ at x_1. We conclude, therefore, that $f_1(x_1) = \phi(x_1)$.

To define the sequence ξ_n, we can take a point ξ_0 of E_2 in (b, x_1) for which

$$f_1(\xi_0) < f_1(x_1) - \tfrac{1}{2}\eta.$$

Now the set in (ξ_0, x_1) for which $f_1(x) \geq f_1(\xi_0)$ has measure less than $(\lambda+\epsilon)(x_1-\xi_0)$, and the set in (ξ_0, x_1) of points of E_1 has measure greater than $k(x_1-\xi_0)$. Hence there is a point ξ_1 of E_1 in the interval

$$\{\xi_0+(k-\lambda-\epsilon)(x_1-\xi_0),\quad x_1\}$$

for which

$$f_1(\xi_1) \leqslant f_1(\xi_0) < f_1(x_1) - \tfrac{1}{2}\eta.$$

Repeating the argument for the interval (ξ_1, x_1), we obtain a point ξ_2 lying in an interval of length $(1-k+\lambda+\epsilon)^2(x_1-\xi_0)$ to the left of x_1, and so on.

Since $k-\lambda-\epsilon$ is greater than 0, ξ_n tends to x_1 as n tends to infinity, and we have the required sequence.

If now we write $f(x, E_2)$ for the function equal to $f(x)$ in E_2 and undefined elsewhere, we have proved that, if E is covered by intervals of length less than $\min(h, \gamma)$, then, in any such interval (b, c) containing points of E_2, $f_1(x, E_2)$ is a decreasing function in the set E_2. Hence, at almost all points of E_2, $f_1(x, E_2)$ has a finite derivative on E_2, and so $f(x, E_2)$ has a finite derivative, $f'(x, E_2)$ say, on E_2.

At any point of density of E_2 for which $f'(x, E_2)$ exists, $f(x)$ has an approximate derivative equal to $f'(x, E_2)$.

Thus we have shown that $f(x)$ has a finite approximate derivative at points of a set of measure greater than $mE-3\epsilon$, or, since ϵ is arbitrary, almost everywhere in E.

We have the following immediate consequences of the theorem.

COROLLARY 1. *The set in which* $AD^+ f(x) = -\infty$ *has measure zero.*

COROLLARY 2. *The theorem is true if* λ *depends on* x.

For if $AD^+(f, x, \lambda) < +\infty$ in E, given ϵ, we can fix λ_0 such that $AD^+(f, x, \lambda_0) < +\infty$ in a set of measure greater than $mE-\epsilon$.

COROLLARY 3. *Except in a set of* x *of measure zero,*

$$AD^+ f(x) = AD^+(f, x, \lambda)$$

for all λ *in the interval* $0 < \lambda < 1$.

This follows from Corollary 2.

5. Mr. Besicovitch has pointed out to us that an alternative proof of Theorem 1 can be derived from our lemmas.

As in § 4, given ϵ [which we now suppose to be less than $\frac{1}{4}(1-\lambda)$], we can choose a, sufficiently large, such that

$$AD^+(f, x, \lambda) < a$$

in a set F of measure greater than $mE-\epsilon$. Writing, as before,

$$f_1(x) = f(x)-a,$$

we show that the set F', consisting of the points of this set at which $f_1(x)$ [and, therefore, $f(x)$] has not an approximate derivative, is of measure zero. The set F' is measurable by Lemma 3. We suppose that it is of measure k, greater than zero, and establish a contradiction.

By choosing h (sufficiently small) we can find a set F_1, of measure greater than $mF-\frac{1}{2}k$, such that, for x in F_1,

$$f_1(x') < f_1(x)$$

for all points x' in $(x, x+l)$, with the possible exception of a set of measure less than $(\lambda+\epsilon)l$, when l may take any value in the interval $0<l\leqslant h$. The product set F_1F' has measure at least $\frac{1}{2}k$. Therefore we can find an interval (b, c), of length less than h, containing a part of F_1F' of measure greater than $(1-\epsilon)(c-b)$.

Taking, for convenience of reference, perpendicular axes of x and y, we define an interval $I(\eta)$ of the line $y=\eta$ for every value of η. Let (x, η) belong to $I(\eta)$ if there is a point ξ belonging to F_1 such that

$$b \leqslant \xi \leqslant x \leqslant c, \quad f_1(\xi) \leqslant \eta.$$

Let $e(\eta)$ be the set of points x for which $\{x, f(x)\}$ lies vertically above $I(\eta)$: that is, for which (x, η) belongs to $I(\eta)$ and $f_1(x)$ is greater than η. We now show that, if η_1 and η_2 are any two values of η,

$$m\{e(\eta_1)+e(\eta_2)\} \leqslant (\lambda+\epsilon)(c-b).$$

Suppose that η_1 is greater than η_2, and let x_2 be the left-hand end point of $I(\eta_2)$. If x_2 is not a point of both $I(\eta_2)$ and F_1, then, if δ is positive, there is a point ξ_2 of F_1 such that

$$x_2 < \xi_2 < x_2+\delta < c, \quad f_1(\xi_2) \leqslant \eta_2. \tag{5.1}$$

If x_2 is a point of both $I(\eta_2)$ and F_1, let ξ_2 be equal to x_2. Similarly choose ξ_1 corresponding to $I(\eta_1)$.

Since ξ_2 belongs to F_1 and $(c-\xi_2)$ is less than h, the set of points x in the interval (ξ_2, c) for which

$$f_1(x) > f_1(\xi_2)$$

is of measure less than $(\lambda+\epsilon)(c-\xi_2)$. Therefore, by (5 . 1), the measure of $e(\eta_2)$ in the interval (ξ_2, c) is less than $(\lambda+\epsilon)(c-\xi_2)$. Similarly, the part of $e(\eta_1)$ in the interval (ξ_1, ξ_2) is of measure less than $(\lambda+\epsilon)(\xi_2-\xi_1)$. Hence

$$m\{e(\eta_1)+e(\eta_2)\} < (\lambda+\epsilon)(c-x_1)+2\delta,$$

and, since δ is arbitrary, we have

$$\begin{aligned} m\{e(\eta_1)+e(\eta_2)\} &\leqslant (\lambda+\epsilon)(c-x_1) \\ &\leqslant (\lambda+\epsilon)(c-b). \end{aligned}$$

The argument extends to any finite number of intervals

$$I(\eta_1),\ I(\eta_2),\ \ldots,\ I(\eta_n),$$

and, as a limiting case, to an enumerable infinity of intervals. We have

$$m\{e(\eta_1)+e(\eta_2)+\ldots\} \leqslant (\lambda+\epsilon)(c-b).$$

If we take $\eta_1, \eta_2, \ldots$, to be the set of all rational numbers, the intervals $I(\eta_1), I(\eta_2), \ldots$, cover an interval (β, c), where

$$\beta-b \leqslant \epsilon(c-b).$$

For, by our assumption, the set of F_1F' in (b, c) is of measure at least $(1-\epsilon)(c-b)$, and so there are points of F_1 in this interval whose distance from b is not greater than $\epsilon(c-b)$.

Hence the set G of points x for which $\{x, f(x)\}$ lies in or below every $I(\eta)$ is of measure at least $(1-\lambda-2\epsilon)(c-b)$. The set of points belonging both to G and to F_1 in (b, c) is therefore of measure at least $(1-\lambda-3\epsilon)(c-b)$. In this set it is clear that $f_1(x)$ is decreasing. Therefore, as in Section 4, we can show that at almost all points of the set $f_1(x)$ has an approximate derivative. The set of F' in (b, c), then, is of measure not greater than $(\lambda+3\epsilon)(c-b)$, which is less than $(1-\epsilon)(c-b)$, since

$$\epsilon < \tfrac{1}{4}(1-\lambda).$$

This is contrary to our hypothesis concerning the measure of F_1F' in (b, c).

We conclude that the set F' is of measure zero; that is, that $f_1(x)$ has an approximate derivative almost everywhere in F. Finally, since ϵ is arbitrary, it follows that $f_1(x)$, and consequently $f(x)$, has an approximate derivative at almost all points of E.

6. The following theorem shows the connection between the approximate derivative and ordinary derivates.

THEOREM 2. *If D^+f is finite in E, then, almost everywhere in E,*

$$D^+f = ADf = D_-f.$$

In E, AD^+f is less than $+\infty$, and so, from Theorem 1, a finite ADf exists at almost all points of E.

If the set in which D^+f is greater than ADf has measure greater than zero, there are positive numbers B, b and a set E_0, of measure greater than zero, in which

$$D^+f < B, \quad ADf > -B, \quad D^+f - ADf > b.$$

Let h_1 be greater than 0, and let E_1 be the set of points x of E_0 such that

$$f(x')-f(x) < B(x'-x) \tag{6.1}$$

for all x' in the interval $x < x' \leqslant x+h_1$. Then E_1 may be shown to be measurable* and we may take h_1 so small that $mE_1 > 0$.

Let x_0 be a point of density of E_1 and let ϵ, η be small positive numbers. We can find h_0, less than or equal to h_1, such that the set E_2 of points x of E_1 in (x_0, x_0+h) for which

$$f(x)-f(x_0) < \{ADf(x_0)+\eta\}(x-x_0) \tag{6.2}$$

has measure at least $(1-\epsilon)h$ whenever $h \leqslant h_0$.

But there are points x', arbitrarily near to x_0, for which

$$f(x')-f(x_0) > \{D^+f(x_0)-\eta\}(x'-x_0). \tag{6.3}$$

If x' lies in the interval (x_0, x_0+h_0), there is a point x of E_2 such that

$$x'-x < \epsilon(x'-x_0).$$

Combining (6.2) and (6.3), we obtain

$$\begin{aligned} f(x')-f(x) &> \{D^+f(x_0)-ADf(x_0)-2\eta\}(x'-x_0)+\{ADf(x_0)+\eta\}(x'-x) \\ &> \left(\frac{b-2\eta}{\epsilon}-B\right)(x'-x). \end{aligned}$$

For sufficiently small ϵ, this inequality is inconsistent with (6.1).

Hence the set in which D^+f is greater than ADf is of measure zero; that is

$$D^+f = ADf$$

almost everywhere in E.

---

* S. Banach, "Sur les fonctions dérivées des fonctions mesurables", *Fundamenta Math.*, 3 (1922), 128-132.

We may establish, by a similar argument, the falsity of the assumption that the set in which D_-f is less than ADf has positive measure. We find values x', x, where $x < x' < x_0$ and $x'-x$ is small, such that the line joining the corresponding points of the curve $y = f(x)$ has a large positive gradient, in contradiction to (6 . 1).

7. We now deduce from Theorems 1 and 2, the possible dispositions of the derivates of a measurable function.

Theorem 1 and its corollaries show that almost everywhere either

(7 . 1) *A finite AD exists, or*

(7 . 2) $$AD^+ = AD^- = +\infty, \quad AD_+ = AD_- = -\infty,$$

and indeed in (7 . 2) every upper λ-derivate is $+\infty$ and every lower λ-derivate is $-\infty$.

Theorem 2 shows that, except for a set of measure zero, the points of the class (7 . 1) are of one of three types:

(7 . 11) *A finite ordinary derivative exists,*

(7 . 12) $$D_+ = D^- = AD, \quad D^+ = +\infty, \quad D_- = -\infty$$

$$(or \quad D^+ = D_- = AD, \quad D_+ = -\infty, \quad D^- = +\infty),$$

(7 . 13) $$D^+ = D^- = +\infty, \quad D_+ = D_- = -\infty.$$

The analysis of Besicovitch is more powerful than the Denjoy-Young analysis in that it reveals

(1) the existence of the approximate derivative in (7 . 12);

(2) that a point for which $D^+ = D^- = +\infty$, $D_+ = D_- = -\infty$ may be of either of two kinds (7 . 2), (7 . 13), and for the class (7 . 2) the λ-derivates take their extreme values.

DIVISOR FUNCTIONS: SOLUTIONS OF THE DIFFERENTIAL EQUATION

By S. W. P. STEEN.

[Received 17 April, 1930.—Read 24 April, 1930.]

Introduction.

The μ-th divisor function of orders $\alpha_1, \ldots, \alpha_\nu$ is defined by the equation

$$Y(\alpha_1, \ldots, \alpha_\nu; \mu; z) = -\int_C z^s e^{\mu\pi i s} \Gamma(\alpha_1 - s) \ldots \Gamma(\alpha_\nu - s)\, ds,$$

where $-\pi < \arg z < \pi$, and where the contour starts from $+\infty$, encircles all the poles of the integrand, and returns to $+\infty$.

This function may be expressed as a multiple integral thus† :

$$Y(\alpha_1, \ldots, \alpha_\nu; 0; z) = 2\pi i z^{\alpha_\nu} \int_0^\infty e^{-t_1} t_1^{\alpha_1 - \alpha_\nu - 1} \int_0^\infty e^{-t_2} t_2^{\alpha_2 - \alpha_\nu - 1} \ldots$$

$$\int_0^\infty e^{-t_{\nu-1}} t_{\nu-1}^{\alpha_{\nu-1} - \alpha_\nu - 1} e^{-z/t_1 \ldots t_{\nu-1}} dt_{\nu-1} \ldots dt_1.$$

Since‡

$$Y(\alpha_1, \ldots, \alpha_\nu; \mu; z) = z^{\alpha_\nu} e^{\mu\pi i \alpha_\nu} Y(\alpha_1 - \alpha_\nu, \ldots, \alpha_{\nu-1} - \alpha_\nu, 0; \mu; z),$$

there is no loss of generality in dealing with a divisor function in which $\alpha_\nu = 0$; we shall accordingly write

$$2\pi i Y_{n+1}(z) = Y(\alpha_n, \ldots, \alpha_1, 0; 0; z).$$

The object of this paper is to give the differential equation satisfied by $Y_{n+1}(z)$, and to find the development of the solutions relative to the singular points of the differential equation. There are two such singular points, 0 and ∞, the first gives rise to a power series and the second to

† Steen, "Divisor functions", *Proc. London Math. Soc.* (2), 31 (1930), 47–80.

‡ *Loc. cit.*

the asymptotic expansion of the solutions. A differential equation has already been given for $Y_{n+1}(z)$ in the paper referred to above, but this equation appears in a different form here.

Paragraph I gives the differential equation, paragraph II the solutions relative to the origin, and paragraph III the solutions relative to infinity.

I. *The differential equation.*

We have $$Y_1(z) = e^{-z}, \quad Y_2(z) = \int_0^\infty e^{-t}\, t^{a_1-1}\, Y_1(z/t)\, dt,$$

and generally

$$Y_{n+1}(z) = \int_0^\infty e^{-t}\, t^{a_n-1}\, Y_n(z/t)\, dt$$

$$= \int_0^\infty e^{-t_n} t_n^{a_n-1} \int_0^\infty e^{-t_{n-1}} t_{n-1}^{a_{n-1}-1} \dots \int_0^\infty e^{-t_1} t_1^{a_1-1}\, e^{-z/t_1 \dots t_n}\, dt_1 \dots dt_n.$$

Also $$Y_1'(z) = -Y_1(z).$$

Suppose that

$$L_n(Y_n) \equiv Y_n^{(n)}(z) + \frac{{}_nA_1}{z}\, Y_n^{(n-1)}(z) + \dots + \frac{{}_nA_{n-1}}{z^{n-1}}\, Y_n'(z) = \frac{(-)^n}{z^{n-1}}\, Y_n(z),$$

where the A's are constants. The result is true for $n = 1$; we shall prove it true generally by induction. We have

$$Y_{n+1}^{(\nu)}(z) = \int_0^\infty e^{-t}\, t^{a_n-1}\, \frac{d^\nu}{dz^\nu}\, Y_n(z/t)\, dt,$$

hence

$$L_n(Y_{n+1}) \equiv \int_0^\infty e^{-t}\, t^{a_n-1} \left\{ \frac{d^n}{dz^n}\, Y_n(z/t) + \frac{{}_nA_1}{z}\, \frac{d^{n-1}}{dz^{n-1}}\, Y_n(z/t) + \dots \right.$$

$$\left. + \frac{{}_nA_{n-1}}{z^{n-1}}\, \frac{d}{dz}\, Y_n(z/t) \right\} dt.$$

If in the equation $$L_n\{Y_n(z)\} = \frac{(-)^n}{z^{n-1}}\, Y_n(z)$$

we write z/t for z, we obtain

$$\frac{d^n Y_n(z/t)}{dz^n} + \frac{{}_nA_1}{z}\, \frac{d^{n-1}}{dz^{n-1}}\, Y_n(z/t) + \dots + \frac{{}_nA_{n-1}}{z^{n-1}}\, \frac{d}{dz}\, Y_n(z/t) = \frac{(-)^n}{tz^{n-1}}\, Y_n(z/t);$$

hence $$L_n\{Y_{n+1}(z)\} = \frac{(-)^n}{z^{n-1}} \int_0^\infty e^{-t}\, t^{a_n-2}\, Y_n(z/t)\, dt.$$

Again,

$$\frac{d}{dz}[L_n\{Y_{n+1}(z)\}] = (-)^n \int_0^\infty e^{-t}\, t^{a_n-2}\, \frac{d}{dz}\left\{\frac{Y_n(z/t)}{z^{n-1}}\right\} dt$$

$$= \frac{(-)^{n+1}(n-1)}{z^n}\int_0^\infty e^{-t}\, t^{a_n-2}\, Y_n(z/t)\, dt + \frac{(-)^{n+1}}{z^n}\int_0^\infty e^{-t}\, t^{a_n-1}\, \frac{d}{dt}\{Y_n(z/t)\}\, dt$$

$$= \frac{(-)^{n+1}}{z^n}(n-a_n)\int_0^\infty e^{-t}\, t^{a_n-2}\, Y_n(z/t)\, dt + \frac{(-)^{n+1}}{z^n}\int_0^\infty e^{-t}\, t^{a_n-1}\, Y_n(z/t)\, dt$$

$$= -\frac{(n-a_n)}{z} L_n\{Y_{n+1}(z)\} + \frac{(-)^{n+1}}{z^n} Y_{n+1}(z);$$

hence $Y_{n+1}(z)$ satisfies the differential equation

$$L_{n+1}(Y_{n+1}) = \frac{(-)^{n+1}}{z^n} Y_{n+1},$$

where the operator L_{n+1} satisfies the relation

$$L_{n+1} = \left(\frac{n-a_n}{z} + \frac{d}{dz}\right) L_n.$$

Hence

$$L_{n+1} = \sum_{j=0}^{n} \frac{{}_{n+1}A_j}{z^j} \frac{d^{n+1-j}}{dz^{n+1-j}}$$

with ${}_{n+1}A_0 = 1$. This is of the same form as L_n, hence the differential equation satisfied by $Y_n(z)$ is the one given above.

The coefficients ${}_nA_j$ may be calculated as follows.

$$L_{n+1} = \sum_{j=0}^{n-1}\left\{\frac{n-a_n}{z^{j+1}}\, {}_nA_j \frac{d^{n-j}}{dz^{n-j}} - \frac{j}{z^{j+1}}\, {}_nA_j \frac{d^{n-j}}{dz^{n-j}} + \frac{{}_nA_j}{z^j}\frac{d^{n+1-j}}{dz^{n+1-j}}\right\},$$

hence

$${}_{n+1}A_j = {}_nA_j + (n-j+1+a_n)\, {}_nA_{j-1},$$

together with ${}_nA_0 = 1$, ${}_nA_{-1} = 0$, ${}_nA_n = 0$, $n \geqslant 1$. Thus

$$L_2 = \frac{d^2}{dz^2} + (1+a_1)\frac{d}{dz},$$

$$L_3 = \frac{d^3}{dz^3} + (3+a_1+a_2)\frac{d^2}{dz^2} + (1+a_1)(1+a_2)\frac{d}{dz},$$

$$L_4 = \frac{d^4}{dz^4} + (6+a_1+a_2+a_3)\frac{d^3}{dz^3} + \{7+3(a_1+a_2+a_3)+a_1a_2+a_2a_3+a_3a_1)\}\frac{d^2}{dz^2}$$
$$+ (1+a_1)(1+a_2)(1+a_3)\frac{d}{dz}, \text{ etc.}$$

We may write

$$L_{n+1} = \left(\frac{n-\alpha_n}{z} + \frac{d}{dz}\right) \dots \left(\frac{1-\alpha_1}{z} + \frac{d}{dz}\right)\frac{d}{dz}.$$

Since $L_1 = d/dz$, we have

$$\left(\frac{n-\alpha_n}{z} + \frac{d}{dz}\right) f = \frac{1}{z^{n-\alpha_n}} \frac{d}{dz}(z^{n-\alpha_n} f),$$

hence we may write

$$L_{n+1} = \frac{1}{z^{n-\alpha_n}} \frac{d}{dz} z^{n-\alpha_n} \frac{1}{z^{n-1-\alpha_{n-1}}} \frac{d}{dz} z^{n-1-\alpha_{n-1}} \dots \frac{1}{z^{1-\alpha_1}} \frac{d}{dz} z^{1-\alpha_1} \frac{d}{dz}$$

$$= \frac{1}{z^{n-\alpha_n}} \frac{d}{dz} z^{1-\sigma_n+\alpha_{n-1}} \frac{d}{dz} z^{1-\alpha_{n-1}+\alpha_{n-2}} \dots z^{1-\alpha_2+\alpha_1} \frac{d}{dz} z^{1-\alpha_1} \frac{d}{dz}.$$

It is easily seen that L_{n+1}, like Y_{n+1}, is symmetrical in $\alpha_1, \dots, \alpha_n$, for

$$\left(\frac{n-\alpha_n}{z} + \frac{d}{dz}\right)\left(\frac{n-1-\alpha_{n-1}}{z} + \frac{d}{dz}\right) = \left(\frac{n-\alpha_{n-1}}{z} + \frac{d}{dz}\right)\left(\frac{n-1-\alpha_n}{z} + \frac{d}{dz}\right).$$

This remark does not, of course, imply that the factors of L_{n+1} are permutable; in fact, in general they are not.

The differential equation satisfied by $Y_{n+1}(z)$ may thus be written in the form

$$\frac{d}{dz} z^{1+\alpha_{n-1}-\alpha_n} \frac{d}{dz} z^{1+\alpha_{n-2}-\alpha_{n-1}} \dots \frac{d}{dz} z^{1-\alpha_1} \frac{d}{dz} Y_{n+1}(z) = (-)^{n+1} z^{-\alpha_n} Y_{n+1}(z);$$

from this the adjoint equation may be written down, it is

$$\frac{d}{dz} z^{1+\alpha_1} \frac{d}{dz} z^{1+\alpha_1-\alpha_2} \dots \frac{d}{dz} z^{1+\alpha_{n-1}-\alpha_n} \frac{d}{dz} \bar{Y}_{n+1}(z) = (-)^{n+1} z^{-\alpha_n} \bar{Y}_{n+1}(z).$$

Thus $Y(\alpha_n, \dots, \alpha_1, 0; 0; z)$ and $Y(0, \alpha_n-\alpha_{n-1}, \dots, \alpha_n-\alpha_1, \alpha_n; 0; z)$ are adjoint. Generally we shall call

$$Y(\alpha_n, \dots, \alpha_1; 0; z) \quad \text{and} \quad Y(\beta_n, \dots, \beta_1; 0; z)$$

adjoint if $$\alpha_n+\beta_n = \alpha_{n-1}+\beta_{n-1} = \dots = \alpha_1+\beta_1.$$

The singular points of the differential equation are 0 and ∞. The latter becomes evident on making the transformation $z = 1/y$, when the

differential equation becomes

$$\frac{d}{dy}y^{1+\alpha_n-\alpha_{n-1}}\frac{d}{dy}y^{1+\alpha_{n-1}-\alpha_{n-2}}\cdots\frac{d}{dy}y^{1+\alpha_1}\frac{d}{dy}Y_{n+1}=y^{\alpha_n-2}Y_{n+1},$$

from which it is also apparent that the singularity at ∞ is irregular.

II. *Solutions relative to the origin.*

If we try for a solution of the form

$$z^\rho(a_0+a_1z+\ldots),$$

we obtain for the indicial equation

$$\rho(\rho-\alpha_1)(\rho-\alpha_2)\ldots(\rho-\alpha_n)=0.$$

Hence the origin is a regular singularity, the coefficients a_m being given by

$$(\rho+m)(\rho+m-\alpha_1)\ldots(\rho+m-\alpha_n)a_m=(-)^{n+1}a_{m-1}$$

provided α_ν is not a positive integer, and that $\alpha_\nu-\alpha_\mu$ is not an integer. In these circumstances we obtain the $n+1$ independent solutions

$$y_i(z)=\sum_{m=0}^{\infty}\frac{(-)^{(n+1)m}z^{m+\alpha i}}{\Gamma(\alpha_i+m+1)\,\Gamma(\alpha_i+m+1-\alpha_1)\ldots\Gamma(\alpha_i+m+1-\alpha_n)}$$

for $i=0, 1, \ldots, n$ and where $\alpha_0=0$. Thus

$$Y_{n+1}(z)=\sum_{i=0}^{n}c_iy_i(z).$$

We can determine the coefficients c_i as follows. When $z=0$,

$$Y_{n+1}(z)=\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)$$

if $\alpha_i>0$, hence

$$c_0=\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)\,\Gamma(1-\alpha_1)\ldots\Gamma(1-\alpha_n)$$

$$=\frac{\pi^n}{\sin\pi\alpha_1\ldots\sin\pi\alpha_n}=\frac{(-\pi)^n}{\sin\pi(\alpha_0-\alpha_1)\ldots\sin\pi(\alpha_0-\alpha_n)},$$

and, since $\quad 2\pi iz^\alpha Y_{n+1}(z)=Y(\alpha_1+\alpha,\ \ldots,\ \alpha_n+\alpha,\ \alpha;\ 0;\ z)$

is symmetric in its orders, we may let any one of these orders play the part of α_0 above, hence

$$c_i=\frac{(-\pi)^n}{\sin\pi(\alpha_i-\alpha_0)\ldots*\ldots\sin\pi(\alpha_i-\alpha_n)},$$

where the asterisk denotes that the term $\sin\pi(\alpha_i-\alpha_i)$ has been omitted.

Thus we have

$$Y_{n+1}(z) = (-\pi)^n \sum_{i=0}^{n} \frac{y_i(z)}{\sin \pi(a_1 - a_0) \dots * \dots \sin \pi(a_i - a_n)}.$$

When two of the a's are equal or differ by an integer, we obtain the extra solutions in the usual way. $Y_{n+1}(z)$ in these cases is obtained as the limit of letting unequal orders become equal, etc.; this follows since $Y_{n+1}(z)$ is a continuous function of its orders; in fact it is an integral function of each of its orders. Since the process of obtaining $Y(a_1, \dots, a_\nu; \mu; z)$ from $Y(a_1, \dots, a_\nu; 0; z)$ is by allowing the argument of z to increase by $\mu\pi$, we obtain

$$Y(a_1, \dots, a_n, 0; \mu; z) = 2\pi i(-\pi)^n \sum_{i=0}^{n} \frac{y_i(e^{\mu\pi i} z)}{\sin \pi(a_i - a_0) \dots * \dots \sin \pi(a_i - a_n)}.$$

When μ is even these functions clearly satisfy the same differential equation as $Y_{n+1}(z)$, and for $\mu = 0, 2, 4, \dots, 2n$ provide a set of $n+1$ independent solutions. For $\mu \neq 0$ we also obtain the solution when two a's are equal, etc., by the limit process mentioned above. For μ odd, $\mu = 1, 3, 5, \dots, 2n+1$, we have a set of $n+1$ independent solutions of the differential equation

$$L_{n+1}(f) = \frac{(-)^n f}{z^n}.$$

The functions $y_j(z)$ may be expressed in terms of the divisor functions. Since

$$y_j(e^{2\mu\pi i} z) = e^{2\mu\pi a_j i} y_j(z),$$

we obtain

$$2\pi i \frac{(-\pi)^n y_j(z)}{\sin \pi(a_j - a_1) \dots * \dots \sin \pi(a_j - a_n)} = \sum_{\mu=0}^{n} \frac{d_{\mu j} Y(a_1, \dots, a_n, 0; 2\mu; z)}{D},$$

where D is the determinant $\| e^{2\pi i \mu a_j} \|$ $(\mu = 0, \dots, n;\ j = 0, \dots, n)$, and where $d_{\mu j}$ is the minor of $e^{2\pi i \mu a_j}$.

Now
$$D = \prod_{n \geq j > i \geq 0} \sin \pi(a_i - a_j) . e^{n\pi i (a_0 + a_1 + \dots + a_n)}$$

and
$$d_{\mu j} = (-)^{\mu + j} \prod_{n \geq k > j \geq 0}^{*} \sin \pi(a_k - a_j)\, e^{(n-1)\pi i (a_0 + \dots * \dots + a_n)} C_{n-\mu},$$

where $C_{n-\mu}$ is the symmetric function of order $n-\mu$ formed from $e^{2\pi i a_0}, \dots * \dots, e^{2\pi i a_n}$; in all cases the asterisk indicates that the term cor-

responding to a_j has been omitted. Hence we obtain

$$2\pi i\, y_j(z) = \left(-\frac{1}{\pi}\right)^n \sum_{\mu=0}^{n} e^{-\pi i a_j} C_{n-\mu}\, Y(a_1, \ldots, a_n, 0;\ 2\mu;\ z).$$

III. *Solutions relative to infinity.*

In the differential equation for $Y_{n+1}(z)$ write $z = \zeta^m$, and it becomes

$$\frac{d}{d\zeta}\zeta^{1+m(a_n-1-a_n)}\frac{d}{d\zeta}\zeta^{1+m(a_{n-2}-a_{n-1})}\cdots\frac{d}{d\zeta}\zeta^{1-ma_1}\frac{d}{d\zeta}Y_{n+1}(\zeta^m)$$
$$= (-m)^{n+1}\zeta^{m-1-a_n m}\, Y_{n+1}(\zeta^m).$$

Now write $$Y_{n+1}(\zeta^m) = e^{a\zeta} g(\zeta).$$

Since $$\frac{d}{d\zeta}\{e^{a\zeta}h(\zeta)\} = e^{a\zeta}\left(a+\frac{d}{d\zeta}\right)h(\zeta),$$

we obtain

$$\left(a+\frac{d}{d\zeta}\right)\zeta^{1+m(a_n-1-a_n)}\cdots\left(a+\frac{d}{d\zeta}\right)\zeta^{1-ma_1}\left(a+\frac{d}{d\zeta}\right)g(\zeta)$$
$$= (-m)^{n+1}\zeta^{m-1-a_n m}\, g(\zeta).$$

Now write $\zeta = 1/\xi$ and $g(\zeta) = \xi^\sigma k(\xi)$, then the equation becomes

$$\left(a-\xi^2\frac{d}{d\xi}\right)\xi^{-1-m(a_n-1-a_n)}\cdots\left(a-\xi^2\frac{d}{d\xi}\right)\xi^{-1+ma_1}\left(a-\xi^2\frac{d}{d\xi}\right)\xi^\sigma k(\xi)$$
$$= (-m)^{n+1}\xi^{1+a_n m-m}\,\xi^\sigma k(\xi).$$

Now choose a, m, and σ so that the two lowest powers of ξ in the coefficient of $k(\xi)$ vanish; to do this we require

$$a^{n+1} = (-m)^{n+1}, \quad -n+ma_n+\sigma = 1+a_n m-m+\sigma,$$
$$\sum_{i=0}^{n}\{ma_i+\sigma-i\} = 0.$$

The first two equations imply that the lowest power of ξ on the left-hand side of the equation is equal to the lowest power of ξ on the right-hand side, and that the coefficients are equal; the third equation is the coefficient of the next highest power of ξ equated to zero. From these equations we obtain

$$a = -(n+1)\,\omega, \quad m = n+1, \quad \sigma = \tfrac{1}{2}n - A_1,$$

where ω is an $(n+1)$-th root of unity and $A_1 = a_0+a_1+\ldots+a_n$.

We shall now assume $k(\xi) = \sum_{\nu=0}^{\infty} k'_\nu \xi^\nu$ and find the coefficients k'_ν formally; afterwards we shall prove that

$$k(\xi) = \sum_{\nu=0}^{N} k'_\nu \xi^\nu + \epsilon_N \xi^{N+1},$$

where $\epsilon_N \to 0$ as $\xi \to 0$. The solution is thus asymptotic. Lastly, we shall identify $Y_{n-1}(z)$ with a certain combination of the $n+1$ solutions relative to ∞ which we shall obtain.

Since $$\left(a - \xi^2 \frac{d}{d\xi}\right) \xi^\lambda f(\xi) = \xi^\lambda \left(a - \lambda\xi - \xi^2 \frac{d}{d\xi}\right) f(\xi),$$

we may rewrite the differential equation for $k(\xi)$ as

$$\left(1 - \{\sigma - n + \alpha_n(n+1)\}\xi - \xi^2 \frac{d}{d\xi}\right) \dots \left(1 - \{\sigma - 1 + \alpha_1(n+1)\} - \xi^2 \frac{d}{d\xi}\right)$$
$$\times \left(1 - \sigma\xi - \xi^2 \frac{d}{d\xi}\right) k(a\xi) = \{-(n+1)\}^{n+1} k(a\xi).$$

Let $B_h = \Sigma \beta_{k_h} \dots \beta_{k_1}$, the summation being over all possible combinations of $\beta_0, \dots, \beta_n$, such that $k_h > k_{h-1} > \dots > k_1$; it is the h-th symmetric function with a particular notation. Define β_h^m, the result of substituting $m+j-1$ for β_{h_j} and of doing this for any number β_{k_j} in β_h, and of adding the results for all possible ways of making the substitutions. Then we have, if $\beta_i = \sigma - i + \alpha_i(n+1)$,

$$\left(1 - \{\sigma - n + \alpha_n(n+1)\}\xi - \xi^2 \frac{d}{d\xi}\right) \dots \left(1 - \sigma\xi - \xi^2 \frac{d}{d\xi}\right) \xi^m$$
$$= \xi^m - B_1^m \xi^{m+1} + B_2^m \xi^{m+2} - \dots + (-)^{n+1} B_{n+1}^m \xi^{m+n+1}.$$

Now, if $k(a\xi) = \sum_{\nu=0}^{\infty} c_\nu \xi^\nu$, we obtain, to calculate the coefficients,

$$\sum_{i+j=m} (-)^i c_i B_j^i = 0 \quad (n+1 \geqslant j \geqslant 1).$$

c_0 is arbitrary, and the first two equations are

$$c_0 B_2^0 - c_1 B_1^1 = 0, \quad c_0 B_3^0 - c_1 B_2^1 + c_2 B_1^2 = 0.$$

Now

$$B_2^0 = \sum_{1 \leqslant i \leqslant n,\ 0 \leqslant j < i} \{\sigma - i + \alpha_i(n+1)\}\{\sigma - j + \alpha_j(n+1)\}$$
$$+ \sum_{1 \leqslant i \leqslant n,\ 0 \leqslant j < i} \{\sigma - j + \alpha_j(n+1)\}.$$

In this the coefficient of $a_i^1 a_j$ is $(n+1)^2$, the coefficient of a_i is

$$(n+1)\sum_{0\leq j<i}(\sigma-j)+(n+1)\sum_{n\geq j>i}(\sigma-j)+(n+1)\sum_{n\geq j>i}1$$
$$=(n+1)^2\sigma-(n+1)^2\tfrac{1}{2}n+(n+1)n$$
$$=(n+1)n-(n+1)^2A_1,$$

the term independent of the a's is

$$\sum_{1\leq i\leq n,\ 0\leq j<i}\sum(\sigma-i)(\sigma-j)+\sum_{1\leq i\leq n,\ 0\leq j<i}\sum(\sigma-j)$$
$$=A_1^2\frac{n(n+1)}{2}-A_1\frac{n(n+1)}{2}+\frac{n(n+1)(n+2)}{24}.$$

Hence

$$B_2^0=(n+1)^2A_2-\tfrac{1}{2}(n+1)(n+2)A_1^2+\tfrac{1}{2}n(n+1)A_1+\tfrac{1}{24}n(n+1)(n+2).$$

Again $B_1^1=\sum_{0\leq i\leq n}\{\sigma-1+a_i(n+1)\}+\sum_{0\leq i\leq n}1$

$$=(n+1)\sigma-\tfrac{1}{2}n(n+1)+(n+1)A_1+n+1$$
$$=n+1.$$

Hence $c_1=[(n+1)A_2-\tfrac{1}{2}(n+2)A_1^2+\tfrac{1}{2}nA_1+\tfrac{1}{24}n(n+2)]c_0.$

Similarly the other coefficients may be found. Here $A_2=\Sigma a_i a_j$.

The solutions that we have found are thus of the form

$$x_\mu(z)=\exp[-(n+1)(e^{2\mu\pi i}z)^{1/(n+1)}](e^{2\mu\pi i}z)^{A_1/(n+1)-n/[2(n+1)]}\sum_{m=0}\frac{c_m}{(e^{2\mu\pi i}z)^{m/(n+1)}},$$

where $c_0=1$, and c_1 has the value given above. The $n+1$ independent solutions are given by $\mu=0, 1, 2, \ldots, n$; the general solution is thus

$$\sum_{\mu=0}^{n}b_\mu x_\mu(z),$$

To find the values of the constants in the case $Y_{n+1}(z)$ we notice that, when $a_1=1/(n+1)$, $a_2=2/(n+1)$, ..., $a_n=n/(n+1)$, the differential equation reduces to

$$\frac{d}{dz}z^{1-[1/(n+1)]}\frac{d}{dz}z^{1-[1/(n+1)]}\cdots\frac{d}{dz}z^{1-[1/(n+1)]}\frac{d}{dz}f(z)=(-)^{n+1}z^{-1+[1/(n+1)]}f(z).$$

If in this we write $z=\zeta^{n+1}$, the equation becomes

$$\frac{d^{n+1}}{d\zeta^{n+1}}f=(-)^{n+1}(n+1)^{n+1}f,$$

the solution of which is $f(z) = \exp[-(n+1)(e^{2\mu\pi i}z)^{1/(n+1)}]$. In this case

$$y_i(z) = \sum_{m=0}^{\infty} \frac{(-)^{(n+1)m}\, z^{m+i/n+1}\, (n+1)^{\frac{1}{2}}\, (n+1)^{(n+1)m+i}}{(2\pi)^{\frac{1}{2}n}\, \Gamma\{m(n+1)+i+1\}},$$

also
$$\sin\frac{\pi}{n+1}\sin\frac{2\pi}{n+1}\dots\sin\frac{n\pi}{n+1} = (n+1)\,2^{-n};$$

hence
$$Y_{n+1}(z) = \frac{(2\pi)^{\frac{1}{2}n}}{(n+1)^{\frac{1}{2}}}\exp[-(n+1)\,z^{1/(n+1)}].$$

This may also be found directly from the integral. Thus

$$Y_{n+1}(z) = \frac{(2\pi)^{\frac{1}{2}n}}{(n+1)^{\frac{1}{2}}}\, x_0(z)$$

$$= \frac{(2\pi)^{\frac{1}{2}n}}{(n+1)^{\frac{1}{2}}}\exp[-(n+1)\,z^{1/(n+1)}]\, z^{A_1/(n+1)-n/[2(n+1)]}\sum_{m=0}^{\infty}\frac{c_m}{z^{m/(n+1)}}.$$

It now remains to prove that the formal solutions $x_\mu(z)$ are asymptotic solutions. By writing $\eta = 1/\xi$ in the differential equation for $k(\alpha\,\xi)$ we obtain the equation

$$\left(1+\frac{d}{d\eta}\right)^{n+1} h(\eta) - h(\eta) = M\{h(\eta)\},$$

where

$$M(u) = -\left(1-\frac{\{\sigma-n+\alpha_n(n+1)\}}{\eta}+\frac{d}{d\eta}\right)\dots\left(1-\frac{\sigma}{\eta}+\frac{d}{d\eta}\right)u+\left(1+\frac{d}{d\eta}\right)^{n+1}u.$$

Now the equation

$$\left(1+\frac{d}{d\eta}\right)^{n+1} h(\eta) - h(\eta) = f(\eta)$$

can be rewritten in the form

$$\frac{d^{n+1}}{d\eta^{n+1}}\{e^{\eta}\,h(\eta)\} - e^{\eta}\,h(\eta) = e^{\eta}f(\eta),$$

and if $f(\eta)$ is a regular function of η in $0 \leqslant \arg\eta \leqslant 2\pi$, and if $f(\eta) = O(\eta^{-2})$ as $|\eta| \to \infty$, then it is easily verified that the solution of the last differential equation which tends to β as $|\eta| \to \infty$ in $0 \leqslant \arg\eta \leqslant 2\pi$ is given by

$$h(\eta) = \sum_{\mu=0}^{n}\frac{\omega_\mu}{n+1}\int_{-\infty}^{\eta} e^{(\omega_\mu-1)(\eta-t)}f(t)\,dt+\beta \quad (\omega_\mu = e^{2\mu\pi i/(n+1)}).$$

The same is still true in $2m\pi \leqslant \arg\eta \leqslant 2(m+1)\pi$ if $f(\eta)$ satisfies the above requirements in this angle, where m is an integer. The convergence to β is uniform in any angle $2m\pi+\delta \leqslant \arg\eta \leqslant 2(m+1)\pi-\delta$ ($\delta > 0$), for taking the line of integration parallel to the real axis we have, if $\eta = |\eta| e^{i\theta}$,

$$|f(t)| \leqslant \frac{C}{|\eta|^2 \sin^2\theta}$$

and

$$\left|\int_{-\infty}^{\eta} e^{(\omega_\mu-1)(\eta-t)} f(t)\,dt\right| \leqslant C\int_{-\infty}^{K(\eta)} \frac{e^{-\tau(1-R\omega_\mu)}\,d\tau}{|\eta_1|^2\sin^2\theta} \leqslant \frac{C}{|\eta|^2\sin^2\theta} \quad (\omega_\mu \neq 1)$$

and

$$\left|\int_{-\infty}^{\eta} f(t)\,dt\right| \leqslant \frac{C}{|\eta|},$$

so that

$$h(\eta) = \beta + O(\eta^{-1}+\sin^{-2}\theta\,.\,\eta^{-2}),$$

and the convergence to β is thus uniform in the angle stated.

Let $\bar{M}(u)$ be the form adjoint to $M(u)$, and form a sequence of functions $h_1(\eta)$, $h_2(\eta)$, ... as follows† :

$$h_1(\eta) = \beta,$$

$$h_{i+1}(\eta) = \sum_{\mu=0}^{n} \frac{\omega_\mu}{n+1}\int_{-\infty}^{\eta} e^{(\omega_\mu-1)(\eta-t)}\,M\{h_i(t)\}\,dt+\beta$$

$$= -\sum_{\mu=0}^{n}\frac{\omega_\mu}{n+1}\,e^{(\omega_\mu-1)\eta}\int_{-\infty}^{\eta} h_i(t)\,\bar{M}(e^{(1-\omega_\mu)t})\,dt+\beta,$$

thus

$$h_{i+1}(\eta)-h_i(\eta) = -\sum_{\mu=0}^{n}\frac{\omega_\mu}{n+1}\,e^{(\omega_\mu-1)\eta}\int_{-\infty}^{\eta}\{h_i(t)-h_{i-1}(t)\}\,\bar{M}(e^{(1-\omega_\mu)t})\,dt.$$

Suppose that $|h_i(t)-h_{i-1}(t)| < H_i$ for $|t| > |\eta_0|$, then

$$|h_{i+1}(\eta)-h_i(\eta)| < \frac{H_i K}{|\eta|\sin\theta},$$

for $\bar{M}(e^{(1-\omega_\mu)t})$ consists of a finite number of terms of the form $a_m e^{(1-\omega_\mu)t}t^{-m}$ with $m \geqslant 1$, if $\omega_\mu \neq 1$ and $m > 1$ if $\omega_\mu = 1$, and

$$\left|e^{(\omega_\mu-1)\eta}\int_{-\infty}^{\eta}\{h_i(t)-h_{i-1}(t)\}\,e^{(1-\omega_\mu)t}\,t^{-m}\,dt\right| \leqslant \frac{H_i K}{|\eta|\sin\theta}.$$

† *Cf.* Ince, *Ordinary differential equations* (1927), 140 and 169.

Thus, if $|\eta|$ is sufficiently large,

$$h(\eta) = \sum_{i=1}^{\infty} \{h_i(\eta) - h_{i-1}(\eta)\} \quad [h_0(\eta) = 0]$$

is absolutely and uniformly convergent in $\delta \leqslant \arg \eta \leqslant 2\pi - \delta$. Multiplying the series by $-\frac{\omega_\mu}{n+1} e^{(\omega_\mu - 1)\eta} \bar{M}(e^{(1-\omega_\mu)t})$, summing for $\mu = 0, \ldots, n$, and integrating from $-\infty$ to η, we obtain

$$h(\eta) - \beta = -\sum_{\mu=0}^{n} \frac{\omega_\mu}{n+1} e^{(\omega_\mu - 1)\eta} \int_{-\infty}^{\eta} h(t)\, \bar{M}(e^{(1-\omega_\mu)t})\, dt$$

$$= \sum_{\mu=0}^{n} \frac{\omega_\mu}{n+1} \int_{-\infty}^{\eta} e^{(\omega_\mu - 1)(\eta - t)}\, M\{h(t)\}\, dt.$$

Thus $h(\eta)$ satisfies the differential equation

$$\left(1 + \frac{d}{d\eta}\right)^{n+1} h(\eta) - h(\eta) = \bar{M}\{h(\eta)\}.$$

Now suppose that

$$h_i(\eta) - h_{i-1}(\eta) = \frac{c_{i,\,i-1}}{\eta^{i-1}} + \frac{c_{i,\,i}}{\eta^i} + \ldots + \frac{c_{i,\,j} + \epsilon_{i,\,j}}{\eta^j},$$

where the c's are constants and $\epsilon_{i,\,j} \to 0$ as $|\eta| \to \infty$, then

$$h_{i+1}(\eta) - h_i(\eta) = -\sum_{\mu=0}^{n} \frac{\omega_\mu}{n+1} e^{(\omega_\mu - 1)\eta} \int_{-\infty}^{\eta} \left(\frac{c_{i,\,i-1}}{t^{i-1}} + \ldots + \frac{c_{i,\,j}}{t^j}\right) \bar{M}(e^{(1-\omega_\mu)t})\, dt$$

$$- \sum_{\mu=0}^{n} \frac{\omega_\mu}{n+1} e^{(\omega_\mu - 1)\eta} \int_{-\infty}^{\eta} \frac{\epsilon_{ij}}{t^j} \bar{M}(e^{(1-\omega_\mu)t})\, dt$$

$$= \frac{c_{i+1,\,i}}{\eta^i} + \frac{c_{i+1,\,i+1}}{\eta^{i+1}} + \ldots + \frac{c_{i+1,\,j}}{\eta^j} + \frac{\epsilon_{i+1,\,j}}{\eta^j}.$$

Hence, for $i \geqslant 1$,

$$h_i(\eta) - h_{i-1}(\eta) = \frac{c_{i,\,i-1}}{\eta^{i-1}} + \frac{c_{i,\,i}}{\eta^i} + \ldots + \frac{c_{i,\,j} + \epsilon_{i,\,j}}{\eta^j};$$

thus, firstly,

$$h(\eta) = \beta + \frac{c_1}{\eta} + \ldots + \frac{c_j}{\eta^j} + \frac{\epsilon_j}{\eta^j};$$

where $\epsilon_j \to 0$ as $|\eta| \to \infty$ in $\delta \leqslant \arg \eta \leqslant 2\pi - \delta$, and, secondly, $M\{h(\eta)\}$ has the properties of the function $f(\eta)$ which occurred previously.

The original differential equation has thus asymptotic solutions at ∞ and the coefficients are found by the recurrence formulae which have been

given. Thus

$$Y(\alpha_1, \ldots, \alpha_n, 0\,;\ \mu\,;\ z)$$

$$= 2\pi i\,\frac{(2\pi)^{\frac{1}{2}n}}{(n+1)^{\frac{1}{2}}}\exp\left[-(n+1)(e^{2\mu\pi i}z)^{1/(n+1)}\right]$$

$$\times (e^{2\mu\pi i}z)^{A_1/(n+1)-\frac{1}{2}+1/2(n+1)}\left\{\sum_{m=0}^{N}\frac{(-)^m c_m}{(ze^{2\mu\pi i})^{m/(n+1)}(n+1)^m}+\frac{\epsilon_N}{z^{N/(n+1)}}\right\},$$

where $\epsilon_N \to 0$ as $|z| \to \infty$ uniformly in

$$\delta \leqslant \arg(z+2\mu\pi) \leqslant 2(n+1)\pi-\delta$$

and in

$$2(n+1)m\pi+\delta \leqslant \arg(z+2\mu\pi) \leqslant 2(n+1)(m+1)\pi-\delta$$

for any integer m.

ON SURFACES OF INTERSECTION OF CUBIC PRIMALS, AND CUBIC CREMONA TRANSFORMATIONS OF FOUR-DIMENSIONAL SPACE

By J. G. Semple.

[Received 16 June, 1930.—Read 19 June, 1930.]

1. The first object of this paper is to consider and classify all the general types of surfaces which can be obtained as partial intersections of two cubic primals in space of four dimensions, and to consider also the systems of cubic primals through such surfaces, and certain also of the manifolds in higher space which represent such systems. We obtain most of the surfaces as residuals to known surfaces of low order, degenerate or otherwise, and we consider altogether in this way sixteen pairs of residual surfaces, the results being given in a table at the end of the paper. A rather remarkable feature of many of the surfaces which arise is their quite unexpected generality—of the three septimic surfaces considered, for example, two are rational and absolutely general of their type—and another advantage of the present method of generation is that it often forces the attention to certain of their special features, such as nodes and plane curves, which would otherwise be easily overlooked.

The second object of the paper is to obtain and discuss some of the simpler types of cubic Cremona transformations of S_4. We employ, for this purpose, the method described by the author in a previous paper dealing with the quadric transformations* of S_4, *i.e.* we start with a given rational surface F and find, as there described, all the H. systems† Φ of cubic primals which have F as general Φ-surface‡. We obtain in this way all those transformations in which the Φ-surfaces are the most general of their type.

---

* *Phil. Trans. Royal Soc.*, 228A (1929), 331–376. We shall refer to this paper as Q.T.

† *H. system* is an abbreviation for *homaloidal system*.

‡ A *Φ-surface* of a system of primals Φ in S_4 is a variable surface of intersection of two primals of Φ.

2. We begin by dividing all surfaces which are partial intersections of two cubic primals into two classes according as they have order $r \leqslant 4$ or > 4. Those of the first class are very well known, and they include, in fact, all surfaces whose order r is less than or equal to 4, excepting only that which consists of four planes of which no two meet in a line; those of the second class are obtained in this paper as residuals to those of the first.

If two cubic surfaces F_2^r and F_2^{9-r} form the complete intersection of two cubic primals, and if we know the properties of the first surface F_2^r, then we can immediately deduce certain properties of the residual surface F_2^{9-r}, namely, (*a*) its order $9-r$, and (*b*) the genus of its prime sections; and, furthermore, we can also obtain immediately the order of the curve of intersection of F_2^r and F_2^{9-r}, which curve we shall always denote by the symbol δ. We obtain these numbers, of course, simply by taking a section of the figure by a solid; for such a section consists of two cubic surfaces whose intersection consists of a curve of order r ($r \leqslant 4$) and of a residual curve of order $9-r$, and by representing one of the cubic surfaces on a plane, we can easily find the genus of the second curve from that of the first, and also the number of points of intersection of the two curves*, which number is the order of δ. This method leads, in fact, to a general relation between the genera, p, p' say, of prime sections of the surfaces F_2^r and F_2^{9-r} respectively, namely

$$p' = 9-2r+p,$$

and it leads also to a formula for the order d, say, of the curve δ, namely

$$d = 2r-2p+2.$$

Moreover, if F_2^r happens to be degenerate, these formulae still hold if we use for p the degenerate genus of the prime sections, calculated according to the formula

$$p = p_1+p_2+\ldots+p_i+t+1-i,$$

where p_1, p_2, ..., p_i are the genera of the various curves, i in number, composing one prime section, and t is the total number of intersections, supposed all distinct. Finally, if two or more of these curves composing one of these prime sections coincide (as happens, for example, if F_2^r is taken to consist partly of a plane of contact), we have to fall back, in order to find p', on the theory of residual curves on a cubic surface, and we can then, for the sake of completeness, calculate the degenerate

* *Cf.* Baker, *Proc. London Math. Soc.* (2), 11 (1912), 287.

genus p from the formula for p'. All the work involved in this kind of calculation, however, is clearly of an elementary character, and we therefore omit further details and give the results only arranged in the form of a table. Before doing this we insert a short note on certain points of notation.

Notation. We shall find it convenient for purposes of identification to introduce the notation ${}^pF_2^r$ to denote a surface of order r whose prime sections are of genus p; and generally we shall use the notation ${}^pV_m^r$ to denote a manifold whose *linear* sections (*i.e.* sections by such spaces as meet it in curves) are of genus p. In addition, we shall use the word *section genus* to denote the genus of the linear sections of a manifold, *e.g.*, the section genus of a surface is the genus of its prime sections; and we shall use the word *characteristic genus* in reference to a system of primals to denote the genus of residual curves of intersection of triads of the primals. Thus any system of primals in $[4]$ of freedom r, grade n, and characteristic-genus p, can be represented, if $r \geqslant 4$, on the prime sections of a ${}^pV_4^n$ of $[r]$; we shall often make use of such representations in the sequel*.

We can now form a preliminary table (p. 372) of the surfaces F_2^r $(r \leqslant 4)$ and of their residual surfaces F_2^{9-r}, and we tabulate also for reference the order d of their curve of intersection.

3. *General procedure.* Each of the sixteen cases in the table is considered separately as follows.

We examine first the residual surface F_2^{9-r} and identify it, if possible, with some known surface, or with a projection of some kind of such a surface. If it is rational, we give its plane representation.

We consider next one or both of the systems of cubic primals, Φ_0 and Φ_0' say, which have the surfaces F_2^{9-r} and F_2^r as base surfaces respectively, noting that for the systems in question the *characteristic surfaces*†, *i.e.* the surfaces of intersection of pairs of primals of the system, are of the types F_2^r and F_2^{9-r} respectively‡. If it happens now in either case that these characteristic surfaces are rational, then we can obtain all the

* For the similar manifolds ${}^pV_3^n$, which represent systems of cubic surfaces in [3], see Semple, *Proc. Camb. Phil. Soc.*, 25 (1929), 145.

† I have elsewhere called these surfaces *Φ-surfaces*, where Φ is the system of primals under consideration.

‡ There is one exception to this rule in the case when the surface F_2^r is a double plane, *cf.* § 8.

properties of the system and all the H. systems of primals which it contains from a certain associated system of plane curves, which we shall call the *index system* of the system of primals. This is obtained, for the

TABLE OF RESIDUAL INTERSECTIONS.

No.	Surface F_2^r $(r \leqslant 4)$	p	p'	Residual Surface	d
i	Segre quartic F_2^4	1	2	${}^2F_2^5$	8
ii	Rational ruled F_2^4	0	1	${}^1F_2^5$	10
iii	Veronese F_2^4	0	1	${}^1F'^5_2$	10
iv	Cubic scroll and a plane	−1	0	${}^0F_2^5$	8+4
v	Double plane...	−1	0	${}^0F_2^5$	4
vi	Cubic surface in [3]	1	4	${}^4F_2^6$	6
vii	Cubic scroll in [4]	0	3	${}^3F_2^6$	8
viii	Quadric surface and a plane ...	−1	2	${}^2F_2^6$	6+4
ix	Three general planes	−2	1	${}^1F_2^6$	4+4+4
x	Three concurrent planes	−2	1	${}^1F'^6_2$	4+4+4
xi	Plane of contact and another plane	−3	0	${}^0F_2^6$	5+4
xii	Plane of osculation	−5	−2	3 F_2^2's	3+3+3
xiii	Quadric surface	0	5	${}^5F_2^7$	6
xiv	Two planes	−1	4	${}^4F_2^7$	4+4
xv	Plane of contact	−2	3	${}^3F_2^7$	5
xvi	Plane	0	7	${}^7F_2^8$	4

We shall refer to this table as T.

system Φ_0 say, by representing a characteristic surface F_2^r on a plane ϖ, and by defining as the index system of Φ_0 the system* of curves Λ_0 of ϖ, which represent complete intersections of F_2^r with primals of Φ_0 We note, however, that each curve Λ_0 breaks up into a base curve Δ, representing the intersection δ of F_2^r with the base surface F_2^{9-r}, together with variable curve Λ_1, describing a system with only base *points*. We shall also speak of this reduced system Λ_1 then as being the index system of Φ_0. It has clearly the following properties :—

(i) The grade of Λ_1 is equal to that of Φ_0; its freedom is less by two than that of Φ_0; and its curves have genus equal to the characteristic genus of Φ_0.

(ii) Any H. net of curves of Λ_1 leads to an H. system of cubic primals Φ. (For any H. net of the complete curves Λ_0 represents the intersection of such a system with F_2^r.)

Having definitely determined this system Λ_1 then, for which purpose we have first to identify the curve δ, we obtain from it what properties of Φ_0 we require.

We obtain finally all the H. systems of primals which have Φ-surfaces of the type of F_2^r simply by picking out all the H. nets of curves of the system Λ_1. The reverse systems Ψ of the corresponding cubic transformations can then be obtained by the usual considerations of exceptional elements†.

N.B.—I shall use the symbol $\{3, r(p), n\}$ to denote a cubic transformation whose reverse is of order n and whose surfaces are of order r and section-genus p.

The Segre quartic surface and a rational quintic surface.

4.1. We consider first the residual intersection of two cubic primals G_1, G_2 which have in common a Segre quartic intersection of two quadrics F_2^4. On turning to the Table (T. i) we obtain the information that the residual surface in question is a quintic of section-genus two, ${}^2F_2^5$, and that it meets F_2^4 in a curve of order eight δ^8. We identify it immediately then with the only non-ruled quintic surface of this section-genus in

* Compare the account in Q.T., p. 340, where the properties of Λ_0 are more fully explained.

† *Cf.* Q.T., p. 341.

[4], namely *the rational normal* ${}^2F_2^5$ *representable on a plane by means of quartic curves with a fixed node* D *and seven simple base points** $A_1, \ldots, A_7$, and we can easily verify that it is the *general* surface of this type simply by showing that such a surface lies on more than one cubic primal†. It lies, in fact, on seven such primals, all independent.

We remark next that *the surface* ${}^2F_2^5$ *lies on one quadric cone* K. For if $Q_1 = 0$, $Q_2 = 0$ be the equations of the two quadrics which intersect in F_2^4, then the equations of G_1, G_2 will be of the form

$$G_1 \equiv lQ_1 + mQ_2 = 0$$

and

$$G_2 \equiv l'Q_1 + m'Q_2 = 0,$$

where l, m, l', m' are linear, and ${}^2F_2^5$ will clearly lie on the cone K whose equation is

$$K = lm' - l'm = 0.$$

Thus *the curve* δ^8 *is a complete intersection of three quadrics*, namely Q_1, Q_2, and K. We note also that any cubic primal which passes through ${}^2F_2^5$ must meet K in this surface and, residually, in a plane of one the two systems on K‡.

4.2. *The system* Φ_0 *of cubic primals through* ${}^2F_2^5$. The first system of cubic primals which we consider is that which has the surface ${}^2F_2^5$ as base surface and the Segre quartic surface F_2^4, consequently, as general characteristic surface; and, since the latter surface is certainly rational, we can form an index system Λ_1 for the system. We represent F_2^4, for this purpose, on a plane ϖ by means of cubic curves through five points $A_1, \ldots, A_5$, and we remark then that the curve δ^8, being the section of the surface by the quadric cone K, will be represented on ϖ by a sextic curve Δ^6 having double points at $A_1, \ldots, A_5$. A complete *cubic* section of the surface, moreover, will be represented on ϖ by a C^9 having triple points at the A's, and a section then by a cubic primal through ${}^2F_2^5$ will break up into δ^8 together with a curve which is represented on ϖ by

---

* *Cf.* Castelnuovo, *Rend. di Palermo*, 4 (1890), 79.

† In the plane representation of the general surface, any line of the plane clearly represents a quartic curve of ${}^2F_2^5$. The intersection of the surface with a cubic primal will be represented therefore by a C^{12} with a 6-fold point at D and triple points at the A's. Such curves have freedom 27. Thus, if we make a cubic primal contain 28 points of the surface, it will contain it completely. But cubic primals in [4] have freedom 34; *i.e.* there are ∞^6 of them which contain ${}^2F_2^5$ completely.

We shall often use this method to find the freedom of cubic primals through a surface.

‡ Planes of one system of K will clearly meet ${}^2F_2^5$ in cubic curves and planes of the other system will meet it in conics. It is clearly a plane of the first system that lies on any cubic primal through ${}^2F_2^5$.

a cubic through $A_1, \ldots, A_5$, *i.e.* a prime section of F_2^4. Thus the index system Λ_1 consists of elliptic cubic curves through $A_1, \ldots, A_5$, and these have freedom four and grade four. It follows then that Φ_0 *has freedom six*, grade four, and characteristic genus one.*

The manifold ${}^1V_4^4$. We can interpret the result just obtained by saying that cubic primals through the surface ${}^2F_2^5$ can be represented on the prime sections of a manifold ${}^1V_4^4$ in [6]. This manifold is nothing more complicated, however, than the complete intersection of two general quadrics in [6]; for, if we project such an intersection from a line λ of itself, the projection is clearly birational, and prime sections project into cubic primals through a surface of the type ${}^2F_2^5$ whose points are projections of lines of the manifold which meet λ. Thus *cubic primals through* ${}^2F_2^5$ *can be represented on the prime sections of a quartic intersection* ${}^1V_4^4$ *of two quadrics in* [6].

4.3. *The cubic-cubic transformation* {3, 4(1), 3}. The index system Λ_1, obtained above, contains three types of H. nets; but these all lead to the same H. system of cubics Φ, the reason being that they are *equivalent* in the sense that we can transform any one of them into any other by a Cremona transformation of the plane which leaves the plane representation of F_2^4 unaltered. Consider then the simplest type, namely that which consists of the lines of ϖ, each augmented by the fixed conic k^2 which passes through $A_1, \ldots, A_5$. This last clearly meets Δ^6 in two free* points, and it corresponds therefore to a line, l say, on F_2^4, meeting ${}^2F_2^5$ twice. We deduce then that: *In any* [4] S, *cubic primals which pass through a surface* ${}^2F_2^5$ *and a chord* l *of this surface form an* H. *system* Φ. They define therefore a cubic transformation of S into another [4] Σ.

The Φ-surfaces in this transformation are, of course, Segre quartic surfaces, and so also, in fact, are the Ψ-surfaces; for any one of the latter is represented on a plane π of S by cubics through the five points in which π meets ${}^2F_2^5$.

The Φ-curves, moreover, are twisted cubics; for certain of them are represented on ϖ by the lines of ϖ; and, since any line of ϖ meets Δ^6 in six points and k^2 twice, the curves in question must meet ${}^2F_2^5$ in six points and k^2 twice. The fact, moreover, that the Φ-curves are cubics shows also that the reverse transformation is cubic, and it is clear by now that we are dealing with a symmetrical cubic-cubic transformation of [4].

---

* *Cf.* § 4.1, foot-note.

† By *free* intersections we mean intersections which are not at base points.

The points of the base line l' of the system Ψ arise from E-planes (*i.e.* exceptional planes) in S, namely from those planes of the cone K which meet ${}^2F_2^5$ in cubic curves, and so impose only one condition on Φ-primals to contain them. The points, similarly, of the base surface, ${}^2F_2'^5$ of Ψ arise from E-lines of S, which meet ${}^2F_2^5$ in two points and l once. Five of these latter are represented on ϖ by the five fundamental points $A_1, \ldots, A_5$, agreeing with the fact the base surface of Ψ is of order five.

Finally we notice that every Φ or Ψ-primal is represented on a solid of Σ or S by means of cubic surfaces through a quintic curve of genus two and a point, the point being the homologue of the unique plane which the primal possesses.

In conclusion then, *cubic primals in* [4] *through a rational normal* ${}^2F_2^5$ *and one of its chords give a symmetrical cubic-cubic transformation with Segre quartic* Φ*-surfaces.*

This transformation could also have been derived in another way, namely by projecting the quartic intersection ${}^1V_4^4$ of [6] (*cf.* § 4.2) first from a line of itself into a [4] S, and then from a different line of itself into a [4] Σ. This method has the advantage of bringing out the symmetry of the transformation.

4.4. *The system* Φ_0' *of cubics through a Segre* F_2^4. We consider next the other system of cubics Φ_0' which has F_2^4 as base surface and rational characteristic surfaces of the type ${}^2F_2^5$; and in order to find the corresponding index system we now represent ${}^2F_2^5$ on a plane ϖ by means of quartics with a fixed node D and seven base points $A_1, \ldots, A_7$.

The surface ${}^2F_2^5$ contains ∞^1 plane cubic curves lying in planes of one system of the cone K, and these are represented on ϖ by cubic curves passing through the eight points D and $A_1, \ldots, A_7$. These latter have therefore a ninth point in common, and *this is clearly the point of* ϖ *which represents the vertex of* K. Planes of the other system of K meet the surface in ∞^1 conics, namely those represented on ϖ by lines through D.

The curve δ^8 lies, as we have seen, on ∞^2 quadrics, of which one is K, and such quadrics will meet ${}^2F_2^5$ residually in ∞^1 conics, which must be those just mentioned. Now a complete quadric section of the surface is represented* on ϖ by a C^8 $(D^4, A_1^2, \ldots, A_7^2)$, and δ^8 then must be

* We shall often indicate the known singularities and base points of a curve by writing them, as above, in brackets after the symbol for the curve, and we shall indicate their multiplicities by indices.

represented by a septimic curve Δ^7 $(D^3, A_1^2, \ldots, A_7^2)$, and this has the correct genus, namely five.

Since, then, a complete *cubic** section is represented by a C^{12} $(D^6, A_1^3, \ldots, A_7^3)$, a section by a cubic primal through F_2^4 must be represented by Δ^7 together with a C^5 $(D^3, A_1, \ldots, A_7)$. Thus *the index system* Λ_1 *consists, in this case, of quintics with* D *as triple and* $A_1, \ldots, A_7$ *as simple base points;* and we deduce that the system of primals Φ_0' has freedom nine, grade nine, and characteristic genus three.

The primals of Φ_0' can be represented, then, on the prime sections of a manifold ${}^3V_4^9$ of [8], of section genus three. This is easily seen to contain ∞^1 quadrics V_3^2, one through any point of itself, corresponding to the ∞^1 quadrics through F_2^4; and it projects, in fact, from any one of these quadrics on to [4], prime sections projecting into cubic primals through a Segre quartic F_2^4. It has also other projections on to [4].

4.5. *The transformations* {3, 5(2), 4} *and* {3, 5(2), 5}. The index system Δ_1 just obtained comprises two kinds of H. nets which are not *equivalent* in the sense explained in § 4.3. The first may be represented by that which consists of the lines of the plane ϖ each augmented by a C^4 $(D^3, A_1, \ldots, A_7)$, and this gives a cubic quartic transformation in which Φ *consists of cubic primals through the Segre surface* F_2^4 *and a rational cubic curve quinquesecant of* F_2^4; while the second type of H. net of Λ_1 is represented by the net which consists of conics through D, A_1, and a fixed point P_1, each augmented by a C^3 $(D^2, A_2, \ldots, A_7)$, and this leads to a cubic quintic transformation in which Φ *consists of cubics through* F_2^4, *a trisecant conic and a base point.* In each of these cases the Φ surfaces are of the type ${}^2F_2^5$ and the Ψ surfaces are rational and quintic with elliptic prime sections†. There are no E-surfaces in S, but only systems of E curves giving rise to base surfaces of Ψ; and the multiplicity of any one of these latter is given, as usual, by the order of the corresponding E curves, while its order is equal to the number of such E curves which lie on F and which reveal themselves in ϖ as curves which meet curves of the associated H. net of Λ_1 in base points only (*cf.* Q.T. 340). We give the results of this work as follows.

---

* *I.e.* section by a cubic primal.

† In almost all the transformations which we obtain in this paper, the Ψ surfaces are rational with elliptic prime sections and of order equal to $9-r$, where r is the order of the base surface of Φ. This is because any one of them is represented on a plane of S by means of cubic curves through the r points in which the plane meets the base surface.

Transformation	H. net of curves of Λ_1	Base elements of Φ	Base surfaces of Ψ
$\{3, 5(2), 4\}$	*Lines of plane* + base quartic $C^4(D^3, A_1 \ldots, A_7)$	(1°) Segre surface F_2^4 (2°) Quinquesecant cubic curve	(1°) Septimic surface F_2^7 (2°) Double plane
$\{3, 5(2), 5\}$	*Conics through* O, A_1, *and a fixed point* P_1 + base cubic $C^3(D^2, A_2, \ldots, A_7)$	(1°) Segre surface F_2^4 (2°) Trisecant conic (3°) Base point	(1°) Septimic surface F_2^7 (2°) Double plane (3°) Triple plane

4.7. Conclusion. *The intersection of two cubic primals can break up into a Segre quartic surface* F_2^4 *and a rational quintic surface* ${}^2F_2^5$ *of section genus two, and each of the primals then contains one plane. Cubic primals through* F_2^4 *form a system of freedom nine, grade nine, and characteristic genus three; and those through* ${}^2F_2^5$ *form a system of freedom six, grade four, and characteristic genus one.*

The rational ruled F_2^4 *and a rational quintic surface.*

5.1. We consider next the residual intersection of two cubic primals which have in common a rational ruled* F_2^4 representable on a plane ϖ by means of ∞^4 cubic curves which have a fixed node D and which pass through a fixed point A. The residual intersection is a quintic surface ${}^1F_2^5$ with elliptic prime sections (*cf.* T. ii), and it is easily identified with the only such non-ruled surface† which exists in [4], namely *the rational* ${}^1F_2^5$ *which is representable on a plane* ϖ *by means of* ∞^4 *cubic curves through four base points* $A_1, \ldots, A_4$. This lies, in fact, on ∞^3 cubic primals, as we can easily verify by utilizing the plane representation‡. The surfaces F_2^4 and F_2^5 meet in a curve of order ten δ^{10}. We note also that the surface F_2^4 has a unique improper node§ O, *i.e.* it has a point O at which there are two distinct tangent planes not meeting in a line. Such a point of a rational surface is represented on the plane by a pair of *conjugate* points, *i.e.* by a pair of points which together impose only

* For properties of this surface, see Segre, *Encyk. der Math. Wiss.*, III C 7, 918, Note 452.

† *Cf.* Castelnuovo, *Rend. Lincei* (5), 3 (1894), 59.

‡ We use the same method as in the foot-note to § 4.1. Namely, we note that cubic sections of F_2^5 are represented on the plane by curves $C^9(A_1^3, \ldots, A_4^3)$ having freedom 30 only; so that ∞^3 of them can be made to contain 31 free points of the surface, and these will then contain it entirely.

§ See Segre, *loc. cit.*

one condition on the curves which represent prime sections of the surface; and any primal through the surface has necessarily a node at the point, for no single tangent prime can contain the two tangent planes there. Thus the two cubic primals through F_2^4 have nodes there, and F_2^5 then must likewise have a node at O. Finally, any cubic primal through either surface must meet the other in a curve having a 4-ple point at O, two branches of the curve lying in each of the two tangent planes at O.

5.2. *The system of cubics through* ${}^1F_2^5$. We consider first the system Φ_0 of ∞^3 cubic primals through ${}^1F_2^5$. Any two of these meet residually in a rational quartic scroll F_2^4, represented on a plane ϖ by cubic curves (D^2, A), and this scroll meets ${}^1F_2^5$, as we have seen, in a curve δ^{10}. Now any generator of F_2^4 is a trisecant line of δ^{10}—this is easily verified by considering a section of the figure by a solid through the generator; it follows then that every cubic primal of Φ_0 meets every generator of a characteristic surface in three points on ${}^1F_2^5$, and if therefore such a primal contain any point, not on F_2^5, of such a generator, it must contain the generator entirely. Thus the intersection, other than δ^{10}, of the cubic primal with the surface F_2^4 must consist of a pair of generators; and the complete section of the cubic primal with F_2^4 will be represented on ϖ by a C^9 (D^6, A^3), which breaks up into a pair of lines through O and a curve Δ^7 (D^4, A^3) representing δ^{10}. *The index system* Λ_1 *therefore consists of pairs of lines through* D, and since it can only be of freedom one, Φ_0 being of freedom three, it must consist of pairs of an involution. Moreover, the curve δ^{10} must have a 4-ple point at O, and Δ^7 then will have double points at each of the conjugate pair of points which represents O (§ 5.1). *Thus the curve* δ^{10} *is seen to be of genus four.*

5.3. *The system* Φ_0' *of cubics through* F_2^4. It has already been shown elsewhere* by the author that the system Φ_0', of cubic primals through a rational quartic scroll F_2^4, can be transformed to the system of quadrics through a rational cubic curve, and it was deduced from this that such primals have freedom seven, and that they are all of the seven-nodal variety, one of the nodes being fixed at the node O of F_2^4. We can obtain the index system for Φ_0' as follows. We represent ${}^1F_2^5$ on a plane ϖ by cubics C^3 $(A_1, \ldots, A_4)$, noting that the double-point O of the surface is represented as usual on ϖ by a pair of conjugate points O_1, O_2. The curve

* Q.T., 348.

δ^{10}, being the complete section of the surface by the quadric cone K which projects F_2^4 from its node, will be represented on ϖ by a curve Δ^6 $(A_1^2, \ldots, A_4^2)$, and since this has clearly additional nodes at O_1, O_2, and genus therefore equal to four, we again verify that δ^{10} has genus four. Finally, the section of the surface by a cubic primal through F_2^4 will be represented on ϖ by a C^9 $(A_1^3, \ldots, A_4^3)$, which consists partly of Δ^6 $(A_1^2, \ldots, A_4^2)$ and partly of a C^3 $(A_1, \ldots, A_4)$. Thus *the index system* Λ_1 *consists of cubic curves through* $A_1, \ldots, A_4$, and, since all such cubics have freedom five and Φ_0' has freedom seven, Δ_1 must be the *complete* system of the curves in question. We deduce then that Φ_0' has grade five and characteristic genus one.

The manifold ${}^1V_4^5$ of [7] whose prime sections represent primals of Φ_0' is already known as the section by a [7] of the well known M_6^5 of [9] which represents all the lines* of [4]. It is known to project birationally from a plane of itself, prime sections projecting into quadrics through a cubic curve, and it projects also from a conic of itself, prime sections now projecting into cubics through a rational quartic scroll F_2^4. It has also other kinds of birational projections on to [4].

5.4. *The three transformations* $\{3, 5(1), n\}$. The index system Λ_1 obtained in § 5.3 contains three irreducible types of H. net which we may take as represented by

(1) the lines or ϖ, each augmented by a fixed conic C^2 $(A_1, \ldots, A_4)$;

(2) conics through A_3, A_4, and a fixed point P, each augmented by the line A_1, A_2;

(3) cubic curves which pass through $A_1, \ldots, A_4$, and have a node at a fixed point P_1 of the plane ϖ.

These lead to H. systems Φ in which the cubic primals, besides having F_2^4 as base surface, have

(1) a base conic, quadrisecant of F_2^4;

(2) a base line, bisecant of F_2^4 and a base point P;

(3) a fixed tangent plane at a fixed point P.

The Φ-surfaces are of the type ${}^1F_2^5$ and the Ψ surfaces, for the reason given in the foot-note to § 4.5, are exactly similar. The reverse of the

* *Cf. Proc. London Math. Soc.* (2), 30 (1929), 505 and 543.

first transformation is cubic, and the transformation is, in fact, a symmetrical cubic-cubic transformation. *Thus cubic primals through a rational ruled F_2^4 and a quadrisecant conic give a symmetrical cubic-cubic transformation of* [4]. In this transformation any prime of the first space S meets the base conic in two points, and these correspond evidently to the two planes of the corresponding cubic Ψ-primal in Σ.

The details of the transformations are given in the following table, the reverse systems being obtained from exceptional elements in the usual way. In particular, Ψ has always a base conic whose points correspond to ∞^1 E-planes which meet F_2^4 in cubics nodal at O.

Transformation	H. net of curves of Λ_1	Base elements of Φ	Base elements of Ψ
$\{3, 5(1), 3\}$	*Lines of plane* + base conic $C^2(A_1, \ldots, A_4)$	(1°) Rational ruled F_2^4 (2°) Quadrisecant conic	(1°) Rational ruled F_2^4 (2°) Quadrisecant conic
$\{3, 5(1), 4\}$	*Conics through* A_3, A_4, *and point* P + line A_1A_2	(1°) Rational ruled F_2^1 (2°) Chord of F_2^4 (3°) Point P	(1°) Base conic (2°) Cubic surface (3°) Double F_2^2
$\{3, 5(1), 5\}$	*Cubic curves* $(A_1, \ldots, A_4, P_1^2)$	(1°) Rational ruled F_2^4 (2°) Tangent plane at fixed point	(1°) Conic (2°) Double F_2^5

These three transformations can also be obtained by projecting the manifold ${}^1V_4^5$ of [7], which represents the system Φ_0' (§ 5.3). We project ${}^1V_4^5$, namely, first from a conic of itself on to a [4], S, and then from

(*a*) another conic of itself,

(*b*) a line and a point of the locus,

(*c*) a tangent plane,

on to a second [4], Σ, and the three transformations so obtained are identical with those given above.

In conclusion: *The intersection of two cubic primals can break up into a rational ruled F_2^4 and a general rational quintic surface with elliptic prime sections. The primals each contain two planes and seven nodes; and cubic primals through F_2^1 form a system of freedom seven, grade five, and characteristic genus one; while those through F_2^2 have freedom three and grade zero.*

The projected Veronese surface and an elliptic quintic scroll.

6. The case in which the intersection of two cubic primals breaks up into a projected Veronese surface F_2^4 and an elliptic quintic scroll ${}^1F'^5_2$ (the dash distinguishes the surface from that discussed in § 5) has already been considered in detail by the author*. We need only then state the results here for completeness. F_2^4 and ${}^1F'^5_2$ meet in a decimic curve δ^{10}; cubics through ${}^1F'^5_2$ form a homaloidal system whose reverse system is that of quadrics through an elliptic quintic curve; cubics through F_2^4 form a system of freedom six, of which any three primals meet residually in five trisecants of F_2^4; finally all the primals concerned are Segre tennodal cubics.

The rational quintic scroll ${}^0F_2^5$ residual to a cubic scroll and a plane.

7.1. From case iv of Table T we see that any two cubic primals which have in common a cubic scroll Γ and a plane σ have as residual intersection a quintic surface ${}^0F_2^5$, which has rational prime sections and which meets Γ in an octavic curve δ^8 and σ in a quartic curve δ^4. This surface ${}^0F_2^5$ is ruled, for it has rational prime sections and its order exceeds four. It is, in fact, the *general rational quintic scroll*, ${}^0R_2^5$, *of* [4]; for ${}^0R_2^5$ is known to possess a plane trinodal quartic curve† δ^4, and any two cubic primals through it must intersect residually in the plane of δ^4 and in a cubic scroll Γ. We notice also that Γ meets σ in three points A, B, and C, and that cubics through Γ and σ must have nodes at these points, as must also have ${}^0F_2^5$ and both the curves δ^8 and δ^4.

7.2. *The system Φ_0 of cubics through ${}^0F_2^5$.* Cubic primals Φ_0, through ${}^0F_2^5$, all contain, as we have seen, the plane σ, and these two surfaces make up a degenerate ${}^3F_2^6$ residual for cubic primals to a cubic scroll. Now the freedom of cubics through ${}^3F_2^6$ is found elsewhere in this paper to be three (§ 10.1). It follows then that Φ_0 is of freedom three. Some of the primals Φ_0, however, have σ as double plane, and have consequently no further residual intersection. For a cubic primal will have a plane as double plane if it contain five nodes in it, and if we

* Q.T., 337. See however, also, Severi, *Mem. Torino*, 52 (1903), § 9, who obtains the two surfaces as complementary intersections of two cubic primals.

† A ruled surface ${}^pR_2^n$ in [4] has $(n-2)_2-3p$ double points (*cf.* Segre, *Encyk. der Math. Wiss.*, III C7, 913): for ${}^0R_2^5$, this number is three; and it can easily be shown that the plane joining the three double points meets the surface in a trinodal quartic curve. I am indebted to Mr. W. L. Edge for this remark.

make primals of Φ_0 have nodes at two given points of δ^4, this requiring only two conditions, they will have A, B, C also as nodes, and they will therefore contain the plane σ doubly. Thus ∞^1 *of the primals* Φ_0 *contain* σ *as double plane.*

7.3. *Cubic primals* Φ_0' *through* Γ *and* σ. Cubic primals through the cubic scroll Γ have freedom twelve*, and seven further conditions are then required to make them contain the plane σ, *i.e.* Φ_0' must be of freedom five. Moreover, exactly one chord of Γ can be drawn through any point to meet also the plane σ, and any primal of Φ_0' is therefore generated completely by such chord secants. In particular, any three primals of Φ_0' must intersect residually in three chords of Γ, which also meet σ. Each of the primals Φ_0', moreover, is of the seven-nodal variety; for any cubic primal through Γ must have six nodes on that surface†, and a primal of Φ_0' therefore has six nodes on Γ, three of which are at A, B, C, while the remaining three lie in a second plane of the primal which meets σ in a fourth node in that plane. In conclusion then: *A pair of cubic primals which have in common a cubic scroll* Γ *and a plane* σ *intersect residually in a general rational quintic scroll* ${}^0F_2^5$ *and the primals are of the seven-nodal variety. Cubic primals through* ${}^0F_2^5$ *are of freedom three, and* ∞^1 *of them have* σ *as double plane; and cubic primals through* Γ *and* σ *have freedom five, and any three of them meet residually in three lines.*

Cubic primals with a fixed double plane.

8.1. We come now to case v of the table, in which we consider the residual intersection of two cubics which have a double plane σ in common. We deduce, as in the case immediately preceding this, that the residual intersection in question is a rational quintic scroll, and from § 7.2 we see, in addition, that this scroll is again the *general* rational quintic scroll of [4]. It is met by the plane σ in a trinodal‡ quartic curve δ^4, and any cubic primal which has σ as double plane meets the surface in δ^4 counted twice and residually in a curve of order seven.

8.2. *The system* Φ_0' *of cubics which have a fixed double plane.* We now represent the general quintic scroll ${}^0F_2^5$ on a plane ϖ by means of

* All cubic primals through a cubic scroll are determinantal and therefore six-nodal. Q.T., p. 354.

† *Cf.* § 10.1.

‡ The curve δ^4 is clearly a directrix curve on ${}^0F_2^5$, and it will have nodes at the three nodes of the surface, which lie, as we have seen, in σ.

an ∞^4 system of cubic curves K which have a fixed node D. The three double points of the surface are represented as usual by conjugate pairs of points (A_1, A_2), (B_1, B_2), (C_1, C_2), which each impose only one condition on curves of K to contain them, and the curve δ^4 is obviously represented by a conic through D and all the three pairs of points mentioned. A general cubic section of the surface, moreover, is represented by a C^9 having D as 6-ple point, and a section then by a cubic having σ as double plane will break up into δ^4 counted twice and a curve represented on ϖ by a C^5 having D as 4-ple point. Thus *the index system* Λ_1 *consists in this case of quintic curves with a* 4*-ple point* D*;* and since such curves have freedom ten, and cubics with a double plane are known to have freedom twelve, the system Λ_1 is complete. We deduce then that three primals of Φ_0' intersect residually in a rational septimic curve, and that four such primals intersect residually in nine points. Thus Φ_0' *has freedom twelve, grade nine, and characteristic genus zero.*

8.3. *The representative manifold* ${}^0V_4^9$ *of* [12]. The manifold ${}^0V_4^9$ of [12], which represents all cubics with a fixed double plane, is of a very simple character and very analogous to the ${}^0V_3^7$ of [9] already discussed by the author*, which represents cubic surfaces in [3] with a fixed double line. In fact, by immediate analogy with the latter, we can write down most of the important properties of ${}^0V_4^9$ as follows :—

(i) ${}^0V_4^9$ contains ∞^1 generating solids corresponding to solids through the plane σ.

(ii) ${}^0V_4^9$ contains ∞^6 directrix cubic curves corresponding to lines of [4], and it contains also a directrix ${}^0V_3^6$ of planes corresponding to the complete neighbourhood of the plane σ. The latter manifold is itself a locus of planes which join the generators of a rational quartic scroll to the corresponding points of a conic whose points are projectively related to the generators in question.

(iii) ${}^0V_4^9$ is completely defined as the locus of solids which join corresponding points and planes of a conic and of a ${}^0V_3^6$ of planes whose points and planes respectively are projectively related.

(iv) ${}^0V_4^9$ projects birationally on to [4] from any pair of solids of itself, prime sections projecting into cubic primals with a fixed double plane.

---

* *Proc. Camb. Phil. Soc.*, 25 (1929), 163.

The manifold obviously projects also from any eight points of itself which do not all lie in a [6] or on ${}^0V_3^6$; and the general projection theorem is as follows :—

${}^0V_4^9$ projects birationally on to [4] *from* q *solids,* r *planes,* s *lines, and* t *points of itself, where* $4q+3r+2s+t=8$, *prime sections projecting into primals of order* $9-s-2r-3q$ *which have in common an* $(8-s-2r-3q)$*-ple plane* σ, t *simple planes meeting* σ *in lines,* s *simple lines meeting* σ, *and* q *other points.*

8.4. *Cubic transformations with a base double plane.* We get all possible cubic transformations of this kind by projecting ${}^0V_4^9$ first from a pair of solids of itself on to a [4], S, and then from r planes, s lines and t points of itself $(3r+2s+t=8)$ on to another [4], Σ; and to find the systems Φ and Ψ we use the projection theorem given above. The result is as follows :—

There is a cubic transformation of [4] *in which the first homaloidal system* Φ *consists of cubic primals which have in common a double plane* σ, r *planes meeting* σ *in lines,* s *lines meeting* σ, *and* t *points, where* $3r+2s+t=8$; *and in which* Ψ *consists of primals of order* $7-2r-s$ *which have in common a* $(6-2r-s)$*-ple plane* σ', t *planes meeting* σ' *in lines,* s *lines meeting* σ', *and* r *other points.*

There are also other non-cubic transformations derivable by projecting ${}^0V_4^9$ birationally from more general vertices of projection.

Symmetrical cubic-cubic transformations. We note, in conclusion, that we obtain three distinct *symmetrical* cubic transformations from the general formula by taking the triad (r, s, t) to be (2, 0, 2), (1, 2, 1), and (0, 4, 0) respectively. In these cases the H. systems of cubics have base elements as follows :—

(*a*) a double plane σ, two planes meeting σ in lines, and two general points;

(*b*) a double plane σ, one plane meeting σ in a line, two lines meeting σ, and one other point; and

(*c*) a double plane σ and four lines meeting σ.

N.B.—So far we have given explicitly all the H. systems Φ obtained by taking any particular surface as base surface; but, as the operation of finding these is purely mechanical once the auxiliary system of plane

curves Λ_1 is obtained, and as the lists of transformations in the remaining cases are rather formidable, we shall limit ourselves henceforth to the determination the system Λ_1 in each case and to an indication of the simplest H. system Φ arising from each such system.

The sextic surface of section genus four.

9.1. The next case is that of the residual intersection of two cubic primals which have in common an ordinary cubic surface lying in a [3]. The residual intersection is easily seen to be a sextic surface ${}^4F_2^6$ with prime sections of genus four, a complete intersection of a quadric with a cubic primal. This case need not detain us long, for it is exactly analogous to that obtained by taking a section of the figure by a [3], and this has been considered at length elsewhere*.

In particular, *all the cubic primals through* ${}^4F_2^6$ *can be represented on the prime sections of a cubic primal* V_4^3 *in* [5], and this primal has generally a unique node corresponding to the whole of the quadric which contains ${}^4F_2^6$. And, conversely, if a nodal cubic primal in [5] be projected from its node on to [4], prime sections project into cubic primals through a surface ${}^4F_2^6$.

As in the case referred to, moreover, if the primal V_4^3 be taken to have two nodes, we can project it in turn from each of these, and so generate a cubic-cubic transformation of one [4], S, into another [4], Σ. In such a procedure, however, the base surfaces ${}^4F_2^6$ in S and Σ respectively will each have a proper node, and the systems Φ and Ψ will each consist of cubic primals passing through a *nodal* ${}^4F_2^6$ and having themselves a node at the node of that surface. There is clearly no need to enter into further detail here.

The cubic scroll and the projectively-generated sextic surface.

10.1. If two cubic primals be made to contain a cubic scroll they will intersect residually in a sextic surface ${}^3F_2^6$ of section genus three, and the two surfaces intersect each other in an octavic curve δ^8 (*cf.* T. vii). The surface ${}^3F_2^6$ will be immediately recognized as *the general determinantal sextic surface*, which is representable on a plane ϖ by means of quartic curves through ten points† $A_1, \ldots, A_{10}$; for such a

* *Proc. Camb. Phil. Soc.*, 25 (1928), 147.

† White, *Proc. Camb. Phil. Soc.*, 21 (1922), 216.

surface has equations of the form

$$\begin{Vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{Vmatrix} = 0,$$

the elements of the matrix being linear functions of the coordinates, and it obviously lies on ∞^3 determinantal cubic primals, any two of which intersect residually in a cubic scroll. The cubic primals in question are all known to have six nodes.

10.2. We consider first now the system of ∞^3 cubic primals through ${}^3F_2^6$. Any cubic primal of this system, since it meets Γ already in the curve δ^8, will meet it further in a generator only. On representing Γ, then, on a plane by means of conics through a point O, the line will be represented by a line through O, and δ^8 will be represented by a quintic curve with a node at O. We infer therefore that δ^8 *is of genus five.* It lies incidentally on the ∞^2 quadrics which contain Γ.

10.3. *The system Φ_0' of cubic primals through Γ.* Consider next the more important system Φ_0' of cubics through Γ. In the representation of ${}^3F_2^6$ on ϖ, any quadric section of the surface is represented on ϖ by a C^8 $(A_1^2, \ldots, A_{10}^2)$; and since δ^8 is of genus five and forms part of ∞^2 quadric sections, the complementary curve in each case being a quartic, it must be represented on ϖ by a septimic curve Δ^7 $(A_1^2, \ldots, A_{10}^2)$, for such a curve is of genus five and forms a C^8 $(A_1^2, \ldots, A_{10}^2)$, with any line of the plane. Since then any section of ${}^3F_2^6$ by a cubic primal is represented by a C^{12} $(A_1^3, \ldots, A_{10}^3)$, a section by one which passes through Γ will be represented by Δ^7 $(A_1^2, \ldots, A_{10}^2)$ together with a quintic curve C^5 $(A_1, \ldots, A_{10})$. Thus *the index system Λ_1 for Φ_0' consists of quintic curves through $A_1, \ldots, A_{10}$, and it is a complete* system.* We deduce that Φ_0' has grade fifteen and characteristic genus six. In conclusion then: *The intersection of two cubic primals may consist of a cubic scroll Γ and a determinantal sextic surface ${}^3F_2^6$, and in such case the primals are of the six-nodal variety. The system of cubics through ${}^3F_2^6$ has freedom three and grade zero, and that of cubics through Γ is of freedom twelve, grade fifteen, and characteristic genus six.*

10.4. *Transformations with ${}^3F_2^6$ as Φ-surface.* We can distinguish five cubic transformations which arise from the system Λ_1. The simplest

* There are easily seen to be ∞^{12} cubic primals through a cubic scroll; *cf.*, for example, Q.T., 354.

of these is that which arises from the lines of the plane with a quartic through $A_1, \ldots, A_{10}$ as base curve, and the corresponding transformation is that in which Φ *consists of cubic primals which have in common a cubic scroll and a sextic curve of genus three meeting the scroll in eight points.* The reverse system Ψ consists of quartics which have a base surface of order ten F_2^{10} and the primals of Ψ have also a fixed triple line λ which is sextuple for F_2^{10}, and they each contain six planes through λ, the locus of such planes being a line cone of genus three. These details are obtained by considering the exceptional elements in the transformation.

The rational ${}^2F_2^6$ residual to a quadric surface and a plane.

11.1. We consider next the case of two cubic primals which have a plane σ and a quadric surface Q in common, and we see from the table (T. viii) that the residual intersection is a sextic surface ${}^2F_2^6$, of section-genus two, meeting σ in a quartic curve δ^4 and Q in a sextic δ^6. The latter curve is clearly the section of ${}^2F_2^6$ by the solid containing Q. Both the cubic primals will have nodes at the two points A, B, in which σ meets Q, and ${}^2F_2^6$, then, will likewise have nodes at these points, as will also each of the curves δ^4 and δ^6. Thus ${}^2F_2^6$ contains a plane binodal quartic curve.

The surface ${}^2F_2^6$, being not ruled and of section genus two, is known to be representable on a plane ϖ by means of uninodal quartics C^4 which, in addition to having a fixed node D, have six other points $A_1, \ldots, A_6$ in common*. The *normal* surface of this type is the ${}^2N_2^6$ of [5]; we shall show now that ${}^2F_2^6$ *is the general projection of this surface from a point outside itself.* This we do as follows.

Of the ∞^5 quartics C^4 $(D^2, A_1, \ldots, A_6)$ which represent prime sections of ${}^2N_2^6$, we can choose six independent representatives to be l_1K_1, l_1K_2, l_1K_3, l_2K_1, l_2K_2, l_2K_3, where l_1 and l_2 are any two lines through D and K_1, K_2, K_3 are three linearly independent cubic curves which can be drawn through the seven points $D, A_1, \ldots, A_6$. These cubic curves represent elliptic quartic curves ${}^1K^4$ belonging to a net of such curves on ${}^2N_2^6$. We can take the system of the ∞^5 curves C^4 then to be given by the equation

$$l_1(\lambda_1K_1+\lambda_2K_2+\lambda_3K_3)+l_2(\lambda_4K_1+\lambda_5K_2+\lambda_6K_3) = 0,$$

where the λ's are free coefficients. On projecting the surface now from

* *Cf.* Castelnuovo, *loc. cit.*

a general point, which is equivalent to assuming a general relation

$$a_1\lambda_1+a_2\lambda_2+a_3\lambda_3+a_4\lambda_4+a_5\lambda_5+a_6\lambda_6 = 0$$

between the λ's, there is exactly one of the curves ${}^1K^4$ which projects to a *plane* elliptic quartic of the projected surface ${}^2N_2^6$. For a curve $\lambda_1K_1+\lambda_2K_2+\lambda_3K_3$ will represent a *plane* curve of ${}^2N_2$ if it belong to ∞^1 curves of the *restricted* system of curves C^4, *i.e.* if the curve

$$(l_1+\mu l_2)(\lambda_1K_1+\lambda_2K_2+\lambda_3K_3) = 0$$

belongs to the restricted system for every value of μ, and this leads to the equations

$$a_1\lambda_1+a_2\lambda_2+a_3\lambda_3 = a_4\lambda_1+a_5\lambda_2+a_6\lambda_3 = 0$$

determining the ratios $\lambda_1 : \lambda_2 : \lambda_3$ uniquely. Thus *the general projection* ${}^2N_2'^6$ *of* ${}^2N_2^6$ *contains a unique plane binodal quartic curve* δ^4. This enables us to show immediately that ${}^2N_2'^6$ is a partial intersection of cubic primals. For cubic primals which contain the plane of δ^4 have freedom twenty-four, and yet they meet ${}^2N_2^6$ residually in curves* C^9 $(D^5, A_1^2, \ldots, A_6^2)$ having only freedom twenty-one, so that ∞^2 of them must contain the surface completely. Thus ${}^2N_2'^6$ is a partial intersection of any two of the latter, and it is therefore essentially the surface ${}^2F_2^6$.

11.2. *The system* Φ_0' *of cubics through* Q *and* σ. The curve δ^4 of ${}^2F_2^6$ is represented on ϖ, as we have just seen, by a cubic Δ^3 $(D, A_1, \ldots, A_6)$ and the curve δ_6, being a prime section of the surface, is represented by a quartic Δ^4 $(D^2, A_1, \ldots, A_6)$. It is easy to see then that cubic primals Φ_0' through Q and σ meet ${}^2F_2^6$ residually in curves represented on ϖ by quintics C^5 $(D^3, A_1, \ldots, A_6)$, and since these have freedom eight, and Φ_0 is easily seen to have freedom ten, the quintics in question form a system which is complete. Thus the index system Λ_1, of Φ_0', is the complete system of quintics C^5 $(D^3, A_1, \ldots, A_6)$, and we deduce that Φ_0' has grade ten and characteristic genus three.

In conclusion then: *Two cubic primals which have in common a plane* σ *and a quadric surface* Q *intersect further in a general rational sextic surface of section genus two; and cubic primals through* Q *and* σ *form a system of freedom ten, grade ten, and characteristic genus three. The primals are all of the seven-nodal variety.*

11.3. *Associated transformations.* There are in the present instance four cubic transformations which arise from H. nets of the system Λ_1,

* Complete cubic sections of the surface are represented by curves C^{12} $(D^6, A_1^3, \ldots, A_6^3)$, and if we fix the cubic curve K which represents δ^4 we are left with curves C^9 $(D^5, A_1^2 \ldots, A_6^2)$.

and which have Φ-surfaces consequently of the type of ${}^2F_2^6$. The simplest of these is that which arises from the H. net consisting of the lines of ϖ each augmented by a $C^4\,(D^3, A_1, \ldots, A_6)$, and it is by means of *cubic primals Φ which have in common a quadric surface Q, a plane σ, and a rational normal quartic curve meeting σ in three points and Q in four.* The reverse system Ψ consists of quartic primals which have in common a sextic surface and a double plane.

The elliptic sextic scroll residual to three planes.

12.1. Consider now the residual intersection of two cubic primals which have three general planes σ_1, σ_2, σ_3 in common. We see from the table, case ix, that the surface so arising is a sextic ${}^1F_2^6$ with elliptic prime sections, and that it meets the planes σ_1, σ_2, σ_3 in quartic curves δ_1^4, δ_2^4, δ_3^4, respectively. Each of these latter will clearly have nodes at two of the points A_{23}, A_{31}, A_{12} in which the planes σ_1, σ_2, σ_3 intersect.

Take now a section of the figure by a solid through the plane σ_1. This solid will meet the two cubic primals residually in quadric surfaces having in common two skew lines, of which one lies in σ_2 and the other in σ_3; and since two such quadrics intersect further in two lines meeting the former two, the solid in question must meet ${}^1F_2^6$ in the latter pair of lines. Thus any solid through one of the three given planes meets ${}^1F_2^6$ residually in two lines. ${}^1F_2^6$ then is an elliptic sextic scroll.

It is, moreover, the general elliptic sextic scroll in [4]; for the latter is known to have three nodes* and three plane quartics as above†, and any two cubic primals through such a surface would have to intersect further in the planes of the quartic curves.

Thus *two cubic primals which have three general planes in common intersect residually in a general elliptic sextic scroll, having nodes at the intersections of the planes and meeting each of them in an elliptic quartic curve.*

Three cubic primals through the planes will intersect further in a sextic curve consisting of six lines, for any cubic primal through the planes and any point P must contain the unique transversal of the three planes which passes through P.

---

* *Cf.* the general formula quoted in foot-note to § 7.1.

† ${}^1F_2^6$ is the projection, from a point O, of the normal ruled surface ${}^1R_2^6$ of [5]; and the three nodes arise from three chords OD_1D_2, OE_1E_2, OF_1F_2 of ${}^1R_2^6$ which pass through O. A [4] which is drawn to contain the first two chords and the generators through D_1, E_1 say, will obviously meet the surface residually in an elliptic quartic curve through D_1, D_2, E_1, E_2, and this will project into one of the three elliptic plane quartics on ${}^1F_2^6$. I owe this remark also to Mr. W. L. Edge.

A rational sextic surface with elliptic prime sections.

13.1. We consider now the intersection of two cubic primals which contain three planes σ_1, σ_2, σ_3 meeting in a point O. The residual surface ${}^1F_2'^6$ has, as in the last case, elliptic prime sections; but the residual section of it by a solid through one of the planes is now a conic instead of two skew lines, and the surface is therefore not ruled. It can only then be a special type of the rational sextic surface which is represented on a plane ϖ by cubics through three points* A_1, A_2, A_3. The two cubic primals, moreover, have now not only nodes at O, but they have also *the same nodal cone* there, for this is completely determined by the three planes through O; and the three quartic curves δ_1^4, δ_2^4, δ_3^4 in which σ_1, σ_2, σ_3 meet ${}^1F_2^6$ will now all have triple points at O, as can easily be verified by taking a section of the figure by a solid through O. It appears then that ${}^1F_2'^6$ is exactly the surface obtained by projecting the rational normal surface of order six in [6] from a line lying in a trisecant plane $P_1P_2P_3$ of that surface; for one curve of each of the three systems of rational quartics on that surface passes through P_1, P_2, P_3, and, on projection, the plane $P_1P_2P_3$ projects into a triple point O of the projected surface, and the three quartics through P_1, P_2, P_3 project into three plane quartics on the surface, all trinodal at O.

13.2. *The system of cubics through three concurrent planes.*

Let us represent ${}^1F_2^6$ now on a plane ϖ by means of cubics through three points A_1, A_2, A_3. Complete cubic sections of the surface are represented on ϖ by curves C^9 (A_1^3, A_2^3, A_3^3), and the curves δ_1^4, δ_2^4, δ_3^4 are represented by conics through the pairs of points (A_2, A_3), (A_3, A_1), and (A_1, A_2) respectively. Cubic primals through σ_1, σ_2, and σ_3, then, meet ${}^1F_2^6$ residually in sextic curves represented on ϖ by cubics through A_1, A_2, A_3; and since these have freedom six and the freedom of cubic primals through σ_1, σ_2, σ_3 is easily seen to be eight, the system of cubic curves in question is complete. Thus the index system Λ_1 is in this case the complete system of cubic curves through A_1, A_2, A_3, and it has grade six and genus one. We deduce then that:

Two cubic primals which pass through three concurrent planes σ_1, σ_2, σ_3 intersect residually in a surface of order six with elliptic prime sections, general of its kind, except that it has a TRIPLE *point instead of three double points. All cubic primals through the three planes form a system of freedom eight, grade six, and characteristic genus one.*

---

* It is a projection of the Del Pezzo surface of order six in [6].

13.3. *Cubic transformations.* There are a number of cubic transformations having ${}^1F_2^6$ as Φ-surface; but we mention only the simplest, namely that obtained from the system Λ_1 by considering the H. net of its curves, which consist of the conics through A_1, A_2, A_3, each augmented by a fixed line of the plane. The derived system Φ consists of cubic primals which contain three concurrent planes and a rational cubic curve which meets each of the planes twice; and since the conics A_1, A_2, A_3 represent cubic Φ curves, the reverse transformation must also be cubic. The system Ψ, moreover, must be such that two of its cubic primals intersect residually in a surface represented on a plane of the first space by cubics through three points, *i.e.* in a sextic surface with elliptic prime sections. Thus the transformation is symmetrical. In conclusion then: *Cubic primals through three concurrent planes and a rational cubic curve meeting each of the planes twice give a symmetrical cubic-cubic transformation of* S_4.

13.4. *The manifold* ${}^1V_4^6$ *in* [8]. Cubic primals through three concurrent planes are represented on the prime sections of a manifold ${}^1V_4^6$ in [8], which is clearly the Segre ${}^1V_4^6$ which represents the pairs of points of two planes*; and since cubics through the three planes and a properly chosen cubic curve form a homaloidal system, the representation in question must be obtainable by projecting ${}^1V_4^6$ from a cubic curve of itself. If, then, we project it from two such cubic curves on to spaces S_4 and S_4', we get the symmetrical cubic-cubic transformation already described.

We note, moreover, that there is another representation of ${}^1V_4^6$ on [4], namely by means of quadrics through two lines†. This means, clearly, that *all cubic primals through three concurrent planes can be transformed into quadrics through two lines.* The second representation referred to is obtainable by projecting ${}^1V_4^6$ from a quadric surface; and if we project it from such a surface and then from a cubic curve, we get a particular case of a quadric cubic transformation of [4], already discussed.

A rational sextic scroll.

14.1. The next case on our list (case xi of the table) is that of the intersection of two cubic primals which have one plane σ in common and which touch each other at every point of another plane σ'. The

* This is the manifold in [8] obtained by making all the first minors of a three-rowed determinant of linear forms vanish.

† This was first noticed by P. du Val.

residual surface is seen from the table to be a sextic with rational prime sections—obviously a sextic scroll, ${}^0F_2^6$—and it meets σ in a quartic curve δ^4 and σ' in a quintic curve δ^5. These two curves are easily seen to be directrix curves on the surface so that they are rational curves. The two cubic primals, moreover, will have nodes at the point of intersection, O say, of the planes, and, as in the last case, they will also have *the same nodal cone at* O, for this cone is determined completely by the plane σ and the plane of contact σ'. Thus their complete intersection has a sextuple point at O, so that ${}^0F_2^6$ has a triple point there. The curves δ^4 and δ^5, then, must also have triple points at O, and this can be verified rigorously by considering a section of the figure by a prime through O. Finally the curve δ^5, since it is rational, must have three further double points in σ', *i.e.* the surface ${}^0F_2^5$ has, in all, one triple and three double points.

14.2. *The system of cubics containing* σ *and having* σ' *as plane of contact.* The surface ${}^0F_2^6$ being a projection of the most general type of rational normal sextic scroll* in [7], is represented on a plane ϖ by quartic curves which have a base triple point D and a simple base point A. The curve δ^4 will be represented by a conic through D and A, and similarly δ^5 will be represented by a cubic nodal at D and passing through A. Complete intersections of the surface, moreover, by cubic primals will be represented by curves $C^{12}(D^9, A^3)$; and since cubic primals which pass through σ and which have σ' as plane of contact contain the curve δ^4 and *touch* ${}^0F_2^6$ along the curve δ^5, such primals must meet ${}^0F_2^6$ residually in curves represented on ϖ by quartic curves having D as 4-ple point, *i.e.* by tetrads of lines through D. Thus the system of cubic primals in question is such that any three primals of it meet residually in four lines. In conclusion then :

Two cubic primals which contain one plane σ *and which touch over another plane* σ' *meet residually in a rational sextic ruled surface with one triple and three double points. Three such cubic primals meet residually in four lines, and the complete system is easily seen to have freedom five.*

---

* ${}^0F_2^6$ is exactly the surface obtained by projecting the most general rational *normal* sextic scroll in [7], ${}^0N_2^6$, from a plane which (*a*) lies in a [5] containing a rational normal C^5 of ${}^0N_2^6$, and (*b*) meets in a line a trisecant plane of the C^5 in question. The C^5 projects into the plane curve δ^5 in σ', and the other plane curve δ^4 of ${}^0F_2^6$ is the projection of the unique rational quartic curve of ${}^0N_2^6$ which passes through the three points of the surface which lie in the trisecant plane mentioned. ${}^0N_2^6$ contains ∞^1 directrix cubic curves, and it is represented on a plane ϖ by means of quartics with one triple and one simple base point.

A plane of osculation.

15. If two cubic primals have osculating contact* at every point of a fixed plane σ, they will have nodes at the same four points of σ, and they will have also the same nodal cones at those points. They will intersect residually, moreover (*cf.* table, case xii), in three quadric surfaces, and these will meet σ in conics through the four common nodes. The complete system of such cubics is of freedom five.

The surface ${}^5F_2^7$ residual to a quadric surface.

16. If two cubic primals have a quadric surface Q in common they intersect residually in a septimic surface ${}^5F_2^7$ with prime sections of genus five (*cf.* table, case xiii), and Q meets this surface in a sextic curve C^6—obviously a sextic of genus four, the complete intersection of Q by a cubic primal. The cubic primals, moreover, are met residually in planes by the solid in which the surface Q is immersed, and these planes intersect in a line l of ${}^5F_2^7$, bisecant of C^6. Beyond the fact, however, that any solid through l meets ${}^5F_2^7$ residually in a sextic of genus four, there do not seem to be any further remarkable properties of that surface : it is certainly not rational, and I have not been able to transform it to any much simpler surface.

The surface is among those considered by Roth, *Proc. Camb. Phil. Soc.*, 25 (1929), 394. This paper came to my notice after the completion of the present one ; the author considers several of the surfaces here taken from an analytical point of view It may be remarked incidentally that the surface in question is the projection of the general surface of intersection of three quadrics in [5] from a point of itself.

The rational septimic surface of section genus four, residual to two planes.

17.1. Two cubic primals which have two general planes σ_1, σ_2 in common meet residually (*cf.* T. xiv) in a septimic surface ${}^4F_2^7$ with prime sections of genus four. Each of the primals has, of course, a node at the point of intersection, O say, of the two planes, and ${}^4F_2^7$ then, has like-

* If the equation of σ be $x = y = 0$, then cubics which osculate at every point of it will have an equation of the form

$$0 = ax^3 + 3bx^2y + 3cxy^2 + dy^3 + (x^2, xy, y^2, x, y)(w_1, w_1', w_1'', w_2, w_2'),$$

where only a, b, c, d are variable coefficients, and the w's are polynomials in z, t, u.

wise an improper node at this point, and it meets the planes in quartic curves, δ_1^4 and δ_2^4, also nodal at O.

The first thing to notice is that the surface ${}^4F_2^7$ is rational. This follows from the fact that it contains a *rational pencil of rational curves**; for any prime through σ_1, for example, meets the two cubic primals residually in quadric surfaces which have a line of σ_2 in common, *i.e.* the prime in question meets ${}^4F_2^7$ residually in a rational cubic curve belonging to a rational pencil of such curves. Knowing, then, that the surface is rational, it should not be difficult to identify its plane representation This we do as follows.

We begin by trying the simplest system of plane curves of genus four we can think of which has grade seven and freedom four. Such a system is defined by all the quintic curves C^5 in a plane ϖ which have in common two double points D_1, D_2, and ten simple points A_1, ..., A_{10}; and such a system therefore defines a surface ${}^4H_2^7$ normal in [4]. This happily is exactly our surface ${}^4F_2^7$. For there is one quartic curve Δ_1^4 in ϖ which has a node at D_1 and which passes through D_2 and A_1, ..., A_{10}, and one similar curve Δ_2^4 which has a node at D_2 and which passes through D_1 and A_1, ..., A_{10}, and these two curves clearly represent a pair of plane uninodal quartic curves, δ_1^4 and δ_2^4, on the surface ${}^4H_2^7$. Moreover, the two curves Δ_1^4 and Δ_2^4 have just two residual intersections, K_1 and K_2 say, and these both correspond, as we shall see, to the same point of the surface ${}^4H_2^7$, *i.e.* they are a pair of *conjugate* points† in the representation. For it is known that all quintic curves which pass through all but one of the intersections of two quartic curves Δ_1^4 and Δ_2^4 also pass of their own accord through the remaining intersection‡, and this means then, in our case, that all the curves C^5 which pass through K_1 pass also through K_2. Thus the pair of points K_1, K_2 together represent an improper double point O of ${}^4H_2^7$, and the curves δ_1^4 and δ_2^4 have also each a node at O. (We remark incidentally that H_2^7 is a good example of a normal surface possessing an improper double point.) Finally we note that ${}^4H_2^7$ lies on two cubic primals. For cubic primals which contain the curves δ_1^4 and δ_2^4 contain the planes of these curves and have therefore freedom fifteen, and, on the other hand, they meet ${}^4H_2^7$ in curves represented on ϖ by septimic curves§ C^7 $(D_1^3, D_2^3, A_1, \ldots, A_{10})$ which have freedom only

* Noether, *Math. Annalen*, 3 (1871), 173.

† *Cf.* § 5.1.

‡ The reference is to the theorem that if a C^{m+n-3} pass through all but one of the intersections of a C^m and a C^n, it will pass through the remaining one.

§ Complete cubic sections of ${}^4H_2^7$ are represented on ϖ by curves $C^{15}(D_1^6, D_2^6, A_1^3, \ldots, A_{10}^3)$. and clearly if we remove Δ_1^4 and Δ_2^4, we are left with septimics $C^7(D_1^3, D_2^3, A_1, \ldots, A_{10})$.

thirteen, so that there must be two of them which contain the surface entirely. We have therefore firmly established the identity of the surfaces ${}^4F_2^7$ and ${}^4H_2^7$, and ${}^4F_2^7$ is therefore representable on a plane ϖ by means of a general linear ∞^4 system of quintics having two fixed nodes D_1, D_2 and ten simple base points A_1, ..., A_{10}.

We have shown incidentally, however, that all cubic primals Φ_0' through σ_1 and σ_2 meet ${}^4F_2^7$ residually in curves represented on ϖ by septimics C^7 $(D_1^2, D_2^3, A_1, \ldots, A_{10})$. *Thus the index system of* Λ_1 *of* Φ_0' *consists of all such septimics*, and we deduce that Φ_0' has grade twenty-one and characteristic genus nine.

In conclusion then : *Two cubic primals which have two planes* σ_1, σ_2 *in common intersect residually in a* GENERAL *rational septimic surface of section genus four, of the type that is represented on a plane by means of binodal quintic curves. All cubic primals through the two planes form a system of freedom fifteen, grade twenty-one, and characteristic genus nine.*

17 . 2. *Properties of* ${}^4F_2^7$. We have seen already that ${}^4F_2^7$ has a unique improper node O, and that it contains two plane quartic curves nodal at O. It possesses, in addition, however, a symmetrical system of eleven lines L_i $(i = 1, \ldots, 11)$, represented on the plane ϖ by the line D_1, D_2 and the ten base points A_1, ..., A_{10} respectively ; and it has also two symmetrical systems of eleven conics, of which those of one system K_i arise from the base point D_1 and the ten lines D_2A_i, while those of the other system K_i' arise from D_2 and the ten lines D_1A_i. *These two sets of eleven conics clearly form a double eleven of conics on the surface, i.e.* two conics of the same system do not meet, but any conic K_i of one system meets all except the corresponding conic K_i' of the other. Each line L_i, moreover, meets one conic of each system, namely K_i and K_i' respectively ; and the conics of one system meet one plane quartic of ${}^4F_2^7$ in two points and the other in one point, the position being reversed for those of the other system, while each of the lines L_i meets each of the quartic curves in question in one point. Finally, any conic K_i is met altogether by ten conics K_j $(i \neq j)$ and by one line L_i, and the range of eleven points so obtained on the conic K_i is related to the similar range on any conic of the same system*. *We obtain thus two sets of eleven related ranges, each of eleven points.*

17 . 3. *Transformations.* As usual, any H. net of the curves (§ 17 . 1) leads to a Cremona cubic transformation, and in the present instance

* This is easily seen from the plane representation.

there will evidently be quite a number of these. The simplest is obtained from the H. net, which consists of the lines of the plane each augmented by a sextic curve C^6 (D_1^3, D_2^3, A_1, ..., A_{13}), and this leads to a cubic-quintic transformation in which Φ *consists of cubic primals which have in common two planes* σ_1, σ_2, *and an octavic curve* ${}^4C^8$ *of genus four meeting each of the planes in five points.* The reverse system Ψ consists of quintic primals having as base surfaces two double planes and a simple surface of order ten.

A rational septimic surface of section genus three.

18.1. In this case we consider the residual intersection of two cubic primals which touch each other at every point of a plane* σ. The two primals will clearly have nodes at the same four points of σ, O_1, ..., O_4 say, and their residual intersection will be a septimic surface ${}^3F_2^7$ of section genus three (*cf.* T. xiv) having nodes at each of O_1, ..., O_4. We note also that, as is shown in the foot-note, ${}^3F_2^7$ meets σ in a quintic curve δ^5, also nodal at O_1, ..., O_4.

The surface ${}^3F_2^7$ is clearly rational; for any prime through σ meets the two cubic primals residually in quadric surfaces meeting σ in the same conic through O_1, ..., O_4, and these quadrics clearly meet residually in a conic belonging to a rational pencil of such conics on ${}^3F_2^7$. Thus the surface contains a rational pencil of conics, and it has section genus three, and these two facts enable us to identify it immediately as a projection of a certain surface belonging to the class of surfaces with hyperelliptic prime sections discussed at length by Castelnuovo†. The surface

---

* If the equations of σ be $x = y = 0$, then the equations of the two cubic primals will be of the form

$$0 = xw_2 + yw_2' + \Phi,$$

$$0 = xw_2 + yw_2' + \Phi',$$

respectively, where w_2, w_2' are quadratics in z, t, u, and Φ, Φ' involve x, y to a degree higher than the first. The points where $x = y = w_2 = w_2' = 0$ will clearly be nodes of both primals. By subtracting the equations, moreover, we obtain an equation of the form

$$0 = \Phi - \Phi' = x^2 p_1 + 2xyq_1 + y^2 r_1 + (x, y)_3,$$

and this represents a cubic primal passing through ${}^3F_2^7$ and having σ as double plane. Now if, of the two primes which form the tangent cone to this primal at any point of σ, one coincide with the common tangent prime to the two original primals at that point, then ${}^3F_2^7$ will clearly pass through the point. The locus of such points is easily seen to be the quintic curve δ^5 whose equation in σ is

$$(r_1, -q_1, p_1)(w_2, w_2')^2 = 0,$$

and this is easily seen to have nodes at the points where $w_2 = w_2' = 0$.

† *Loc. cit.*

in question is the rational normal ${}^3N_2^7$ of [5], which is represented on a plane by quintics C^5 which have a triple point D and nine simple points $A_1, \ldots, A_9$ in common, and it is our purpose now to show that ${}^3F_2^7$ is a perfectly general projection of ${}^3N_2^7$ on to [4].

There are six independent curves C^5 of the system mentioned above; and if l_1, l_2 be any two lines through D and K_1, K_2, K_3 be three independent quartic curves nodal at D and passing through the A's, then the six can clearly be taken to be l_1K_1, l_1K_2, l_1K_3, l_2K_1, l_2K_2, l_2K_3. The curves K_1, K_2, K_3 represent quintic curves ${}^2K^5$ of genus two belonging to a net of such curves on ${}^3N_2^7$. The system of curves C^5, then, can be taken to have an equation

$$l_1(\lambda_1K_1+\lambda_2K_2+\lambda_3K_3)+l_2(\lambda_4K_1+\lambda_5K_2+\lambda_6K_3) = 0,$$

where the λ's are free numerical parameters. On projecting the surface now into [4], which is equivalent to assuming a relation

$$a_1\lambda_1+a_2\lambda_2+a_3\lambda_3+a_4\lambda_4+a_5\lambda_5+a_6\lambda_6 = 0$$

restricting the parameters λ, it is easy to see that there is exactly one of the curves ${}^2K^5$ which projects into a *plane* quintic of the projected surface; for a curve $\lambda_1K_1+\lambda_2K_2+\lambda_3K_3$ will represent a plane curve of the projected surface if it belongs to ∞^2 curves C^5 of the *restricted* system of such curves, *i.e.* if the curve

$$(l_1+\mu l_2)(\lambda_1K_1+\lambda_2K_2+\lambda_3K_3) = 0$$

belongs to the restricted system for every value of μ, and this leads to the equations

$$a_1\lambda_1+a_2\lambda_2+a_3\lambda_3 = a_4\lambda_1+a_5\lambda_2+a_6\lambda_3 = 0,$$

determining the ratios $\lambda_1 : \lambda_2 : \lambda_3$ uniquely. Thus *the general projection of* ${}^3N_2^7$ *contains one plane four-nodal quintic curve* δ^5. We can now completely establish the identity of this projected surface with ${}^3F_2^7$ by showing that the former lies on two cubic primals. For cubic primals which contain the curve δ^5 contain the plane σ in which that curve lies, and they have therefore freedom twenty-four, and they meet the projected surface, on the other hand, in curves represented on ϖ by curves C^{11} $(D^7, A_1^2, \ldots, A_9^2)$ which have only freedom twenty-two*, *i.e.* there are two of the primals which contain the surface entirely. Thus the identity of the two surfaces is absolutely established.

* Complete cubic sections of ${}^3F_2^7$ are represented by curves C^{15} $(D^9, A_1^3, \ldots, A_9^3)$, and if Δ^4 be removed once we get curves C^{11} $(D^7, A_1^2, \ldots, A_9^2)$; while if it be removed twice we get curves C^7 $(D^5, A_1, \ldots, A_9)$.

We note, finally, that all cubic primals Φ_0' which touch the original two at every point of σ touch the surface ${}^3F_2^7$ also at every point of δ^5. Their freedom, moreover, is easily found to be thirteen, and they meet ${}^3F_2^7$ in the curve δ^5 counted twice and in curves represented on ϖ by the complete system of septimics $C^7\,(D^5, A_1, \ldots, A_9)$. *Thus the index system* Λ_1 *of* Φ_0' *consists of septimics* $C^7\,(D^5, A_1, \ldots, A_9)$, and we deduce that Φ_0' has grade fifteen and characteristic genus five.

In conclusion then: *Two cubic primals which touch each other at every point of a plane* σ *meet residually in a* GENERAL *rational septimic surface with hyper-elliptic prime sections of genus three. All such cubic primals, which touch at every point of* σ, *form a system of freedom thirteen, grade fifteen, and characteristic genus five.*

18.2. We may mention, in passing, that the surface ${}^3F_2^7$, besides possessing a rational pencil of conics R, possesses eighteen lines consisting of nine degenerate line-pairs of R and 256 isolated cubic curves. It has also, as we have stated, four nodes lying in a plane; but for other properties of the surface we refer to Castelnuovo's paper already quoted.

18.3. *Transformations.* The simplest transformation associated with ${}^3F_2^8$ as Φ-surface is that derived from the H. net of curves of Λ_1, which consists of the lines of the plane each augmented by a curve $C^6\,(D^5, A_1, \ldots, A_9)$. *The corresponding* Φ *system consists of cubic primals which touch each other at every point of a plane* σ, *and which contain, in addition, a rational sextic curve touching the primals at five points* of σ. The reverse system Ψ^r consists of quintic primals having in common a triple plane and a simple surface of order nine.

The surface ${}^7F_2^8$ *residual to a plane.*

19. The only remaining surface to be considered here is the octavic surface ${}^7F_2^7$ with prime sections of genus seven, the residual intersection of two cubic primals which have a plane σ in common. There is not much to be said about this surface from our point of view. It contains one plane quartic curve δ^4, namely that in which it is met by the plane σ, and ∞^1 elliptic quartic curves lying in solids through σ and each meeting δ^4 in four points. We need only remark further that it has certainly no nodes, that it is not rational, and that *it projects from any point of* δ^4 *into a septimic surface* ${}^7F_2^7$ *in ordinary space which has a triple line and a double conic not meeting the line.*

20. We conclude by giving a table indicating the main results obtained in the paper.

TABLE OF RESIDUAL INTERSECTIONS AND TRANSFORMATIONS.

No.	Surface F_2^r $(r \leqslant 4)$	Residual surface F_2^{6-r}	Associated transformations
i	Segre quartic F_2^4	General rational ${}^2F_2^5$ represented by quartics $C^4\,(D^2, A_1, \ldots, A_7)$	$\{3, 4(1), 3\}$ and $\{3, 5(2), n\}$ $(n = 4, 5)$
ii	Rational ruled F_2^4	General rational ${}^1F_2^5$ represented by cubics $C^3\,(A_1, \ldots, A_4)$	$\{3, 5(1), n\}$ $(n = 3, 4, 5)$
iii	Veronese F_2^4	Elliptic ruled ${}^1F_2^5$	$\{3, 4(0), 2\}$
iv	Cubic scroll and a plane	General rational ruled ${}^0F_2^5$, represented by cubics $C^3\,(D^2)$.	—
v	Double plane	General rational ruled ${}^0F_2^5$, as in iv	$\{3, 5(0), n\}$ $(n = 2, \ldots, 7)$
vi	Cubic F_2^3 in [3]	General ${}^4F_2^6$ intersection of a quadric and a cubic primal	—
vii	Cubic scroll in [4]	General determinantal ${}^3F_2^6$ represented by quartics $C^4\,(A_1, \ldots, A_{10})$	$\{3, 6(3), n\}$ $(n = 4, \ldots, 7)$
viii	Quadric surface and a plane	General rational ${}^2F_2^6$ represented by quartics $C^4\,(D^2, A_1, \ldots, A_6)$	$\{3, 6(2), n\}$ $(n = 4, 5, 6)$
ix	Three general planes	General elliptic sextic scroll ${}^1F_2^6$	—
x	Three planes meeting in a point	Rational ${}^1F_2^6$ with a triple point, represented by cubics $C^3\,(A_1, A_2, A_3)$	$\{3, 6(1), n\}$ $(n = 3, \ldots, 6)$
xi	Plane of contact and another plane	Rational sextic scroll ${}^0F_2^6$ with one triple and three double points	—

TABLE OF RESIDUAL INTERSECTIONS AND TRANSFORMATIONS (*continued*).

No.	Surface F_2^r $(r \leqslant 4)$	Residual surface F_2^{9-r}	Associated transformations
xii	Plane of osculation	Three quadric surfaces	—
xiii	Quadric surface	Surface ${}^5F_2^7$	—
xiv	Two planes	General rational ${}^4F_2^7$ representable by quintics $C^5\,(D_1^2,\ D_2^2,\ A_1,\ \ldots,\ A_{10})$	$\{3,\ 7(4),\ n\}$ $(n = 5,\ \ldots,\ 9)$
xv	Plane of osculation	General rational ${}^3F_2^7$ with hyperelliptic sections represented by quintics $C^5\,(D^3,\ A_1,\ \ldots,\ A_9)$	$\{3,\ 7(3),\ n\}$ $(n = 5,\ \ldots,\ 8)$
xvi	Plane	Surface ${}^7F_2^8$ has one plane quartic and ∞^1 elliptic quartics residual to it	—

PARSEVAL'S THEOREM FOR ALMOST-PERIODIC FUNCTIONS

By H. D. URSELL.

[Received 29 October, 1929.—Read 14 November, 1929.—Revised 15 April, 1931.]

In § 1 of the following paper the normality of S_r-a.p. functions is developed for any positive index r and there is a brief discussion of continuous S_r-a.p. functions. In § 2 I discuss convergence in mean in an infinite interval with particular reference to the appropriate "general principle of convergence": this clears the ground for the four equivalent definitions of W-a.p. functions given in § 3. In § 4 I define a perfect set of functions and give a proof of Parseval's theorem for W_2-a.p. functions. In § 5 I show how the methods originally used by Bohr in proving the fundamental theorem may be applied to obtain, in a somewhat different form, results on the convergence of a perfect sequence already obtained by Bohr and Bochner (*loc. cit.*, **1**, **4**).

In the same section the main trend of the paper (for which r is essentially variable) begins with a transformation from an S_r-a.p. function to an S-a.p. function. In § 6 I show that there are two distinct ways (M, J) of forming mean values of (the modulus of) an S-a.p. function, and that, while J always obeys Parseval's theorem, the more natural mean M does not always do so. In §§ 7–9 the position is illustrated by examples and the results are summarized.

1. Bohr's definition, **1**, of an a.p. function is based on his definition of a *relatively dense set* of real numbers: the set E is said to be relatively dense if there is some number l, such that every interval of length l contains at least one point of E. A wider class of functions was introduced by Stepanoff, **2**, and accordingly we shall say that τ is an S_r-*translation number* of $f(x)$ corresponding to ϵ if, for all y,

$$\left[\int_y^{y+1} |f(x+\tau)-f(x)|^r\, dx\right]^{\rho} \leqslant \epsilon, \tag{1}$$

where ρ is equal to unity when $r \leqslant 1$ and is equal to r^{-1} when $r > 1$: the constant r is supposed to be positive. For given values of ϵ and r and a given function f, we denote the set of such numbers τ by $E_r(\epsilon, f)$ or more briefly by $E(\epsilon, f)$ or $E(\epsilon)$. We shall also (given f and r) use $v(\tau)$ to denote the least value of ϵ for which (1) holds. Finally $f(x)$ is said to be S_r-a.p. if it is of class L^r and $E(\epsilon)$ is relatively dense for each positive ϵ.

The definitions depend only trivially on the constant length, here taken to be unity, of the interval of integration in (1). By classical inequalities the *translation-function* $v(\tau)$ satisfies the relations

$$v(0) = 0, \quad v(\tau) = v(-\tau), \quad |v(\tau+\delta) - v(\tau)| \leqslant v(\delta). \tag{2}$$

Now, if $|f|^r$ is summable, we have, by a known theorem,

$$\int_\alpha^\beta |f(x+\delta) - f(x)|^r \, dx \to 0 \quad \text{as} \quad \delta \to 0.$$

This holds in fact uniformly in α, β in a finite range (H, K). We may express this fact by saying that any function of class L^r is S_r-continuous when considered in a finite range. If now $f(x)$ is S_r-a.p. and ϵ is positive, we find that there is a number τ of $E(\epsilon, f)$ in any interval of length $l = l(\epsilon)$. We choose δ_0 so small that

$$\left[\int_\alpha^{\alpha+1} |f(x+\delta) - f(x)|^r \, dx\right]^\rho \leqslant \epsilon \quad \text{for } \delta \leqslant \delta_0 \text{ and } 0 \leqslant \alpha \leqslant l.$$

Then, for any β, we have $\beta = \alpha + \tau$ for an appropriate τ, and hence

$$\left[\int_\beta^{\beta+1} |f(x+\delta) - f(x)|^r \, dx\right]^\rho$$
$$\leqslant \left[\int_\alpha^{\alpha+1} |f(x+\tau) - f(x)|^r \, dx\right]^\rho + \left[\int_{\alpha+\delta}^{\alpha+\delta+1} |f(x+\tau) - f(x)|^r \, dx\right]^\rho$$
$$+ \left[\int_\alpha^{\alpha+1} |f(x+\delta) - f(x)|^r \, dx\right]^\rho \leqslant 3\epsilon \text{ for } \delta \leqslant \delta_0.$$

Thus $\int_\beta^{\beta+1} |f(x+\delta) - f(x)|^r \, dx \to 0$ with δ uniformly in β. It follows from (2) that $v(\tau)$ is a continuous a.p. function, since we have just proved it to be continuous at the origin. This condition is certainly also sufficient for a function (of class L^r) to be S_r-a.p., for, if $v(\tau)$ is merely† a.p., the set $[v(\tau) \leqslant \epsilon]$ is relatively dense [since $v(0) = 0$] for each positive ϵ, and this is the set $E(\epsilon, f)$.

† Using the definition of paper 3.

A sequence $\{f_n\}$ of functions of class L^r is said to be S_r-*convergent* if

$$\int_y^{y+1} |f_m(x)-f_n(x)|^r\,dx \to 0 \quad \text{as} \quad m, n \to \infty,$$

or if there exists a function $F(x)$, necessarily of class L^r, such that

$$\int_y^{y+1} |f_n(x)-F(x)|^r\,dx \to 0 \quad \text{as} \quad n \to \infty :$$

convergence in each case is to be uniform in y. A function $f(x)$ is said to be S_r-*normal*, **4**, if it is of class L^r, and if, given any sequence $\{\sigma_n\}$ of real numbers, there is a subsequence of $\{f(x+\sigma_n)\}$ which is S_r-convergent.

If $f(x)$ is S_r-normal, $v(\tau)$ is a.p.; for otherwise we could choose a sequence $\{\sigma_n\}$ such that $v(\sigma_m-\sigma_n) = \max\{|v(\tau+\sigma_m)-v(\tau+\sigma_n)|\} > k > 0$ unless $m = n$. Thus $f(x)$ is S_r-a.p. and $v(\tau)$ is in fact continuous a.p. Conversely, if $f(x)$ is S_r-a.p. and $\{\sigma_n\}$ is a sequence of real numbers, we can find a subsequence $\{\rho_n\}$ such that $\{v(x+\rho_n)\}$ converges uniformly; then $v(\rho_m-\rho_n) \to 0$ as $m, n \to \infty$, and $\{f(x+\rho_n)\}$ is S_r-convergent. Thus $f(x)$ is S_r-normal. In the Stepanoff case, since continuity (of v) is ensured automatically, there is complete equivalence between the concepts of normality and almost periodicity.

Further generalizations of the a.p. property have been made, and I shall now consider Bochner's theorem as generally as possible. Let us write

$$M^*(|f|) = \text{u. } \lim_{T\to\infty} \frac{1}{T}\int_0^T |f|\,dx \tag{3}$$

and

$$v(\tau) = M^*\{|f(x+\tau)-f(x)|\}. \tag{4}$$

Then $v(\tau)$ is measurable, f being, of course, supposed summable. We say that $f(x)$ has generalized normality if, given any sequence $\{\sigma_n\}$ of real numbers, there exists a subsequence $\{\rho_n\}$ such that $\{f(x+\rho_n)\}$ converges in mean, *i.e.*, such that $v(\rho_m-\rho_n) \to 0$ as $m, n \to \infty$ (independently). This is the necessary and sufficient condition for $v(x+\rho_n)$ to converge uniformly. Hence $f(x)$ has generalized normality if and only if $v(\tau)$ is normal and therefore, **3**, (being measurable) continuous, a.p.

Now, if $f(x)$ is continuous a.p. and if τ is such that

$$\text{u. bd.}\{|f(x+\tau)-f(x)|\} \geqslant k,$$

then there is an interval of length δ, where $\omega(\delta) \leqslant \frac{1}{3}k$, in which

$$|f(x+\tau)-f(x)| \geqslant \tfrac{2}{3}k.$$

If there is a member of $E(\frac{1}{3}k, f)$ in every interval of length $l > \delta$, we have

$$M^*\{|f(x+\tau)-f(x)|\} \geqslant \tfrac{1}{3}k\delta l^{-1}.$$

Thus we find that $|f(x+\tau)-f(x)| < \epsilon$ if $M^*\{|f(x+\tau)-f(x)|\} < \eta(\epsilon)$. So also, if $f(x)$ is S_r-a.p., we may show that

$$\int_y^{y+1} |f(x+\tau)-f(x)|^r\, dx < \epsilon \quad \text{if} \quad M^*\{|f(x+\tau)-f(x)|^r\} < \theta(\epsilon).$$

This shows that the usual forms of Bochner's theorem are in a sense included in the form just stated. So also W-convergence of the sequence $\{f(x+\rho_n)\}$ will be found to follow from its convergence in mean in the case of the W-a.p. functions to be considered later.

As an interesting application of the same sort of argument we shall prove that a uniformly continuous S_r-a.p. function is a.p. For, given ϵ, we can choose $\delta < 1$ so small that

$$\omega(\delta) \leqslant \tfrac{1}{4}\epsilon,$$

where $\omega(\delta)$ denotes the maximum oscillation of $f(x)$ in any interval of length δ.

Now suppose τ such that, for some y,

$$|f(y+\tau)-f(y)| > \epsilon.$$

Then

$$|f(x+\tau)-f(x)| > \tfrac{1}{2}\epsilon$$

for every point x in $(y, y+\delta)$ and hence

$$\int_y^{y+1} |f(x+\tau)-f(x)|^r\, dx > \delta\left(\frac{\epsilon}{2}\right)^r.$$

It follows that every S_r-translation number of f corresponding to $\eta(\epsilon)$ is also a translation number of $f(x)$ corresponding to ϵ: and this completes the proof.

On the other hand there exist continuous S-a.p. functions which are not uniformly continuous, and hence not a.p. For suppose that

$$0 < \epsilon_n < 1, \quad \Sigma\epsilon_n < \infty,$$

and for each n define a function $f_n(x)$ as follows:

$$f_n(x) = \frac{|x-y|}{\epsilon_n} \text{ if } |x-y| < \epsilon_n \text{ for some point,}$$

$$y = (2\nu+1)n$$

(where ν is an integer, positive, negative, or zero),

and

$$f_n(x) = 0 \text{ otherwise.}$$

Then the function $\Sigma f_n(x)$ is S-a.p. and continuous, but not uniformly continuous.

2. It is important in this connection to know when a limit function exists. We shall prove first that, if a sequence of functions $\{f_n\}$ is such that $M^*(|f_m - f_n|) \to 0$ as m and $n \to \infty$, then there exists a function ϕ such that $M^*(|f_n - \phi|) \to 0$ as $n \to \infty$. We choose a subsequence $\{g_n\}$ of $\{f_n\}$ so that

$$M^*(|g_m - g_n|) < \eta_k \quad \text{for} \quad m, n \geqslant k,$$

where $\Sigma\eta_k$ converges. We put $\phi_0 = g_1$ and choose T_1 so large that

$$\frac{1}{T}\int_0^T |g_2 - g_1|\, dx < \eta_1 \quad \text{for} \quad T > T_1.$$

Put $\phi_1 = 0$ for $x \leqslant T_1$, $\phi_1 = g_2 - g_1$ for $x > T_1$, so that

$$\frac{1}{T}\int_0^T |\phi_1|\, dx < \eta_1.$$

Next we choose $T_2 > T_1$ so large that

$$\frac{1}{T}\int_0^T |g_3 - g_2|\, dx < \eta_2 \quad \text{for} \quad T > T_2,$$

and put $\phi_2 = 0$ for $x \leqslant T_2$, $\phi_2 = g_3 - g_2$ for $x > T_1$; then

$$\frac{1}{T}\int_0^T |\phi_2|\, dx < \eta_2.$$

We continue in this manner indefinitely: we may suppose that $T_n \to \infty$ †. We put $\phi = \phi_0 + \phi_1 + \ldots$ so that $\phi = g_n$ for $T_{n-1} < x \leqslant T_n$. Then, for $T \geqslant T_{n-1}$,

$$\frac{1}{T}\int_{T_{n-1}}^T |\phi - g_n|\, dx = \frac{1}{T}\int_0^T |\phi_n + \phi_{n+1} + \ldots|\, dx < \eta_n + \eta_{n+1} + \ldots,$$

and so $M^*(|\phi - g_n|) \to 0$ as $n \to \infty$. It follows at once that $M^*(|\phi - f_n|) \to 0$ as $n \to \infty$, that $M^*(|\phi - g_n|) \leqslant \eta_n$, and that

$$M^*(|\phi - f_n|) \leqslant \epsilon_n \quad \text{if} \quad M^*(|f_m - f_n|) \leqslant \epsilon_k \quad \text{for} \quad m, n \geqslant k.$$

† If this is not the case, $\{g_n\}$ is S-convergent beyond the point $x = \lim T_n$.

We shall now introduce certain restrictions of uniformity. Let $\{f_n\}$ be a sequence of functions such that

$$\frac{1}{T}\int_0^T |f_m - f_n|\,dx \to 0 \quad \text{as} \quad m, n, T \to \infty .$$

Then there exists a function ϕ such that

$$\frac{1}{T}\int_0^T |f_n - \phi|\,dx \to 0 \quad \text{as} \quad n, T \to \infty .$$

For suppose that

$$\frac{1}{T}\int_0^T |f_m - f_n|\,dx \leqslant \epsilon_k \quad \text{for} \quad m, n \geqslant k \text{ and } T \geqslant T_k,$$

where ϵ_k decreases to zero as $k \to \infty$. We put $\phi = f_n$ in

$$R_{n-1} \leqslant x < R_n \quad (R_0 = 0).$$

Then, for $R_n \leqslant T \leqslant R_{n+1}$, we have

$$\frac{1}{T}\int_0^T |f_m - \phi|\,dx \leqslant \frac{1}{T}\int_0^T |f_m - f_{n+1}|\,dx + \frac{1}{R_n}\sum_{\nu=1}^{n}\int_0^{R_\nu} |f_{\nu+1} - f_\nu|\,dx.$$

We choose the (increasing) sequence $\{R_n\}$ so that $R_n \geqslant T_n$, $R_n \to \infty$ with n, and

$$R_n \epsilon_n \geqslant \sum_{k=1}^{n-1} \epsilon_k R_k.$$

Then

$$\frac{1}{T}\int_0^T |f_m - \phi|\,dx \leqslant \frac{1}{T}\int_0^T |f_m - f_{n+1}|\,dx + \frac{1}{R_n}\sum_{\nu=1}^{n} R_\nu \epsilon_\nu$$

$$\leqslant 3\epsilon_k \quad \text{for} \quad m \geqslant k, \; T \geqslant R_k.$$

So far we have been concerned only with the values of the functions considered for positive values of the variable x: there are obvious extensions when the functions are defined for negative values as well. There is, however, an alternative definition (not equivalent) of the upper mean value. We may write

$$M^*(|f|) = \text{u.}\lim_{T\to\infty} \operatorname*{Max}_{(\alpha)} \frac{1}{T}\int_\alpha^{\alpha+T} |f(x)|\,dx.$$

This definition of the upper mean value suggests a corresponding convergence theorem, namely that, if $\{f_n\}$ is a sequence of functions such that

$M^*(|f_m - f_n|) \to 0$ as $m, n \to \infty$, then a function ϕ exists such that $M^*(|f_n - \phi|) \to 0$ as $n \to \infty$. This result is false, as I shall now show. It will be sufficient to confine ourselves to characteristic functions of sets of points.

Let E be a measurable point set on the infinite straight line and $m(E, \alpha, \beta)$ its measure on the interval (α, β). Then I call the quantity

$$m^*E = \text{u. } \lim_{T\to\infty} \operatorname*{Max}_{(\alpha)} \frac{m(E, \alpha, \alpha+T)}{T}$$

the upper mean measure of E. We may also define the lower mean measure

$$m_*E = \text{l. } \lim_{T\to\infty} \operatorname*{Min}_{(\alpha)} \frac{m(E, \alpha, \alpha+T)}{T}.$$

Let F be a subset of E; then we write

$$\rho^*(F) = \text{u. } \lim_{T\to\infty} \operatorname*{Max}_{(\alpha)} \frac{m(F, \alpha, \alpha+T)}{m(E, \alpha, \alpha+T)},$$

and call ρ^* the upper mean density of F on E. Similarly we define the lower mean density ρ_*. We find without difficulty that

$$m^*F \leqslant \rho^* . m^*E, \quad m^*F \geqslant \rho^* . m_*E, \quad m^*F \geqslant \rho_* . m^*E,$$

$$m_*F \geqslant \rho_* . m_*E, \quad m_*F \leqslant \rho^* . m_*E, \quad m_*F \leqslant \rho_* . m^*E.$$

If E is measurable in mean, *i.e.* if $m^*E = m_*E$, then

$$m^*F = \rho^* . mE, \quad m^*F = \rho_* . mE.$$

LEMMA. *Given any sequence of real numbers ϵ_n such that $0 \leqslant \epsilon_n \leqslant 1$, we can construct sets E_n such that $mE_n = \epsilon_n$, while, if F_n be any subset of E_n of upper density on E_n not less than θ, then $m^*\Sigma F_n \geqslant \theta$.*

If $\epsilon_n > 0$, we choose a set E_n consisting of intervals of length L_n "evenly spaced," or say actually periodic; then there are intervals of length L_n on which F_n has mean density exceeding $\theta - \epsilon$. If $\epsilon_n = 0$ we may put a single interval L_n in E_n. If now $L_n \to \infty$, as $n \to \infty$, we get immediately

$$m^*\Sigma F_n > \theta - \epsilon, \quad m^*\Sigma F_n \geqslant \theta.$$

In particular, we notice that the sum of a sequence of null-sets is not necessarily measurable, and it may consist of the whole line. We shall now use the lemma to construct a sequence $\{f_n\}$ which violates the

suggested mean convergence theorem. We put

$$f_1 = 0,$$

$$f_2 - f_1 = 1 \text{ in } E_1, \quad = 0 \text{ elsewhere},$$

$$f_3 - f_2 = 1 \text{ in } E_2, \quad = 0 \text{ elsewhere},$$

and so on. Then we find that

$$M^*(|f_m - f_n|) < \eta_n \text{ for } m > n$$

provided that $$m^*E_n + m^*E_{n+1} + \ldots + m^*E_{m-1} < \eta_n,$$

which is the case if $$\epsilon_n = \eta_n - \eta_{n+1}.$$

Now, if $M^*(|f_n - \phi|) \to 0$, we must have $M^*(|f_n - \phi|) \leqslant \eta_n$. In this case the set in which $|f_n - \phi| > \frac{1}{2}$ has upper measure $\leqslant 2\eta_n$. But, if $5\eta_n \leqslant \eta_{n-1}$, we have $2\eta_n \leqslant \frac{1}{2}m^*E_{n-1}$, and so there is a subset of E_{n-1}, having upper measure not less than $\frac{1}{2}m^*E_{n-1}$, in which $|f_n - \phi| \leqslant \frac{1}{2}$. This subset then has upper density on E_{n-1} not less than $\frac{1}{2}$. But in it $|f_1 - \phi| \geqslant \frac{1}{2}$ since $|f_n - f_1| \geqslant 1$. Hence $|f_1 - \phi| \geqslant \frac{1}{2}$ in all in a set of upper measure not less than $\frac{1}{2}$, and so $M^*(|f_1 - \phi|) \geqslant \frac{1}{4}$. We have only to choose $\eta_1 < \frac{1}{4}$, and then $5\eta_n \leqslant \eta_{n-1}$, and $\epsilon_n = \eta_n - \eta_{n+1}$ and we obtain a contradiction on supposing that a function ϕ exists with the desired property. Thus the suggested mean convergence theorem is false. The above functions f_n are not necessarily characteristic functions, but there is no difficulty, save that of detail, in constructing non-overlapping sets with the property of the lemma, provided that $\Sigma\epsilon_n < 1$, as is the case in our application.

Now suppose that T_k increases to infinity with k, that ϵ_k decreases to zero, and that $\{f_n\}$ is a sequence of functions such that

$$\frac{1}{T}\int_\alpha^{\alpha+T} |f_m - f_n|\,dx \leqslant \epsilon_k \text{ for } m \text{ and } n \geqslant k,\ T \geqslant T_k, \text{ and all } \alpha.$$

We put $\phi = f_1$ in I_1, an interval of length $\geqslant T_1$; $\phi = f_2$ in intervals I_2 and I_2' adjoining I_1 to the right and left respectively and each of length greater than T_2; $\phi = f_3$ in I_3, I_3' intervals adjoining I_2 to the right and I_2' to the left and each of length greater than T_3, and so on. Then we have at once

$$\frac{1}{T}\int_\alpha^{\alpha+T} |\phi - f_n|\,dx \leqslant 3\epsilon_1 \text{ for } T \geqslant T_1.$$

Now the contribution to this integral from the parts of the interval out-

side I_1 does not exceed $3\epsilon_2 T$ for $n \geqslant 2$, $T \geqslant T_2$. Since the contribution from I_1 is bounded, we can choose R_2 so large that

$$\frac{1}{T}\int_a^{a+T} |\phi - f_n| \, dx \leqslant 4\epsilon_2 \quad \text{for} \quad n \geqslant 2, \; T \geqslant R_2;$$

so also

$$\frac{1}{T}\int_a^{a+T} |\phi - f_n| \, dx \leqslant 4\epsilon_k \quad \text{for} \quad n \geqslant k, \; T \geqslant R_k.$$

Thus when we introduce certain restrictions of uniformity we get a limit function with the present definition also, and the convergence to the limit function has the corresponding character of uniformity.

3. We shall say that a function $f(x)$ is W-normal if, given any sequence $\{\sigma_n\}$ of real numbers, there exists a subsequence $\{\rho_n\}$ such that the sequence $\{f(x+\rho_n)\}$ is W-convergent, *i.e.*,

$$\frac{1}{T}\int_0^T |f(x+\rho_m) - f(x+\rho_m)| \, dx \to 0 \quad \text{as} \quad m, n, \text{ and } T \to \infty.$$

Such a function is S-bounded. For suppose that $f(x)$ is not S-bounded; then we put $\sigma_0 = 0$, and choose σ_1 so that

$$\int_0^1 |f(x+\sigma_1)| \, dx = I_1 > I_0 + 1 = \int_0^1 |f(x)| \, dx + 1.$$

We next choose σ_2 so that

$$\tfrac{1}{2} I_2 = \tfrac{1}{2}\int_0^1 |f(x+\sigma_2)| \, dx > 1 + \tfrac{1}{2}(I_0 + I_1),$$

and then σ_3 so that

$$\tfrac{1}{3} I_3 > 1 + \tfrac{1}{3}(I_0 + I_1 + I_2),$$

and so on. For the sequence $\{\sigma_n\}$ so determined, we have, for $m > n$,

$$\frac{1}{R}\int_0^R |f(x+\sigma_n) - f(x+\sigma_m)| \, dx > 1 \quad \text{for} \quad 1 \leqslant R \leqslant m,$$

and no subsequence of $\{f(x+\sigma_n)\}$ is W-convergent.

A function $f(x)$ will be said to be W-a.p. if, for each positive ϵ, the set $E(\epsilon, T)$ of numbers τ such that

$$\frac{1}{R}\int_0^R |f(x+\tau) - f(x)| \, dx < \epsilon \quad \text{for} \quad R > T$$

is relatively dense for $T > T(\epsilon)$. We shall now prove that a W-normal function is W-a.p. For if not, then there is some $\epsilon > 0$ such that no set $E(\epsilon, T)$ is relatively dense. We put $\sigma_0 = 0$ and choose a point σ_1 of $cE(\epsilon, 1)$, we then choose a point σ_2 so that $\sigma_2-\sigma_0$, $\sigma_2+\sigma_1$ are points of $cE(\epsilon, 2)$, a point σ_3 so that $\sigma_3-\sigma_0$, $\sigma_3-\sigma_1$, $\sigma_3-\sigma_2$ are points of $cE(\epsilon, 3)$, and so on. If then $\{\rho_n\}$ is any subsequence of $\{\sigma_n\}$, $(\rho_m-\rho_n)$ lies outside $E(\epsilon, M)$ for $m > n$ (where M is some number $\geqslant m$). But, if $\{f(x+\rho_n)\}$ is W-convergent, we have

$$\frac{1}{T}\int_0^T |f(x+\rho_m)-f(x+\rho_k)|\,dx \leqslant \epsilon_k \quad \text{for} \quad m \geqslant k \text{ and } T \geqslant T_k.$$

Thus

$$\int_0^T |f(x+\rho_m-\rho_k)-f(x)|\,dx \leqslant \epsilon_k(T+\rho_k)+2K(|\rho_k|+1),$$

where $$K = \max \int_a^{a+1} |f(x)|\,dx \quad \text{and} \quad T \geqslant T_k+|\rho_k|.$$

Hence

$$\frac{1}{R}\int_0^R |f(x+\rho_m-\rho_k)-f(x)|\,dx \leqslant 2\epsilon_k \quad \text{for} \quad m \geqslant k,\ R \geqslant R_k,$$

and this is false when $\epsilon_k < \frac{1}{2}\epsilon$ if m and therefore M exceeds R_k.

We could prove also the converse of this result, but instead we shall discuss certain properties of a W-a.p. function. I first prove that

$$\frac{1}{T}\int_0^T |f(x+\delta)-f(x)|\,dx \to v(\delta) \quad \text{as} \quad T \to \infty$$

uniformly in δ in any finite range $|\delta| \leqslant \delta_0$; the limit function $v(\delta)$ is accordingly continuous. For this purpose it is clearly sufficient to suppose δ positive. Let ϵ be a sufficiently small positive number, and let R be such that $cE(\epsilon, R)$ contains no interval of length exceeding $l = l(\epsilon)$. We may suppose R to be as large as we please; we shall fix its value later, but we now suppose that $R > l$. Then there is a number τ_1 of E in $(R-l, R)$, a number τ_2 of E in (τ_1+R-l, τ_1+R), a number τ_3 of E in (τ_2+R-l, τ_2+R), and so on. We now have

$$\frac{1}{R}\int_R^{R+\tau_1} |f(x)-f(x-\tau_1)|\,dx < \epsilon, \quad \frac{1}{R+\delta_0}\int_R^{R+\tau_1} |f(x+\delta)-f(x+\delta-\tau_1)|\,dx < \epsilon,$$

$$\frac{1}{R}\int_{R+\tau_1}^{R+\tau_2} |f(x)-f(x-\tau_2)|\,dx < \epsilon, \quad \frac{1}{R+\delta_0}\int_{R+\tau_1}^{R+\tau_2} |f(x+\delta)-f(x+\delta-\tau_2)|\,dx < \epsilon,$$

and so on. We denote by θ any number, not necessarily the same in any two instances, of modulus not greater than unity. Then

$$\frac{1}{n(R+\delta_0)}\int_0^{R+\tau_{n-1}}|f(x+\delta)-f(x)|\,dx$$
$$=2\theta\epsilon+\frac{1}{n(R+\delta_0)}\sum_{\nu=0}^{n-1}\int_{R+\tau_{\nu-1}-\tau_\nu}^{R}|f(x+\delta)-f(x)|\,dx,$$

where τ_0 stands for zero and τ_{-1} for $-R$. The lower limit of integration on the right lies in every case in the interval $(0, l)$. Thus we have

$$\int_0^{R+\tau_{n-1}}|f(x+\delta)-f(x)|\,dx-2n\theta\epsilon(R+\delta_0)-n\int_0^R|f(x+\delta)-f(x)|\,dx$$
$$=\theta n\int_0^l|f(x+\delta)-f(x)|\,dx=\theta n\int_0^l\{|f(x+\delta)|+|f(x)|\}\,dx$$
$$=2\theta n\int_0^{l+\delta_0}|f(x)|\,dx=\theta nR\epsilon \quad \text{for} \quad R>R_1(\epsilon).$$

Now τ_n lies between $n(R-l)$ and nR; we choose further $R>\delta_0+l\epsilon^{-1}$. Then

$$\int_0^{R+\tau_{n-1}}|f(x+\delta)-f(x)|\,dx=5n\theta R\epsilon+n\int_0^R|f(x+\delta)-f(x)|\,dx.$$

For $n>5$ and $n\epsilon>1$ we also have

$$\int_0^{R+\tau_n}|f(x+\delta)-f(x)|\,dx=6n\theta R\epsilon+n(1+\theta\epsilon)\int_0^R|f(x+\delta)-f(x)|\,dx,$$

so that, for $$R+\tau_{n-1}\leqslant T\leqslant R+\tau_n,$$

we have

$$\int_0^T|f(x+\delta)-f(x)|\,dx=6n\theta R\epsilon+n(1+\theta\epsilon)\int_0^R|f(x+\delta)-f(x)|\,dx.$$

But now

$$nR(1-\epsilon)\leqslant n(R-l)\leqslant T\leqslant(n+1)R\leqslant n(1+\epsilon)R,$$

and so

$$\frac{1}{T}\int_0^T|f(x+\delta)-f(x)|\,dx=7\theta\epsilon+(1+3\theta\epsilon)\frac{1}{R}\int_0^R|f(x+\delta)-f(x)|\,dx.$$

This relation shows first that the left-hand side is bounded in $T>1$; thus the right-hand side may be replaced by

$$\frac{1}{R}\int_0^R |f(x+\delta)-f(x)|\,dx+K\theta\epsilon,$$

where K depends only on δ_0, not on ϵ or R. Then, since ϵ is arbitrary, the result desired follows.

We may observe that a precisely similar proof shows that

$$\frac{1}{T}\int_{-T}^0 |f(x+\delta)-f(x)|\,dx \to v(\delta) \quad \text{as} \quad T\to\infty$$

uniformly in δ in any finite range. But we can now deduce a much more precise result. Consider quite generally the behaviour of

$$\frac{1}{T}\int_a^{a+T} |f(x+\delta)-f(x)|\,dx.$$

Defining ϵ, l, and R as before, we choose a τ_1 of $E(\epsilon, R)$ in $(a-l, a)$, a number τ_2 in $(a+R-l, a+R)$, a number τ_3 in $(a+2R-l; a+2R)$, and so on. We also choose τ_1' in $(a+\delta-l, a+\delta)$, a number τ_2' in $(a+R+\delta-l, a+R+\delta)$, a number τ_3' in $(a+2R+\delta-l, a+2r+\delta)$, and so on. Then we find that

$$\frac{1}{R+l}\int_a^{a+R} |f(x)-f(x-\tau_1)|\,dx<\epsilon,$$

$$\frac{1}{R+l}\int_a^{a+R} |f(x+\delta)-f(x+\delta-\tau_1')|\,dx<\epsilon,$$

and so

$$\int_a^{a+R} |f(x+\delta)-f(x)|\,dx = 2(R+l)\,\theta\epsilon+\int_{\alpha_1}^{\alpha_1+R} |f(x+\delta_1)-f(x)|\,dx,$$

where $\delta_1=\delta+\tau_1-\tau_1'$, and the lower limit α_1 of integration on the right lies in the interval $(-l, 0)$. In the same way we find that

$$\int_{a+R}^{a+2R} |f(x+\delta)-f(x)|\,dx = 2(R+l)\,\theta\epsilon+\int_{\alpha_2}^{\alpha_2+R} |f(x+\delta_2)-f(x)|\,dx,$$

and so on. Thus, by summation,

$$\int_a^{a+nR} |f(x+\delta)-f(x)|\,dx = 2n(R+l)\,\theta\epsilon+\sum_{r=1}^{n}\int_{\alpha_r}^{\alpha_1+R} |f(x+\delta_r)-f(x)|\,dx.$$

Now $v(\tau_r-\tau_r')\leqslant 2\epsilon$, and so $v(\delta_r)=v(\delta)+2\theta\epsilon$. But $|\delta_r|\leqslant l$, and so, for R sufficiently great, we have

$$\frac{1}{R}\int_{\alpha_r}^{\alpha_r+R}|f(x+\delta_r)-f(x)|\,dx = v(\delta_r)+\theta\epsilon.$$

Hence also, for $R>l$,

$$\begin{aligned}\int_{\alpha}^{\alpha+nR}|f(x+\delta)-f(x)|\,dx &= 4nR\theta\epsilon+R\Sigma[v(\delta_r)+\theta\epsilon]\\ &= 7nR\theta\epsilon+nRv(\delta).\end{aligned}$$

Thus we see that

$$\frac{1}{T}\int_{\alpha}^{\alpha+T}|f(x+\delta)-f(x)|\,dx\to v(\delta)\quad\text{as}\quad T\to\infty$$

uniformly in α, δ. $v(\delta)$ is, of course, continuous a.p., and hence we see that the set $E_*(\epsilon, T)$ of numbers τ such that

$$\frac{1}{R}\int_{\alpha}^{\alpha+R}|f(x+\tau)-f(x)|\,dx<\epsilon\quad\text{for}\quad R>T$$

is relatively dense for $T>T(\epsilon)$, where ϵ is any positive number. A function satisfying this definition may be called W_*-a.p.

We say that $f(x)$ is W_*-normal, if, given any sequence $\{\sigma_n\}$, there exists a subsequence $\{\rho_n\}$ such that $\{f(x+\rho_n)\}$ is W_*-convergent, *i.e.*,

$$\frac{1}{R}\int_{\alpha}^{\alpha+R}|f(x+\rho_m)-f(x+\rho_n)|\,dx\to 0\quad\text{as}\quad m, n, R\to\infty$$

(uniformly in α, of course). We prove now that a W_*-a.p. function is W_*-normal. Let $f(x)$ be W_*-a.p. and $\{\sigma_n\}$ any given sequence. Then we can choose a subsequence $\{\rho_n\}$ so that $\{v(x+\rho_n)\}$ is uniformly convergent, *i.e.* $v(\rho_m-\rho_n)\to 0$ as $m, n\to\infty$. It follows immediately that $\{f(x+\rho_n)\}$ is W_*-convergent, for we have $v(\rho_m-\rho_n)<\epsilon_k$ for $m, n>k$, and

$$\frac{1}{R}\int_{\alpha}^{\alpha+K}|f(x+\delta)-f(x)|\,dx = v(\delta)+\theta\epsilon_k\quad\text{for}\quad R>R_k.$$

Hence

$$\frac{1}{R}\int_{\alpha}^{\alpha+R}|f(x+\rho_m)-f(x+\rho_n)|\,dx<2\epsilon_k\quad\text{for}\quad m, n>k\text{ and }R>R_k.$$

Thus the classes of W-normal, W-a.p., W_*-a.p., and W_*-normal functions all coincide.

I may remark that, if we confine ourselves to *bounded* sequences $\{\sigma_n\}$, we get W-convergence for some subsequence provided only that $v(\tau)$† is continuous‡. We shall prove in fact that, if $v(\tau)$ is continuous, then

$$\frac{1}{R}\int_0^R |f(x+\delta)-f(x)|\,dx \to 0 \quad \text{as} \quad \delta \to 0$$

uniformly in $R \geqslant 1$: it is clear that this last property will be sufficient to ensure the W-convergence of any sequence $\{f(x+\rho_n)\}$ for which $\{\rho_n\}$ converges.

We assume then that $v(\tau)$ is continuous (and in particular finite). Let ϵ be a positive number and choose δ_0 so small that $v(\delta) \leqslant \epsilon$ for $0 < \delta \leqslant \delta_0$. Let $E = E_T$ be the set of points common to $E(2\epsilon, T)$ and the interval $(0, \delta_0)$: let G be the complementary set in this interval. The set E increases with T and its measure tends to δ_0 as $T \to \infty$: we choose T so large that

$$mE \geqslant \tfrac{2}{3}\delta_0.$$

Then, if τ is positive and lies outside $E(4\epsilon, T)$, we find at once that

$$mG \geqslant mE - \tau,$$

$$\tau \geqslant mE - mG \geqslant \tfrac{1}{3}\delta_0,$$

since the difference of τ and a point of E is necessarily outside $E(2\epsilon, T)$. Thus the set $E(4\epsilon, T)$ includes the whole interval $(0, \frac{1}{3}\delta_0)$. Thus

$$\frac{1}{R}\int_0^R |f(x+\delta)-f(x)|\,dx \leqslant 4\epsilon \quad \text{if} \quad R \geqslant T, \quad 0 < \delta < \delta_1 = \tfrac{1}{3}\delta_0,$$

$$\text{and hence also if} \quad R \geqslant 1, \quad 0 < \delta < \delta_2,$$

for a suitable choice of δ_2.

The theory of §§2, 3 can be extended at once with trivial changes when the mean values used are replaced by mean values taken with any positive index r and the definition of $v(\tau)$ is replaced by

$$v(\tau) = [M^*\{|f(x+\tau)-f(x)|^r\}]^\rho.$$

The more powerful W_*-properties will usually be understood without the suffix after the present section.

---

† Defined by (4) of § 1.

‡ Since $v(\tau)$ is automatically measurable it is sufficient, 3, to postulate that it possesses the normal property in a finite interval or that in any sequence $\{\sigma_n\}$ tending to zero there is a subsequence $\{\rho_n\}$ such that $v(\rho_m - \rho_n) \to 0$ as m and n tend to infinity.

4. An aggregate F of functions $f(x)$ of a real variable x is said to form a *perfect set* if the functions are uniformly continuous in their aggregate, while for each positive ϵ the sets $E(\epsilon, f)$ have a product $E(\epsilon, F)$ which is relatively dense. The restriction of continuity may also be expressed by saying that the origin is an internal point of $E(\epsilon, F)$. From the existence of any one such set it is easily seen that there is an absolute constant K such that, for any function of the aggregate and all x_1, x_2,

$$|f(x_1)-f(x_2)| < K,$$

so that the functions $g = f-M(f)$ form a perfect set uniformly bounded in the aggregate. A perfect sequence $\{f_n\}$ such that (for some positive r) $M\{|f_m-f_n|^r\} \to 0$ as m and $n \to \infty$ is necessarily uniformly convergent. The particular case $r = 1$, $f_n = f(x+\rho_n)$ was stated in § 1; the proof there given is applicable with trivial changes to the general case.

An aggregate F is said to form an S_r-perfect set ($r > 0$) if, for each positive ϵ, the set $E(\epsilon, F)$ of S_r-translation numbers corresponding to ϵ common to all functions of the aggregate is relatively dense and contains a neighbourhood of the origin; the functions of the set are uniformly S_r-continuous. So also we say that F is a W_r-perfect set if, for each positive ϵ, the set $E(\epsilon, T, F)$ of numbers τ such that

$$\left[\frac{1}{R}\int_a^{a+R} |f(x+\tau)-f(x)|^r\,dx\right]^{\rho} \leqslant \epsilon \quad \text{for} \quad R \geqslant T \text{ and all } a, f$$

is relatively dense and contains a neighbourhood of the origin for $T \geqslant T(\epsilon)$; the functions of such a set are uniformly W_r-continuous. A W_r-perfect sequence which converges in mean with index r is W_r-convergent: an S_r-perfect sequence which converges in mean with index r is S_r-convergent. These results follow as above by trivial changes of the argument of § 1.

If a sequence $\{f_m\}$ is S_r-convergent and some (sufficiently advanced) term f_μ is S_r-bounded, then all (sufficiently advanced) terms f_m are uniformly S_r-bounded in their aggregate. In the sequel, when we speak of an S_r-convergent sequence we shall understand that the functions are uniformly S_r-bounded. Similarly for a W_r-convergent or a uniformly convergent sequence. Now let $\{f_m\}$ be S_p-convergent and $\{g_n\}$ be S_q-convergent where

$$\frac{1}{p}+\frac{1}{q} = 1.$$

Then the double sequence $\{f_m g_n\}$ is S_1-convergent. Hence, if $f(x)$ is S_p-normal and $g(x)$ is S_q-normal, their product fg is S_1-normal. More gener-

ally, if $\{f_m\}$ is $S_{p\alpha}$-convergent and $\{g_n\}$ is $S_{q\alpha}$-convergent, then $\{f_m g_n\}$ is S_α-convergent and, if $f(x)$, $g(x)$ are respectively $S_{p\alpha}$-a.p. and $S_{q\alpha}$-a.p., then fg is S_α-a.p. Again, if $\{f_m\}$ is S_r-convergent and $\{g_n\}$ is uniformly convergent, then $\{f_m g_n\}$ is also S_r-convergent, so that, if $f(x)$ is S_r-a.p. and $g(x)$ is continuous a.p., then fg is also S_r-a.p. Moreover, we may replace the symbol S by the symbol W throughout these results†. In particular, since, when $f(x)$ is W_1-a.p., the mean value

$$M(f) = \lim_{T\to\infty} \frac{1}{T}\int_x^{x+T} f(t)\,dt$$

exists, uniformly in x, by arguments similar to those of § 3, we infer the existence of $a_\lambda = M\{f(x)\,e^{-i\lambda x}\}$ by putting $g(x) = e^{-i\lambda x}$. The Fourier series of f is then defined in the usual way; it will appear later that it is a *series*. Now let $f(x)$ be W_2-a.p., then so is $g(x) = \overline{f(x+\theta)}$, and hence $f(x)\,g(x)$ is W_1-a.p. We write

$$\phi(\theta) = M_{(x)}(fg).$$

Now
$$\frac{1}{T}\int_a^{a+T} |f|^2 dx < K^2 \quad \text{for} \quad T > 1,$$

and hence

$$\frac{1}{T}\int_a^{a+T} |f(x)\,g(x) - f(x+\tau)\,g(x+\tau)|\,dx$$

$$\leqslant \left[\frac{1}{T}\int_a^{a+T} |f(x)|^2 dx\right]^{\frac{1}{2}} \left[\frac{1}{T}\int_a^{a+T} |g(x)-g(x+\tau)|^2 dx\right]^{\frac{1}{2}}$$

$$+ \left[\frac{1}{T}\int_a^{a+T} |g(x+\tau)|^2 dx\right]^{\frac{1}{2}} \left[\frac{1}{T}\int_a^{a+T} |f(x)-f(x+\tau)|^2 dx\right]^{\frac{1}{2}}$$

$$\leqslant 2K\epsilon \quad \text{for } T > 1 \text{ if } \tau \text{ belongs to } E_2(\epsilon, T, f).$$

† It may be observed that in the last pair of results the restrictions on the functions g_n may be somewhat lessened; it is clearly sufficient that they should be S-equivalent to bounded functions and should converge uniformly at all points outside a set of S-measure zero. The same remark holds for the propositions in which W replaces S, *but this substitution may not be made in the remark itself*. To see this consider the product of a particular W_h-null function f and a particular function g which is W_k-null for every k. We put $f = \alpha_n$, $g = \beta_n$ in the intervals $(n, n+\delta_n)$ and $(-n-\delta_n, -n)$, where $\delta_n < 1$, elsewhere we take f and g to be zero. If we choose α_n, β_n, δ_n, so that

$$\alpha_n^h \beta_n^h \delta_n = 1, \quad \delta_n = e^{-\beta_n} \to 0 \text{ as } n \to \infty,$$

then $M\{(fg)^h\} = 1$, although fg vanishes except in a W-null set. Thus fg cannot be W_h-a.p. It is not difficult to see from this example that the suggested generalisation of the remark is false.

Thus, whatever be the value of θ, the set $E_1(2K\epsilon, T, fg)$ includes all points of $E_2(\epsilon, T, f)$; in other words, the various functions fg form a W_1-perfect set. Hence

$$\frac{1}{T}\int_a^{a+T} f(x)\,g(x)\,dx \to \phi(\theta) \quad \text{as} \quad T \to \infty,$$

uniformly in both a and θ. It follows easily that $\phi(\theta)$ is continuous a.p. and hence it is the uniform limit of its Fejér sums. But

$$\begin{aligned} a_0(\phi) &= M\{\phi(\theta)\} = M_{(\theta)}\,M_{(t)}\{\overline{f(\theta+t)}\,f(t)\} \\ &= M_{(\theta)}\{\overline{f(\theta+t)}\}\,.\,M_{(t)}\{f(t)\} \quad \text{(by the uniformity)} \\ &= a_0(\bar{f})\,.\,a_0(f) = |\,a_0(f)\,|^2, \end{aligned}$$

and, treating $g(x) = f(x)\,e^{-i\lambda x}$ in the same way, we find that

$$a_{-\lambda}(\phi) = |\,a_0(g)\,|^2 = |\,a_\lambda(f)\,|^2.$$

Hence

$$\begin{aligned} M\{|f|^2\} &= \phi(0) = \text{limit of Fejér sums of } \phi(\theta) \text{ for } \theta = 0 \\ &= \Sigma\, a_\lambda(\phi) = \Sigma\,|\,a_\lambda(f)\,|^2. \end{aligned}$$

This is Parseval's theorem for W_2-a.p. functions; it holds, in particular, for S_2-a.p. functions.

5. Let F be a W_2-perfect aggregate of functions f and denote by $v(\tau)$ the least value of ϵ for which $M\{|\,f(x+\tau)-f(x)\,|^2\} \leqslant \epsilon^2$ for every function f. Let $E_2(\epsilon, F)$ be the set $[v(\tau) \leqslant \epsilon]$; it is the set of ϵ-translation numbers of the continuous a.p. function $v(\tau)$, which is its own translation function. Now, given δ, we can find c so that $v(\tau) \leqslant \delta$ for $|\,\tau\,| \leqslant c$; in other words

$$M\{|\,f(x+\tau)-f(x)\,|^2\} \leqslant \delta^2 \quad \text{for} \quad |\,\tau\,| \leqslant c.$$

Hence

$$M\{|\,\phi(x)-f(x)\,|^2\} \leqslant \delta^2$$

if

$$\phi(x) = \frac{1}{c}\int_x^{x+c} f(y)\,dy.$$

ϕ has the same Fourier exponents as f and, if a_λ, α_λ denote the corresponding coefficients, we find that

$$\Sigma\,|\,a_\lambda - \alpha_\lambda\,|^2 \leqslant \delta^2.$$

But $$\phi'(x) = \frac{1}{c}\{f(x+c)-f(x)\} \sim \Sigma i\lambda a_\lambda e^{i\lambda x},$$

and so $$\Sigma\lambda^2|a_\lambda|^2 = M\{|\phi'|^2\} \leqslant \left[\frac{v(c)}{c}\right]^2 = K^2(c).$$

Hence $$\Sigma'|a_\lambda|^2 \equiv \sum_{|\lambda|\geqslant\Omega}|a_\lambda|^2 \leqslant \frac{K^2}{\Omega^2},$$

and

(1) $$\Sigma'|a_\lambda|^2 \leqslant \left(\delta+\frac{K}{\Omega}\right)^2 \leqslant 4\delta^2 \quad \text{for} \quad \Omega \geqslant \frac{K}{\delta}.$$

The sets $E(\epsilon, F)$ possess the a.p. property proved by Bohr (*loc. cit.*). Denoting by $E'(\epsilon)$ the set of integral points of $E(\epsilon, F)$ we know that $E'(\epsilon)$ is relatively dense, and that, if η, θ are any assigned positive numbers, we can find in the interval $(\frac{1}{2}\eta, \eta)$ a number γ, and corresponding numbers $\rho > 0$, $I_0 > 0$, such that, if I is an interval of length at least I_0 and if the set $E'(\gamma).I$ is translated a distance τ where τ belongs to $E'(\rho)$, then a proportion less than θ of its points will cease to belong to $E'(\gamma)$.

We apply this result, taking $\theta = \frac{1}{4}$ for simplicity. An angle β will be said to be in the 0-interval if $|\beta| \leqslant \frac{1}{6}\pi \pmod{2\pi}$, and otherwise to be in the large π-interval; unless $|\beta| \leqslant \frac{1}{3}\pi \pmod{2\pi}$, it will be said to be also in the small π-interval. Having fixed γ and ρ as above, we shall say that λ has the π-property if $E'(\rho)$ contains a number t such that λt is in the small π-interval.

If λ possesses the π-property with a positive t, then, for any N such that

$$N > N_0(\lambda) = N_0,$$

more than a quarter of the first N positive numbers τ of $E'(\gamma)$ are such that $\lambda\tau$ lies in the large π-interval. It is sufficient to take $N_0 > 4t$, $N_0 > I_0$: for let A be the number of angles $\lambda\tau$ which belong to the large π-interval and $B = N-A$ the number which belong to the 0-interval. The number of points τ of the second kind in $\{0, \tau_N - t\}$ is $B_1 \geqslant B-\frac{1}{4}N$. On translating these a distance t we obtain at least $B_1-\frac{1}{4}N \geqslant B-\frac{1}{2}N$ which are still numbers τ. But they are necessarily of the first kind and so $A \geqslant B-\frac{1}{2}N$, $A \geqslant \frac{1}{4}N$.

If λ_0 possesses the π-property, there is a neighbourhood i_0 of λ_0 all points of which possess the π-property with a single number N_0 appropriate to all. Thus the set of numbers λ which possess the π-property is open. But if λ_0 does not possess the π-property, $\lambda_0 t$ is outside the small π-interval for every t of $E'(\rho)$. Now the values of t for which $\lambda_0 t$ lies in

the 0-interval and t in $E'(\rho)$ are relatively dense, and there is one such number in every interval of length L if L is chosen sufficiently great. Enumerate them as $\{t_n\}$ in order of magnitude, so that $t_{n+1}-t_n < L$. Then, if $h_1 = \pi/L$, we find that $|ht_{n+1}-ht_n| < h_1 L = \pi$ if $|h| \leqslant h_1$, and so there is some t_n such that ht_n falls on the arc $(\frac{1}{2}\pi, \frac{3}{2}\pi)$. Since $\lambda_0 t_n$ falls on the 0-interval, $(\lambda_0+h)\,t_n$ falls on the small π-interval: and all points, except λ_0 in $|\lambda-\lambda_0| \leqslant h_1$, possess the π-property. Thus the set of points λ which do not possess the π-property has no limit point.

If δ is an assigned positive number, we choose Ω as above, so that

$$\Sigma' |a_\lambda|^2 \leqslant 4\delta^2,$$

the summation being taken over those values of λ for which $|\lambda| \geqslant \Omega$. We take $\eta = \delta$ and determine $\gamma \leqslant \delta$ and ρ as above; let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the points of the closed interval $(-\Omega, \Omega)$ which do not possess the π-property. Take any positive number ω and enclose the points λ_i in intervals $(\lambda_i-\omega, \lambda_i+\omega)$. If λ lies in $(-\Omega, \Omega)$ and outside these intervals $(\lambda_i-\omega, \lambda_i+\omega)$, it possesses the π-property and lies in an open interval i_0 as above. The whole interval $(-\Omega, \Omega)$ is covered by the intervals $(\lambda_i-\omega, \lambda_i+\omega)$ together with a finite number of intervals i_0. Let N be the greatest value of N_0 for these intervals i_0. Then we find that

$$\Sigma |a_\lambda|^2 \left\{ \frac{1}{N} \sum_{k=1}^{N} |1-e^{i\lambda\tau_k}|^2 \right\} = \frac{1}{N} \sum_{k=1}^{N} M\{|f(x+\tau_k)-f(x)|^2\} \leqslant \gamma^2 \leqslant \delta^2,$$

where $\tau_1, \ldots, \tau_N$ are the first N positive members of $E'(\gamma)$. Hence

$$\Sigma'' |a_\lambda|^2 . \tfrac{1}{4} . \tfrac{1}{4} \leqslant \delta^2, \tag{2}$$

where Σ'' is taken over those values of λ in $(-\Omega, \Omega)$ and outside the intervals $(\lambda_i-\omega, \lambda_i+\omega)$, since for any such point more than a quarter of the angles $\lambda\tau_k$ fall in the large π-interval. Since ω is arbitrary, the summation may be extended at once to cover all values of λ in $(-\Omega, \Omega)$ other than the λ_i themselves. Hence, finally, by (1) and (2),

$$\Sigma^* |a_\lambda|^2 \leqslant 20\delta^2, \tag{3}$$

the summation Σ^* being taken over all exponents of $f(x)$ other than $\lambda_1, \lambda_2, \ldots, \lambda_m$.

The values of λ belonging to the sets $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ obtained by a sequence of such analyses from values of δ decreasing to zero fall into a

single sequence $\{\lambda_n\}$: they are the only possible exponents of any function f of the aggregate, and the result (3) states that *the Fourier sums converge in mean with index 2 to the value of the function, uniformly for all functions of the set.*

A sequence of (continuous a.p., S_1-a.p. or) W_1-a.p. functions

$$f_n \sim \Sigma A_\lambda^{(n)} e^{i\lambda x}$$

is said to be formally convergent if, for each λ, the sequence $\{A_\lambda^{(n)}\}$ converges to a limit A_λ. If the sequence be perfect, then, by (3), it converges in mean with index 2, and hence is uniformly convergent. So also a formally convergent S_2-perfect or W_2-perfect sequence is S_2-convergent or W_2-convergent.

This last result can be extended to any index $r \geqslant 1$ as we shall now show. If F be a W_r-perfect set, then

$$\phi(x) = \phi(x, c) = \frac{1}{c}\int_x^{x+c} f(t)\, dt$$

converges (W_r) to $f(x)$ as $c \to 0$ uniformly for all functions f of F. But, for a fixed c, the aggregate $\phi(x, c)$ is W_r-perfect, and the functions $\psi = \phi - M(\phi)$ are uniformly bounded in their aggregate (*cf.* § 4).† Hence (for a fixed c) the functions ψ, and hence also the functions ϕ, form a W_2-perfect set. Now the Fourier series of $\phi(x, c)$ is obtained by formal integration from that of $f(x)$‡ ; hence a formally convergent sequence $\{f_n\}$ gives rise to a formally convergent sequence $\{\phi_n\}$. Since $M(\phi_n)$ is convergent, the functions ϕ_n are bounded in their aggregate with the functions ψ_n. The formally convergent sequence $\{\phi_n\}$ is W_2-convergent since it is W_2-perfect and hence it is W_r-convergent as being bounded. Since ϕ_n is W_r-convergent to f_n uniformly in n as $c \to 0$, it follows at once that $\{f_n\}$ is W_r-convergent.

An examination of the above argument shows that, given a W_r-perfect set F (and hence its sequence of exponents λ_n) and a positive number ϵ, we can find positive numbers N, T, δ such that

$$\frac{1}{R}\int_a^{a+R} |f(x)|^r\, dx \leqslant \epsilon \quad \text{for} \quad R \geqslant T$$

---

† Incidentally we may infer here that the functions $g = f - M(f)$ are W_r-bounded in their aggregate.

‡ Here (by using *two incommensurable* values of c) we may infer that the possible exponents of all functions of F fall into a single sequence.

for any function of the aggregate

$$f(x) \sim \Sigma A_n e^{i\lambda_n x}$$

for which $|A_n| \leqslant \delta$ for $n \leqslant N$. As regards applications of this result, we remark that, if f_1, f_2 each belongs to a W_r-perfect set, so do the functions $f_1 - f_2$. Throughout the above, of course, the symbol W may be replaced by the symbol S, and the numbers R, T by unity.

Let us write

$$\Pi_n(t) = \sum_{|\nu| \leqslant n} \left(1 - \frac{|\nu|}{n}\right) e^{i\nu t} = \frac{1}{n}\left(\frac{\sin \frac{1}{2}nt}{\sin \frac{1}{2}t}\right)^2,$$

and $$\Pi(t) = \Pi_{n_1, n_2, \ldots, n_k}^{a_1, a_2, \ldots, a_k} = \Pi_{n_1}(a_1 t).\Pi_{n_2}(a_2 t) \ldots \Pi_{n_k}(a_k t),$$

where $a_1, a_2, \ldots, a_k$ are any linearly independent real numbers. Any trigonometric polynomial of the form $\sigma(x) = M_{(t)}\{f(x+t)\,\Pi(t)\}$ is called a (generalized) Fejér sum of $f(x)$, which we suppose W_1-a.p. (at least) in order that the mean value may exist. Since $\Pi(t)$ is real and positive and $M\{\Pi(t)\} = 1$, it is easily seen that $\sigma(x)$ possesses all translation numbers (of whatever kind) of f; thus they form a perfect, S_r-perfect, or W_r-perfect set according as $f(x)$ is continuous a.p., S_r-a.p., or W_r-a.p. If we choose a sequence of Fejér sums which converges formally to f (as we may) it will accordingly be uniformly S_r- or W_r-convergent to the function, as the case may be.

We conclude this section by observing that the necessary and sufficient condition for $f(x)$ to be W_r-a.p. is that

$$\phi(x) = |f(x)|^r \operatorname{sgn} f(x)$$

is W-a.p. where r may now denote any positive number. For, if

$$\xi_1 = |z_1|^r \operatorname{sgn} z_1 \text{ and } \xi_2 = |z_2|^r \operatorname{sgn} z_2,$$

then

$$|\xi_1 - \xi_2| \leqslant A\,|z_1 - z_2|^r \quad (r \leqslant 1), \tag{4}$$

$$|\xi_1 - \xi_2| \leqslant A\,|z_1 - z_2|.\ (|z_1|^{r-1} + |z_2|^{r-1}) \quad (r \geqslant 1), \tag{5}$$

where A depends on r alone. These inequalities are proved most simply by supposing that

$$|z_2| \leqslant |z_1|, \quad z_1 = 1$$

and observing that $|z|^r \operatorname{sgn} z$ is a differentiable function of Rz and Iz, at $z = 1$.

If $\alpha > 0$ and if a sequence $\{f_n(x)\}$ is $W_{r\alpha}$-convergent† then the corresponding sequence $\{\phi_n(x)\}$ is W_α-convergent: this result follows directly from (4) if $r \leqslant 1$ and follows from (5) by means of Hölder's inequality if $r > 1$. Thus, in order that $\{f_n(x)\}$ may be W_r-convergent, it is necessary and sufficient for $\{\phi_n(x)\}$ to be W-convergent: and in order that $f(x)$ may be W_r-normal it is necessary and sufficient for $\phi(x)$ to be W-normal.

In case $r < 1$ we deduce immediately a.p. functions and hence trigonometric polynomials approximating to the W_r-a.p. function $f(x)$. In fact, if $\sigma(x)$ is an approximation to the W-a.p. function $\phi(x)$, the function

$$s(x) = |\sigma(x)|^{1/r} \operatorname{sgn} \sigma(x)$$

is an a.p. function approximating to $f(x)$ to a degree corresponding in a definite manner to the approximation of $\phi(x)$ by $\sigma(x)$: and if

$$\phi(x) = W\text{-}\lim \sigma_n(x)$$

then

$$f(x) = W_r\text{-}\lim s_n(x).$$

Clearly the whole theory applies with obvious changes to the S case.

6. We have proved that, if $f(x)$ is W_2-a.p., then $M(|f|^2) = \Sigma |a_\lambda|^2$. Parseval's theorem for pure periodic functions is sharper than this; it states that the equality holds provided that $f(x)$ is summable, **5**. Writing the equation in the form $M_2^2 = S_2^2$, we know in fact that, for pure periodic summable functions, the condition $S_2 < \infty$ is necessary *and sufficient* for $f(x)$ to be of summable square. I shall now discuss the analogue of Parseval's theorem in more detail, with particular reference to this aspect of the matter. Let $f(x)$ be W-a.p. and consider the sets $[|f| < k]$, $[|f| \leqslant k]$‡. Denote the upper and lower mean measure of the first set by $m_2(k)$ and $m_1(k)$, those of the second set by $m_4(k)$ and $m_3(k)$: *mean measure* is used in the sense of § 2. Then m_1, m_2, m_3, m_4 are non-decreasing functions of k and

$$0 \leqslant m_1 \leqslant m_3 \leqslant m_4, \quad m_1 \leqslant m_2 \leqslant m_4 \leqslant 1. \tag{1}$$

Let ϵ be any positive number; then there is a number τ in every interval of length $l(\epsilon)$ such that, for $T \geqslant T(\epsilon)$ and for all α,

$$\frac{1}{T}\int_\alpha^{\alpha+T} |f(x+\tau) - f(x)|\, dx \leqslant \epsilon^2.$$

† So that by our convention each term of the sequence is $W_{r\alpha}$-bounded.

‡ A parallel treatment may be given in which f and not merely $|f|$ is considered, but this is not required for our purpose.

Consider any interval $(\beta, \beta+R)$. We can find a number $a+\tau_0$ in $(\beta, \beta+C)$, a number τ_1 in (τ_0+T, τ_0+T+l), a number τ_{n+1} in (τ_n+T, τ_n+T+l). We have

$$\frac{1}{T}\int_a^{a+T} |f(x+\tau_n)-f(x)|\,dx \leqslant \epsilon^2.$$

Hence in the interval $(a, a+T)$ we have $|f(x+\tau_n)-f(x)| \leqslant \epsilon$, except at most in a set of measure $T\epsilon$. In this way we show that, for T sufficiently large compared with l, and then for R sufficiently large compared with T, the set $[|f(x)| < k+2\epsilon]$ has mean measure in $(\beta, \beta+R)$ at least equal to that of $[|f(x)| \leqslant k]$ in $(a, a+T)$ diminished by 2ϵ. In this way we find that

$$m_1(k+2\epsilon) \geqslant m_4(k)-2\epsilon,$$

and so
$$m_1(k+0) \geqslant m_4(k). \tag{2}$$

Similarly
$$m_4(k-0) \leqslant m_1(k). \tag{3}$$

The monotonic functions m_1, m_2, m_3, m_4 have only discontinuities of the first kind and at most an enumerable set of these. For all other values of k, all four functions are continuous and

$$m_1(k) = m_2(k) = m_3(k) = m_4(k) = m(k). \tag{4}$$

Clearly $m_i(k) \to 1$ as $k \to \infty$ and is zero for negative values of k. The functions m_i may be defined in a finite interval and for functions not necessarily measurable; measurable functions are defined by $m_1 = m_2$, $m_3 = m_4$ in the case of a finite interval, and, if the function satisfies (2) and (3), it is effectively measurable so far as processes of integration are concerned. For a finite interval we have, by definition,

$$M(|f|) = \int_0^\infty k\,dm(k),$$

the integral being a Stieltjes integral and $m(k)$ being any monotonic function satisfying (4) (for the appropriate values of k). If $f(x)$ is bounded, this relation holds for W-a.p. functions; for, if we denote quantities referring to a finite interval L by primes, we have $m' \to m$ as $L \to \infty$ for values of k for which (4) holds, and hence

$$\int_0^K m'\,dk \to \int_0^K m\,dk, \quad \int_0^K k\,dm'(k) \to \int_0^K k\,dm(k),$$

while, by definition,

$$M = \lim M' = \lim \int_0^K k dm'(k).$$

In all cases we have

$$\int_0^\infty k dm(k) \geqslant M(|f_K|) = \int_0^K (1-m) dk \geqslant \int_0^K k dm(k),$$

and so

$$\int_0^\infty k dm(k) = \lim_{K\to\infty} M(|f_K|) = \int_0^\infty (1-m)\, dk$$

$$\leqslant \underline{\lim} \int_0^\infty (1-m')\, dk = M_*(|f|). \tag{5}$$

Now, if $f(x)$ is W_1-a.p., we can, as usual, find sets $E(\epsilon, T)$ containing a member τ in every interval of length $-l(\epsilon)$ such that, for $R \geqslant T$ and any a,

$$\frac{1}{R}\int_a^{a+R} |f(x+\tau)-f(x)|\, dx \leqslant \epsilon,$$

and we may choose T so large that

$$\frac{1}{T}\int_{-l}^0 |f(x)|\, dx \leqslant \epsilon.$$

Then, if $\phi(x)$ denotes $f(x)-f_K(x)$, we have, whatever the value of K,

$$\frac{1}{R}\int_a^{a+R} |\phi(x+\tau)-\phi(x)|\, dx \leqslant \epsilon \quad (R \geqslant T), \qquad \frac{1}{T}\int_{-l}^0 |\phi(x)|\, dx \leqslant \epsilon.$$

Taking K so large that

$$\frac{1}{T}\int_0^T |\phi(x)|\, dx \leqslant \epsilon,$$

we find at once that $M^*(|\phi|) \leqslant 3\epsilon$. Thus, when $f(x)$ is W_1-a.p.,

$$M(|f|) = \lim_{K\to\infty} M(|f_K|) = \int_0^\infty k dm(k). \tag{6}$$

In fact, $f(x)$ is the W_1-limit of the functions $f_K(x)$ as $K \to \infty$, as follows easily from the construction used to establish convergence in mean.

The definition and properties (1), (2), (3), (4) hold, of course, for any function which is W_s-a.p. with some $s > 0$ (and indeed more generally

still); we have

$$M\{|f(x)|^s\} = \int_0^\infty k^s\, dm(k) \quad \text{if } f(x) \text{ is } W_s\text{-a.p.},$$

$$M_*\{|f(x)|^r\} \geqslant \int_0^\infty k^r\, dm(k) \quad \text{in any case.}$$

Suppose $f(x)$ to be W_1-a.p. and let s_n be any one of its Fourier sums. Then $|s_n|^2 - 2R\{s_n f(x)\}$ is also W_1-a.p. (§ 4), and

$$M\{|s_n|^2 - 2R(fs_n)\} = -\Sigma_n |a_\lambda|^2.$$

Hence

$$M\{|f_K|^2 - |f_K - s_n|^2\} = -M\{|s_n|^2 - 2R(f_K s_n)\} \to \Sigma_n |a_\lambda|^2 \quad \text{as} \quad K \to \infty,$$

since f is the W_1-limit of f_K. But, as in (5) above,

$$\int_0^\infty k^2\, dm(k) = \lim_{K\to\infty} M(|f_K|)^2\dagger,$$

and so

$$\int_0^\infty k^2\, dm(k) \geqslant \Sigma |a_\lambda|^2.$$

On the other hand, since the Fejér sums σ_n converge (W_1) to f, we have $m(\sigma_n, k) \to m(k)$ except for an enumerable set of values of k; hence

$$\int_0^\infty k^2\, dm(k) = \int_0^\infty (1-m)\, d(k^2) \leqslant \underline{\lim} \int_0^\infty \{1 - m(\sigma_n, k)\}\, d(k^2)$$

$$= \underline{\lim}\, \Sigma |a_\lambda(\sigma_n)|^2 = \Sigma |a_\lambda|^2.$$

Combining the two inequalities, we obtain

$$\int_0^\infty k^2\, dm(k) = \Sigma |a_\lambda|^2,$$

which we write

$$J_2^2 = S_2^2.$$

7. This is Parseval's theorem for S-a.p. or W-a.p. functions. It is, in a certain sense, a best possible result as it stands. It is not true that

$$M\{|f(x)|^2\} = S_2^2.$$

† This relation holds, of course, with any index $r > 0$ replacing the index 2 on each side.

That $S_2^2 < \infty$ does not impose any restriction on $M\{|f|^2\}$ for a W_1-a.p. function is fairly obvious; we may, indeed, construct a W_1-a.p. function which is in fact a W_1-null function, *i.e.*

$$M\{|f(x)|\} = 0,$$

but such that $M^*\{|f(x)|^2\}$ and $M_*\{|f(x)|^2\}$ have any assigned values. To do this we put $f(x) = k_n$ in the intervals $(n, n+\delta_n)$ and $f(x) = 0$ elsewhere. If $k_n\delta_n \to 0$ as $|n| \to \infty$, then $M\{|f(x)|\} = 0$; while, if $k_n^2\delta_n \to K$, then $M\{|f(x)|^2\} = K$. We may choose $\delta_n \to 0$ and $k_n \to \infty$ so that these conditions are satisfied for any assigned value, finite or infinite, of K. We may indeed choose k_n, δ_n so that $f(x)$ is a W_r-null function for $r < 2$, *i.e.* $k_n^r\delta_n \to 0$ as $|n| \to \infty$, while $M^*\{f^2\}$, $M_*\{f^2\}$ have any previously assigned values and $M^*\{f^r\} = \infty$ for $r > 2$.

Suppose that a function $F(x)$ is W_h-a.p. and W_l-bounded† and that $h < k < l$. Then we may choose $p > 1$, so that

$$k = \frac{h}{p} + \frac{l}{q} \quad \text{where} \quad 1 = \frac{1}{p} + \frac{1}{q},$$

and we then find that

$$\frac{1}{R}\int_\alpha^{\alpha+R} |\phi(x)|^k\,dx \leqslant \left[\frac{1}{R}\int_\alpha^{\alpha+R} |\phi(x)|^h\,dx\right]^{1/p}\left[\frac{1}{R}\int_\alpha^{\alpha+R} |\phi(x|^l\,dx\right]^{1/q}.$$

Putting $\phi(x) = F(x+\tau) - F(x)$, we see that τ is a translation number of F with index k corresponding to $K\epsilon^h$, R provided that it is a translation number of F with index h corresponding to ϵ^{kp}, R, where $K = K(f; h, k, l)$. Thus $F(x)$ is W_k-a.p. for $h < k < l$ and hence for $k < l$.

We see then that either $F(x)$ is W_k-a.p. for every value of k, or it is so for $k < \lambda$ and not so for $k > \lambda$, where λ is some critical index: and in the latter case $F(x)$ is not even W_k-bounded for any value of $k > \lambda$. Finally, for the critical value $k = \lambda$, it may be a W_λ-null function, a W_λ-a.p. function not W_λ-null, or a function not W_λ-a.p. having any assigned values for $M^*(|F|^\lambda)$, $M_*(|F|^\lambda)$ and $J(|F|^\lambda)$ subject to the conditions

$$+\infty \geqslant M^* \geqslant M_* \geqslant J \geqslant 0, \tag{1}$$

which are certainly necessary in virtue of § 6, (5). All these possibilities may be obtained by adding a continuous periodic function to a function of the form $[f(x)]^{2/\lambda}$, where $f(x)$ is to be defined as above.

† Or, what is the same thing, S_l-bounded.

It is to be observed that

$$J_l = \left[\int_0^\infty k^l \, dm(k)\right]^{1/l}$$

is a function of l without discontinuities of the first kind: in particular

$$J_\lambda = \lim_{l \to \lambda} J_l,$$

whether the limit be finite or infinite. In this respect J_l differs sharply from M_l^*.

Of all the possible determinations of M^*, M_*, and J subject to (1), the only ones which permit of $F(x)$ being W_λ-a.p. are given by

$$+\infty > M^* = M_* = J \geqslant 0 \tag{2}$$

or

$$J = M < \infty. \tag{2.1}$$

If instead of (2) we have

$$+\infty > M^* = M_* = J = 0 \tag{3}$$

or

$$M = 0, \tag{3.1}$$

the function will certainly be W_λ-a.p. and indeed W_λ-null. And the only examples of (2) which we have cited are certainly W_λ-a.p. as being W_λ-equivalent to a continuous periodic function.

THEOREM. *Every solution of* (2) *is* W_λ*-a.p.*

Let $F(x)$ be a solution of (2) and let $F_K(x)$ denote the function obtained by bounding it to be not greater than K. Then

$$M\{|F_K|^\lambda\} \to J_\lambda^\lambda(F) = M\{|F|^\lambda\} \text{ as } K \to \infty.$$

Hence

$$M\{|F|^\lambda - |F_K|^\lambda\} \to 0 \text{ as } K \to \infty,$$

and, by § 5, fin.,

$$M\{(|F| - |F_K|)^\lambda\} \to 0 \text{ as } K \to \infty,$$

$$M\{|F - F_K|^\lambda\} \to 0 \text{ as } K \to \infty.$$

COROLLARY 1. *If* $|F|$ *is* W_λ-a.p., *so also is* F.

COROLLARY 2. *The condition* $J_\lambda < \infty$ *is necessary and sufficient in order that the functions* $F_K(x)$ *should be* W_λ*-convergent to some function*

$f(x)$. [We may suppose that $\lambda = 1$: then $F(x)$ is W_r-equivalent to $f(x)$ and $\Phi = |F|^r \operatorname{sgn} F$ has the same Fourier series as $\phi = |f|^r \operatorname{sgn} f$ for any $r < 1$.]

We may restate the hypotheses from which these conclusions are drawn:

$$(a) \quad J_\lambda = M_\lambda < \infty,$$

$$(b) \quad \text{the functions } F_K(x) \text{ are } W\text{-a.p.}$$

Except for the purpose of the latter part of Corollary 2, they contain no reference to λ being the critical index, nor indeed do they require explicitly that $F(x)$ should be W_l-a.p. for sufficiently small l. But the result is significant only when λ is the critical index of the function.

The condition (b) is closely related to the class of functions which Stepanoff, 2, designates (I). These may be defined by the conditions

(i) that the functions $F_K(x)$ be all S-a.p.,

(ii) that the measure $\mu(K, \alpha)$ of the set of points belonging to the interval $(\alpha, \alpha+1)$ for which $|F| > K$ tends to zero, uniformly in α, as $K \to \infty$.

We get an analogous class of functions by combining (b) with the condition

$$(c) \quad \text{that } m(K) \to 1 \text{ as } K \to \infty,$$

this being evidently equivalent to the condition†

$$(d) \quad \text{that } J(F_K) = o(K) \text{ as } K \to \infty.$$

The conditions (i) and (b) are not alone sufficient to define the classes of functions considered: we shall give two examples to this effect immediately. Divide the range of x from $-\infty$ to $+\infty$ into intervals $(n, n+1)$, where n is an integer (positive, negative or zero), and enumerate them in a single sequence

$$n = n_\nu \quad (\nu = 1, 2, \ldots).$$

We now divide the intervals into classes as follows. The first class contains the interval $\{\nu = \nu(1) = 1\}$ and consists of all intervals for which

$$n \equiv n_{\nu(1)} \pmod{p},$$

† The analogue of (d) would state

(iii) that $\int_\alpha^{\alpha+1} |F_K(x)|\, dx = o(K)$ uniformly in α.

where p is an integer: our two examples are obtained by putting (i) $p = 2$, (ii) $p = 3$.

The q-th class of intervals consists of all those for which

$$m \equiv n_{\nu(q)} \pmod{p^q},$$

where $\nu(q)$ is the least value of ν giving rise to an interval not belonging to the first $q-1$ classes.

We put $F(x) = q$ in intervals of the q-th class. Then $F_\lambda(x)$ is periodic with period p^q, so that (i) is satisfied; but, if $p = 3$,

$$m(q+\tfrac{1}{2}) = \sum_{r=1}^{q} p^{-r} < \tfrac{1}{2},$$

and (c) is not satisfied; while if $p = 2$,

$$m(q+\tfrac{1}{2}) \to 1 \text{ as } q \to \infty,$$

so that in this case (c) holds without (ii) being true. Thus we cannot replace (ii) by (c) in the definition of Stepanoff's class.

The effect produced by the subsidiary conditions (ii) and (c) may also be expressed as follows. If $F(x)$ belongs to Stepanoff's class (I), then a function $\phi(x)$ can be found which increases steadily from zero to $+\infty$ with x†, and is such that

$$\phi\{|F(x)|\} \operatorname{sgn} F(x)$$

is S-a.p. Conversely, if such a function ϕ can be found, then $F(x)$ belongs to (I).

And if $F(x)$ satisfies (i), then a continuous function $\phi(x)$ can be found which‡ increases steadily from zero with x, and is such that

$$\phi\{|F(x)|\} \operatorname{sgn} F(x)$$

is S-a.p. Conversely, if such a function ϕ can be found, then $F(x)$ satisfies (i). The analogous remarks relating to (b) instead of (i) may be omitted.

The theorem of the present section is the only positive result in the field we are now considering, which may be described as the context of (1). In the next section we shall be constructing examples which are S_l-a.p. for $l < \lambda$ (and not for $l > \lambda$), and which exhibit the possible circumstances relating to (1). In every case we shall be able to construct a *positive*

† Or which transforms the interval $x \geqslant 0$ topologically into itself.

‡ Or which transforms the interval $x \geqslant 0$ topologically into an interval $0 \leqslant \phi < \Phi$, where Φ may be finite or infinite.

example: and we may, without loss of generality, suppose that $\lambda = 1$ (*cf.* § 5, fin.). This will remove the distinction between

$$J_\lambda \text{ and } J(|F|^\lambda)$$

and will justify us in omitting the suffix λ.

8. As in the W-case we know that if $f(x)$ is S_l-a.p. for sufficiently small l, then either it is S_l-a.p. for every l or else there is a critical index λ such that for $l < \lambda$ the function is S_l-a.p., while for $l > \lambda$ it is not S_l-bounded. Since an S_l-bounded function is W_l-bounded and conversely, $f(x)$ has the same critical index in relation to its S-properties as in relation to its W-properties: and for $l > \lambda$ it is *not* W_l-a.p. A purely periodic function may be S_λ-a.p. with critical index λ: for the rest of this section we shall be constructing examples *not* S_λ-a.p. They will all be non-negative and have $\lambda = 1$ (*cf.* § 7) and they all have $J > 0$. Indeed, if we were merely given that $f(x)$ satisfied condition (i) above (instead of being S_l-a.p. for small l), we could still infer from

$$J(f) = 0$$

that

$$J(f_K) = 0,$$

$$f_K = 0 \text{ p.p. (since it is an } S\text{-a.p. function)},$$

and

$$f = 0 \text{ p.p.},$$

so that f would have no critical index (other than $+\infty$).

A periodic function with critical index 1 may have

$$J = M = +\infty,$$

but this example (which has points of non-summability) will prove to be of no value to us, and for the present we confine ourselves to functions everywhere summable. Suppose that we have a sequence $\{F_n(x)\}$ with the following properties:

(i) $F_n(x)$ is periodic with period P_n, where P_n is a positive integer and P_n divides P_{n+1};

(ii) $F_n(x)$ vanishes throughout $(0, P_n)$ except in a single interval of the form $\{\nu, \nu+(1/n)\}$, when it takes the value k_n (ν is an integer, positive, negative, or zero);

and (iii) at any particular point x there is at most one function of the sequence different from zero.

Such a sequence can be constructed, when $\{P_n\}$ is given, if and only if

$$\Sigma \frac{1}{P_n} \leqslant 1,$$

as we shall suppose. Write

$$\Phi(x) = \Sigma F_n(x),$$

then Φ is well defined and the functions Φ_K are S-a.p. Also†

$$J(\Phi) = \Sigma J(F_n) = \Sigma \frac{k_n}{nP_n}.$$

Let us first take $k_n = n, \quad P_{n+1} > P_n.$

Then Φ has a critical index $\lambda = 1$ but is *not* S-a.p., and

$$J(\Phi) = \Sigma \frac{1}{P_n} \leqslant \frac{2}{P_1}.$$

Thus, by choice of P_1, we can make J arbitrarily small. When $\{P_n\}$ is given we can so determine the functions $F_n(x)$ that in each interval $(\nu, \nu+1)$ there is exactly one function of the sequence which does not vanish identically, so that

$$\int_\nu^{\nu+1} \Phi(x)\, dx = 1.$$

In fact, we arrange the intervals $(\nu, \nu+1)$ in a single sequence by writing

$$\nu = \nu(m) \quad (m = 1, 2, \ldots),$$

and then, in constructing $F_n(x)$, we "fill in" that particular interval $(\nu, \nu+1)$ which, of all the intervals not yet "filled in", comes first in the sequence. In this way we obtain an example $\Phi(x)$ for which

$$M = 1, \quad J < \epsilon.$$

By another process of choice we can obtain an example for which

$$M^* = 1, \quad J = M_* < \epsilon.$$

---

† If $f_n(x)$ is non-negative and increases to $\phi(x)$, and if each $f_n(x)$ becomes W-a.p. when bounded, then so does $\phi(x)$ and

$$J(f_n) \to J(\phi) \text{ as } n \to \infty.$$

For $$J(f_n) = \int_0^\infty \{1 - m_n(k)\}\, dk, \quad J(\phi) = \int_0^\infty \{1 - \mu(k)\}\, dk \text{ (say)},$$

and $m_n(k)$ decreases to $\mu(k)$, save for an enumerable set of values of k.

First choose P_1 so that $$\frac{2}{P_1} < \epsilon.$$

Then take a (large) interval I_1 and choose a batch of functions $F_n(x)$ so as to "fill in" every interval $(\nu, \nu+1)$ lying in I_1. When this is done choose an interval I_1' so large that the sum of the F_n so far chosen has a mean value on I_1' nearly equal to its mean value over the whole line. Next choose an interval I_2, exterior to I_1' and still larger than I_1, and choose a second batch of functions $F_n(x)$ so as to "fill in" the intervals $(\nu, \nu+1)$ of I_2 not already attended to, but with periods P_n so great that they vanish identically throughout I_1'. Then choose I_2' still larger than I_1' so as to represent adequately the sum of the F_n of the first two batches. Next choose I_3 outside $I_1'+I_2'$, and fill it in with a third batch of functions leaving $I_1'+I_2'$ undisturbed. Continuing this process indefinitely, we obtain a function $\Phi(x)$ with two sequences of intervals $\{I_n\}$, $\{I_n'\}$ such that

$$\underset{(I_n)}{M}\{\Phi(x)\} \to 1, \quad \underset{(I_n')}{M}\{\Phi(x)\} \to J(\Phi),$$

$$|I_n| \to \infty \quad \text{and} \quad |I_n'| \to \infty \quad \text{as} \quad n \to \infty.$$

Secondly let us take

$$k_n = n \log (n+1), \quad P_{n+1} > P_n$$

so that $$J(\Phi) = \Sigma \frac{\log (n+1)}{P_n} \leqslant \frac{A}{P_1}$$

where $$A = \Sigma \frac{\log (n+1)}{2^{n-1}} < \infty,$$

and choose P_1 so great that

$$\Sigma \frac{1}{P_n} \leqslant \frac{2}{P_1} \leqslant \tfrac{1}{2}.$$

A process of choice following the same lines as the last produces an example $\Phi(x)$ for which

$$J = M_* < \infty, \quad M^* = \infty.$$

It will be found that in the present case there is a new reason for making I_n "large" at each stage: this is the only difference in the argument required. On the other hand, suppose that, when $\{F_n\}$ is constructed, every interval $(\nu, \nu+1)$ is "filled in" in its turn. Then for any assigned

value of n we have

$$\int_{\mu}^{\mu+1} (F_1+F_2+\ldots+F_n)\,dx = 0,$$

and hence

$$\int_{\mu}^{\mu+1} \Phi(x)\,dx \geqslant \log(n+2),$$

for precisely

$$P_n\left(1-\frac{1}{P_1}-\frac{1}{P_2}-\ldots-\frac{1}{P_n}\right) > \tfrac{1}{2}P_n$$

values of μ out of any P_n consecutive values. Hence

$$M_*\{\Phi(x)\} \geqslant \tfrac{1}{2}\log(n+2), \quad M\{\Phi\} = +\infty.$$

By taking a linear combination of functions such as constructed above, with the addition of a suitable constant, we may obtain a function with S-critical index $\lambda = 1$ but not S-a.p., everywhere summable and having any assigned values for J, M_*, M^* subject to the conditions

$$0 < J < \infty, \quad J \leqslant M_* \leqslant M^*.$$

In particular we may have

$$0 < J = M < \infty,$$

so that the function is W-a.p. but not S-a.p. This occurs when

$$k_n = n, \quad \Sigma\frac{1}{P_n} = 1.$$

We may also obtain an example, everywhere summable and with critical index $\lambda = 1$, for which

$$J(=M) = +\infty$$

by putting

$$k_n = n\log n$$

and dropping the restriction $P_{n+1} > P_n$. Suppose, in fact, that $P_1, P_2, \ldots, P_n$ have been chosen in such a way that

$$\frac{1}{P_1}+\frac{1}{P_2}+\ldots+\frac{1}{P_n} < 1,$$

while

$$P_m \text{ divides } P_{m+1} \quad (m < n).$$

Write

$$\epsilon = 1-\left(\frac{1}{P_1}+\frac{1}{P_2}+\ldots+\frac{1}{P^n}\right)$$

and choose

$$P_{n+1} = P_{n+2} = \ldots = P_{n+N} = P, \text{ say.}$$

We still have $$\frac{1}{P_1}+\frac{1}{P_2}+\ldots+\frac{1}{P_{n+N}}<1$$

provided that $$N<P_\epsilon.$$

Let us put $$N=P_\epsilon-1$$

and suppose that $$P_\epsilon>2,$$

so that $$N>\tfrac{1}{2}P_\epsilon.$$

Then

$$\frac{\log(n+2)}{P_{n+1}}+\ldots+\frac{\log(n+N+1)}{P_{n+N}}>\frac{N}{2P}\log\frac{N}{2}>\frac{\epsilon}{4}\log\frac{P_\epsilon}{4}.$$

This last is arbitrarily large with P, and it is clear how in this way we may choose a sequence $\{P_n\}$ for which

$$\Sigma\frac{1}{P_n}\leqslant 1,$$

$$P_n \text{ divides } P_{n+1},$$

and $$J=\Sigma\frac{\log(n+1)}{P_n}=+\infty.$$

9. In § 8 we showed by examples everywhere summable that nothing can be added to the inequalities (1) of § 7. In this section we shall show that the set of points of non-summability of a function with S-critical index $\lambda=1$ may be assigned in advance subject to one trivial restriction, and that this affects the domain of variation open to J, M_*, and M^* only in obvious ways. We first construct a function having a single point of non-summability at the origin (so that $M^*=+\infty$) for which

$$J=M_*<\infty.$$

We write $$\phi(x)=\Sigma f_n(x)$$

where $$f_n(x)=1 \quad \text{if} \quad 0<x<\frac{1}{n},$$

$$=0 \quad \text{if} \quad \frac{1}{n}\leqslant x\leqslant p_n,$$

and has period p_n. We shall suppose that p_n is an integer and tends to $+\infty$ with n so fast that

$$\Sigma \frac{1}{np_n} < \infty .$$

Then $\phi(x)$ is certainly well-defined and is bounded in each finite closed interval excluding the origin; but it is not summable near the origin, since

$$\int_0^1 \phi(x)dx = \Sigma \int_0^1 f_n(x)dx = \Sigma \frac{1}{n} = +\infty.$$

In (0, 1) we have

$$\phi(x) = n \quad \text{if} \quad \frac{1}{n+1} \leqslant x < \frac{1}{n}$$

and hence

$$\int_0^1 \phi^l(x)dx = \Sigma n^l \left(\frac{1}{n} - \frac{1}{n+1}\right),$$

which converges if $l < 1$. Hence $\phi(x)$ is of class L^l in (0, 1) for $l < 1$, and the partial sums

$$s_n = f_1 + f_2 + \ldots + f_n,$$

since they increase to ϕ, must converge in mean to ϕ with index l in this interval. We infer at once that the series Σf_n is S_l-convergent to ϕ by comparing an arbitrary interval $(n, n+1)$ with the "most doubtful" case (0, 1). Hence $\phi(x)$ is S_l-a.p. for $l = 1$ and has critical index $\lambda = 1$. Also

$$\frac{1}{N}\int_1^N \phi(x)dx = \Sigma \frac{1}{N}\int_1^N f_n(x)dx,$$

and the series on the right is uniformly convergent since

$$\frac{1}{N}\int_1^N f_n(x)dx < \frac{2}{np_n} \quad (N \geqslant 2).$$

Since

$$\frac{1}{N}\int_1^N f_n(x)dx \to \frac{1}{np_n} \quad \text{as} \quad N \to \infty ,$$

we infer that

$$\frac{1}{N}\int_1^N \phi(x)dx \to \Sigma \frac{1}{np_n} \quad \text{as} \quad N \to \infty .$$

Hence (*cf.* § 8, foot-note)

$$J = \Sigma \frac{1}{np_n} = M_* < \infty .$$

The points of non-summability of a function necessarily form a closed set: and if $M_* < \infty$ the set cannot be relatively dense. Let E be a given closed set not relatively dense: we shall construct a function $\psi(x)$ having S-critical index $\lambda = 1$, $J = M_* < \infty$, and having E as the set of its points of non-summability. If the given set E consists of a finite number of points x_n, then we simply write

$$\psi(x) = \Sigma \phi(x - x_n).$$

If E is infinite, we take a sequence $\{x_n\}$ of points of E such that the whole sequence is dense on E. We also choose a sequence $\{I_n\}$ of intervals each at distance from E at least 1 and such that

$$|I_n| \to \infty \quad \text{with} \quad n.$$

We write

$$\phi_1(x) = \phi(x - x_1)$$

where ϕ is defined as above, the integers p_n, however, being still left undetermined. We have

$$\frac{1}{|I_n|}\int_{I_n} \phi_1(x)\,dx = \sum_{\nu=1} \frac{1}{|I_n|}\int_{I_n} f_\nu(x - x_1)\,dx$$

and

$$\frac{1}{|I_n|}\int_{I_n} f_\nu(x - x_1)\,dx \to \frac{1}{\nu p_\nu} \quad \text{as} \quad n \to \infty.$$

Before choosing $\{p_\nu\}$ we choose an auxiliary sequence $\{q_\nu\}$ such that

$$\Sigma \frac{1}{\nu q_\nu} < \infty.$$

We then choose $p_\nu > q_\nu$ so great that $f_\nu(x - x_1)$ vanishes throughout each interval I_n for which $|I_n| < q_\nu$: when this is done,

$$\frac{1}{|I_n|}\int_{I_n} f_\nu(x - x_1)\,dx < \frac{2}{\nu q_\nu},$$

and so, by uniform convergence,

$$\frac{1}{|I_n|}\int_{I_n} \phi_1(x)\,dx \to \Sigma \frac{1}{\nu p_\nu} = J(\phi_1) \quad \text{as} \quad n \to \infty.$$

We define $\phi_\nu(x)$ for each ν, with an appropriate sequence $\{p_n\}$, so as to have a single point of non-summability at x_ν while

$$\frac{1}{|I_n|}\int_{I_n} \phi_\nu(x)\,dx \to J(\phi_\nu) \quad \text{as} \quad n \to \infty.$$

Now consider the function

$$\psi(x) = \Sigma \epsilon_n \phi_n(x),$$

where the ϵ_n are "small" but positive. Suppose that $l < 1$: then

$$\int_a^{a+1} \phi_\nu^l(x)dx \leqslant \int_0^1 \phi^l(x)dx = M_l \quad \text{(say)},$$

and hence

$$\int_a^{a+1} \left[\sum_\mu^\nu \epsilon_n \phi_n(x)\right]^l dx \leqslant M_l \sum_\mu^\nu \epsilon_n.$$

If

$$\Sigma \sqrt{\epsilon_n} < \infty,$$

then

$$\Sigma \epsilon_n < \infty \quad (\tfrac{1}{2} \leqslant l < 1),$$

and the series $\Sigma \epsilon_n \phi_n$ is S_l-convergent for $\frac{1}{2} \leqslant l < 1$, *a fortiori* for $l < \frac{1}{2}$. $\psi(x)$ is finite p.p. and is the S_l-limit (for $l < 1$) of the partial sums of the series: clearly $\psi(x)$ is non-summable at each of the points x_n, hence at each point of E. Also (*cf.* § 8, foot-note)

$$J(\psi) = \Sigma \epsilon_n J(\phi_n).$$

Now

$$\frac{1}{|I_n|} \int_{I_n} \phi_\nu(x)dx \to J(\phi_\nu) \quad \text{as} \quad n \to \infty,$$

$$\frac{1}{|I_n|} \int_{I_n} \phi_\nu(x)dx < K_\nu \quad \text{for all } n.$$

If

$$\Sigma \epsilon_\nu K_\nu < \infty,$$

then

$$J(\psi) = \Sigma \epsilon_\nu J(\phi_\nu) < \infty,$$

and, by the theory of uniform convergence,

$$\frac{1}{|I_n|} \int_{I_n} \psi(x)dx = \sum_{\nu=1} \frac{\epsilon_\nu}{|I_n|} \int_{I_n} \phi_\nu(x)dx \to \Sigma \epsilon_\nu J(\phi_\nu) = J(\psi)$$

as $n \to \infty$. Thus $\psi(x)$ is such that

$$M_* = J < \infty.$$

Finally, in order to secure that $\phi(x)$ shall be summable in a neighbourhood of each point not belonging to E, we express the complement of E (which is an open set) as a sum of closed intervals L_n and write

$$A_{n\nu} = \int_{L_n} \phi_\nu(x)dx.$$

If we choose $$\epsilon_n = \frac{1}{n^2 \max(n, K_n, A_{1n}, A_{2n}, \ldots, A_{nn})},$$

we obtain the desired result.

The method of condensation of singularities also enables us to construct a function $\chi(x)$ of S-critical index $\lambda = 1$ having $J < \infty$ and any assigned closed set E as the aggregate of its points of non-summability. E may now be relatively dense, but we have discarded restrictions on M^*. The intervals I_n now do not arise, and in forming the functions ϕ_ν we may use any sequences $\{p_n\}$ such that

$$\Sigma \frac{1}{np_n} < \infty .$$

For example we could use the same sequence $\{p_n\}$ for each function ϕ_ν. And in the final determination of ϵ_n we replace K_n by $J(\phi_n)$.

By taking linear combinations of the examples of §§ 8 and 9 we may construct a function with S-critical index $\lambda = 1$, but not S-a.p., with any assigned closed set E of points of non-summability and,

(i) in case E is relatively dense, any assigned value for J subject to

$$0 < J \leqslant +\infty ;$$

(ii) in case E is not relatively dense, any assigned values for J, M_* subject to

$$0 < J \leqslant M_* \leqslant +\infty ;$$

and (iii) in case E is void, any assigned values for J, M_*, M^* subject to

$$0 < J \leqslant M_* \leqslant M^* \leqslant +\infty .$$

It should be observed that we have confined ourselves throughout to examples of "grenz-periodisch" character, *i.e.*, functions for which the appropriate trigonometric approximations are *purely* periodic. And it is perhaps not without interest to observe that for each example $\Phi(x)$ constructed in § 8, save the last, we have

$$J_l < \infty ,$$

not only for $l < 1$ but for all values of l. In fact,

$$J_l(\Phi) = \Sigma \frac{k_n^l}{nP_n}$$

and $$P_n \geqslant 2^{n-1}, \quad k_n \leqslant n \log n.$$

And, if $r > 1$, we have (*cf.* § 8, foot-note)

$$J_r(\phi) = \lim_{n\to\infty} J_r(s_n) \leqslant \Sigma J_r(f_n) = \Sigma(np_n)^{-1/r},$$

$$J_r(\psi) = \lim_{n\to\infty} J_r(\epsilon_1\phi_1 + \ldots + \epsilon_n\phi_n) \leqslant \Sigma \epsilon_n J_r(\phi_n).$$

If for $\phi(x)$ and for each $\phi_\nu(x)$ we choose

$$p_n \geqslant 2^{n-1},$$

we get

$$J_r(\phi) < \infty, \quad J_r(\psi) < \infty.$$

References.

1. Bohr, *Acta Math.*, 45, 46, 47.
2. Stepanoff, *Math. Annalen*, 95 (1926), 473.
3. Ursell, *Journal London Math. Soc.*, 4 (1929), 123.
4. Bochner, *Math. Annalen*, 96 (1926), 119; *Math. Zeitschrift*, 27 (1927), 187.
5. Pollard, *Proc. London Math. Soc.* (2), 25 (1926), 237.

SOME THEOREMS CONCERNING TRIGONOMETRICAL SERIES OF A SPECIAL TYPE

By G. H. HARDY.

[Received 10 May, 1930.—Read 15 May, 1930.]

1. In this note I develop more systematically a group of theorems some of which have been proved already in the *Journal* or *Proceedings*. They concern primarily the series

$$(1.1)\quad \Sigma a_n \cos n\theta = f(\theta), \qquad (1.2)\quad \Sigma a_n \sin n\theta = g(\theta),$$

in which $n = 1, 2, 3, \ldots$, and the a_n are positive and decrease steadily to the limit 0. It is, of course, familiar that the sine series is convergent for all θ, and the cosine series for all θ save multiples of 2π, and that they are the Fourier series of their sums when these are integrable. They also possess a number of interesting and less obvious properties, among which I may mention the following.

(1) It was proved by Chaundy and Jolliffe* that *a necessary and sufficient condition for the uniform convergence of* (1.2) *is that* $na_n \to 0$; *more precisely, the condition is sufficient for the uniform convergence of the series and necessary for the continuity of its sum.*

(2) In a recent note in the *Journal*† I proved that *if*

$$(1.3)\quad a_n \sim A n^{-\alpha},$$

where $A > 0$, $0 < \alpha < 1$, *and the formula is to be interpreted as meaning* $a_n = o(n^{-\alpha})$ *when* $A = 0$, *then*

$$(1.4)\quad f(\theta) \sim A \sin \tfrac{1}{2}\alpha\pi\,\Gamma(1-\alpha)\,\theta^{\alpha-1},$$

$$(1.5)\quad g(\theta) \sim A \cos \tfrac{1}{2}\alpha\pi\,\Gamma(1-\alpha)\,\theta^{\alpha-1}$$

(*with a similar gloss*), *when* $\theta \to 0$ *by positive values.*

* Chaundy and Jolliffe, 2; Jolliffe, 8. Only the first form of the theorem is given in the first paper. In the second Jolliffe considers also series of more general types.

† Hardy, 3.

(3) Littlewood and I have shown still more recently that *a necessary and sufficient condition that* $f(\theta)$ *or* $g(\theta)$ *should belong to the Lebesgue class* L^r, *where* $r > 1$, *is that* $\Sigma n^{r-2} a_n^r$ *should be convergent.*

The last theorem is of a different character and I shall not refer to it again. My object here is to go further in the direction indicated by (1) and (2).

2. THEOREM 1. *Suppose that* a_n *is positive and decreases steadily to the limit* 0, *that* $f(\theta)$ *and* $g(\theta)$ *are defined by* (1.1) *and* (1.2), *and that* $A > 0$, $0 < \alpha < 1$. *Then* (1.3) *is a necessary and sufficient condition for the truth of either* (1.4) *or* (1.5).

I proved in my former note that the condition is sufficient, and it remains only to prove it necessary. The proofs for (1.1) and (1.2) are not quite the same, but we require in either case a lemma of a "Tauberian" character.

LEMMA. *If* a_n *is positive and decreasing, or, more generally, if* $n^{-k} a_n$ *is decreasing for some* k, *and if* (1.3) *holds in some Cesàro sense, then* (1.3) *holds in the ordinary sense.*

By saying that (1.3) holds in some Cesàro sense, say (C, r), where r is a positive integer, we mean that, if

$$a_n^0 = a_n, \quad a_n^1 = a_1^0 + \ldots + a_n^0, \quad a_n^2 = a_1^1 + \ldots + a_n^1, \quad \ldots,$$

then

$$a_n^r \sim A \frac{\Gamma(1-\alpha)}{\Gamma(r+1-\alpha)} n^{r-\alpha}. \tag{2.1}$$

It is obviously sufficient to prove that (1.3) holds $(C, r-1)$ if it holds (C, r).

Suppose then that (2.1) holds for a particular r, and write A_n for a_n^{r-1} and C for the constant on the right-hand side. If $\delta > 0$ and $\nu = [n + \delta n]$, then

$$A_{n+1} + A_{n+2} + \ldots + A_\nu \sim C\{(1+\delta)^{r-\alpha} - 1\} n^{r-\alpha}. \tag{2.2}$$

If $r > 1$, A_n increases with n, so that

$$\delta n A_n \leqslant C\{(1+\delta)^{r-\alpha} - 1\} n^{r-\alpha} + o(n^{r-\alpha}),$$

$$\frac{A_n}{n^{r-1-\alpha}} \leqslant C \frac{(1+\delta)^{r-\alpha} - 1}{\delta} + o(1),$$

$$\overline{l} = \overline{\lim} \frac{A_n}{n^{r-1-\alpha}} \leqslant C \frac{(1+\delta)^{r-\alpha} - 1}{\delta},$$

and so, since δ is as small as we please, $\bar{l} \leqslant (r-\alpha)C$. Similarly, considering a block of A_m to the left of A_n, we find that $\underline{l} \geqslant (r-\alpha)C$. Hence $\bar{l} = \underline{l} = (r-\alpha)C$, and (1.3) holds $(C, r-1)$.

If $r = 1$, (2.2) is

$$a_{n+1} + a_{n+2} + \ldots + a_\nu \sim \frac{A}{1-\alpha}\{(1+\delta)^{1-\alpha} - 1\}\, n^{1-\alpha}. \tag{2.3}$$

We have now (supposing that $k > 0$, obviously the most unfavourable case)

$$a_\mu \leqslant \left(\frac{\mu}{n}\right)^k a_n \leqslant (1+\delta)^k a_n \quad (n < \mu \leqslant \nu),$$

and we deduce that

$$\underline{l} = \underline{\lim}\, n^\alpha a_n \geqslant \frac{A}{1-\alpha}\,\frac{(1+\delta)^{1-\alpha}-1}{(1+\delta)^k \delta},$$

and so $\underline{l} \geqslant A$. Similarly $\bar{l} \leqslant A$, and (1.3) holds in the ordinary sense.

Actually we require only the cases of the lemma in which r is 1 or 2 and k is 0 or 1.

3. (i) Considering now the cosine series, we prove first that

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta\, d\theta, \tag{3.1}$$

i.e. that the series is the Fourier series of its sum*. In fact, since the series is uniformly convergent in (ϵ, π) for any positive ϵ, we have

$$F(\theta) = \int_\theta^\pi f(u)\, du = -\Sigma a_n \frac{\sin n\theta}{n},$$

and the series on the right is uniformly convergent for all θ†. Hence

$$\frac{a_n}{n} = -\frac{2}{\pi}\int_0^\pi F(\theta)\sin n\theta\, d\theta = -\frac{2}{\pi}\lim_{\epsilon\to 0}\int_\epsilon^\pi F(\theta)\sin n\theta\, d\theta$$

$$= \frac{2}{n\pi}\lim_{\epsilon\to 0}\left\{-F(\epsilon)\cos n\epsilon + \int_\epsilon^\pi f(\theta)\cos n\theta\right\} = \frac{2}{n\pi}\int_0^\pi f(\theta)\cos n\theta\, d\theta.$$

The integral here is in the first instance a "Cauchy" integral, but it plainly exists also as a Lebesgue integral.

---

* This is a corollary of general theorems of the "Riemannian" theory. It is more appropriate here to give a direct proof.

† Since $\Sigma \frac{\sin n\theta}{n}$ is boundedly convergent.

We now deduce from (3 . 1) in the usual manner

$$a_n^1 = \frac{1}{\pi}\int_0^\pi f(\theta)\,\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,d\theta, \qquad a_n^2 = \frac{1}{\pi}\int_0^\pi f(\theta)\,\frac{\sin^2\frac{1}{2}(n+1)\theta}{\sin^2\frac{1}{2}\theta}\,d\theta.$$

The kernel in the last integral being positive, the asymptotic value of a_n^2 may be calculated in the obvious way. We find that

$$a_n^2 \sim \frac{4}{\pi}A\,\Gamma(1-\alpha)\sin\tfrac{1}{2}\alpha\pi\int_0^\infty \theta^{\alpha-3}\sin^2\tfrac{1}{2}n\theta\,d\theta = \frac{A}{(1-\alpha)(2-\alpha)}\,n^{2-\alpha};$$

the justification of the process in detail is a straightforward piece of routine which may be left to the reader. Applying the lemma, with $r=2$, $k=0$, we obtain (1 . 3).

(ii) Passing to the sine series, we observe first that the series $\Sigma a_n \sin n\theta \sin m\theta$ is uniformly convergent, since

$$\left|\sum_p^q a_n \sin n\theta \sin m\theta\right| \leqslant |m\theta|\left|\sum_p^q a_n \sin n\theta\right| \leqslant 2m\,\frac{\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\,a_p.$$

It follows that* $$a_m = \frac{2}{\pi}\int_0^\pi g(\theta)\sin m\theta\,d\theta.$$

Hence

$$a_n^1 = \frac{1}{\pi}\int_0^\pi g(\theta)\,\frac{1-\cos(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\,d\theta - \frac{1}{\pi}\int_0^\pi g(\theta)\,\frac{1-\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\,d\theta.$$

The second term is independent of n, and the first, having again a positive kernel, may be evaluated by substituting the asymptotic value of $g(\theta)$ and replacing the upper limit π by ∞. We find that

$$a_n^1 \sim \frac{An^{1-\alpha}}{1-\alpha},$$

and (1 . 3) again follows from the lemma, with $r=1$, $k=0$.

4. The argument fails when $\alpha = 1$, but the result still holds for the *sine* series with the obvious modifications.

THEOREM 2. *If a_n decreases steadily to the limit 0, then $na_n \to A$ is a necessary and sufficient condition that $g(\theta) \to \frac{1}{2}\pi A$, where $\theta \to 0$ by positive values.*

* This again follows from general theorems ; the simple proof given here was suggested to me by Dr. Zygmund. See Zygmund, **10**, 105.

The case $A = 0$ is the theorem of Chaundy and Jolliffe. The general case is not immediately reducible to the special case, since $a_n - A/n$ is not necessarily monotonic.

(i) If $na_n \to A$, we have

$$g(\theta)-g_0(\theta) = \Sigma a_n \sin n\theta - A \Sigma \frac{\sin n\theta}{n}$$

$$= \sum_{n \leqslant c/\theta} a_n \sin n\theta - A \sum_{n \leqslant c/\theta} \frac{\sin n\theta}{n} + \sum_{n \geqslant C/\theta} a_n \sin n\theta$$

$$- A \sum_{n \geqslant C/\theta} \frac{\sin n\theta}{n} + \sum_{c/\theta < n < C/\theta} \left(a_n - \frac{A}{n}\right) \sin n\theta$$

$$= g_1(\theta)+g_2(\theta)+g_3(\theta)+g_4(\theta)+g_5(\theta),$$

say, c and C being positive (small and large respectively) and independent of θ. Then

$$g_1(\theta) = \sum_{n \leqslant c/\theta} O\left(\frac{1}{n}\right) O(n\theta) = O\left(\theta \cdot \frac{c}{\theta}\right) = O(c),$$

$$g_3(\theta) = O\left(a_{[C/\theta]} \operatorname*{Max}_{q > p \geqslant C/\theta} \left| \sum_{p}^{q} \sin n\theta \right|\right) = O\left(\frac{\theta}{C} \cdot \frac{1}{\theta}\right) = O\left(\frac{1}{C}\right),$$

where the constants of the O's are independent of c and C. There are plainly similar upper bounds for $g_2(\theta)$ and $g_4(\theta)$; and we may therefore choose c and C so that

$$|g(\theta)-g_0(\theta)-g_5(\theta)| < \epsilon$$

for all θ. Since $g_0(\theta) \to \frac{1}{2}\pi A$ when $\theta \to 0$, and

$$g_5(\theta) = \sum_{c/\theta < n < C/\theta} o\left(\frac{1}{n}\right) = o\left(\log \frac{C}{c}\right) = o(1)$$

when c and C are fixed, it follows that $g(\theta) \to \frac{1}{2}\pi A$.

(ii) To prove the condition necessary, it seems most convenient to use Poisson's integral. We have

$$\sum_{1}^{\infty} a_n r^n \sin n\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-r^2) g(\phi)\, d\phi}{1-2r\cos(\phi-\theta)+r^2}$$

for $r < 1$, and so, differentiating with respect to θ and putting $\theta = 0$,

$$\sum_1^\infty n a_n r^n = \frac{2r}{\pi} \int_0^\pi \frac{(1-r^2) g(\phi) \sin \phi \, d\phi}{(1-2r\cos\phi + r^2)^2}$$

$$\sim 2A \int_0^\infty \frac{(1-r)\,\phi\, d\phi}{\{(1-r)^2+\phi^2\}^2} = \frac{A}{1-r}$$

when $r \to 1$*. Since $a_n \geqslant 0$, it follows† that

$$a_1 + 2a_2 + \ldots + n a_n \sim An, \qquad b_n = n a_n \to A \quad (C, 1).$$

Since b_n/n decreases, it follows from the lemma (with $r = 1$, $k = 1$) that $b_n \to A$.

There is no precise analogue of Theorem 2 for the *cosine* series. It is easy to prove that $n a_n \to A$ implies that

$$f(\theta) \sim A \log \frac{1}{\theta},$$

but the converse is false. Thus, if we take

$$a_n = 2^{-\nu} \quad (2^\nu \leqslant n < 2^{\nu+1}, \ \nu = 0, 1, \ldots),$$

we find without difficulty that $f(\theta)$ differs from

$$\psi(\theta) = \frac{1}{\theta} \sum_1^\infty 2^{-\nu} \sin 2^\nu \theta$$

by a bounded function, and that $\psi(\theta) \sim \log(1/\theta)$ when $\theta \to +0$. But plainly $n a_n$ does not tend to a limit.

5. There is a theorem "reciprocal" to Theorem 1 in which a_n is not restricted to be monotonic but the series are supposed to be Fourier series of monotonic functions.

THEOREM 3. *If the series* (1.1) *is the Fourier series of* $f(\theta)$, *or* (1.2) *of* $g(\theta)$, *and* $f(\theta)$, *or* $g(\theta)$, *decreases steadily as* θ *increases from* 0, *then* (1.3) *is a necessary and sufficient condition for the truth of* (1.4), *or* (1.5).

The proof is very much like that of Theorem 1 (the rôles of θ and n being roughly interchanged), and I need hardly write it out at length. One half of the theorem, that (1.4), or (1.5), implies (1.3), is nearly, but not exactly, the same as a theorem of Bromwich generalised by Haslam-

* Here again I omit some details which the reader will easily supply.

† Hardy and Littlewood, **4**; Hobson, **7**, 185; Landau, **9**, 50.

Jones* in a recent note in the *Journal*. Bromwich supposes, for example, that $f(\theta) = \theta^{-\alpha}\phi(\theta)$, where $\phi(\theta)$ is of bounded variation. If $\phi(\theta)$ decreases steadily, the result is included in Theorem 3. We could replace the hypothesis that $f(\theta)$ decreases by the hypothesis that its variation in (θ, π) is $O(\theta^{-\alpha})$, and then our result would include the more general form of Bromwich's theorem.

We may also modify the theorem by assuming that (1 . 1) converges to $f(\theta)$, or to $\bar{f}(\theta) = \frac{1}{2}\{f(\theta+0)+f(\theta-0)\}$. It is then easily proved that the series is the Fourier series of $f(\theta)$.

6. Theorem 2 reduces to the Chaundy-Jolliffe theorem when $A = 0$. The most sophisticated part of the theorem is that which asserts that $na_n \to 0$ is a *necessary* condition for the continuity of $g(\theta)$. It may be worth while, in conclusion, to reproduce Jolliffe's ingenious proof, which does not depend in any way on the theory of Fourier series, in its simplest form, since the point is rather obscured in his paper by the greater generality of the series considered.

Let $\theta = \pi/(2m+1)$, where m is integral, and write

$$g(\theta) = \sum_1^m + \sum_{m+1}^{2m} + \sum_{2m+1}^{\infty} = g_1(\theta)+g_2(\theta)+g_3(\theta).$$

Then

$$g_1(\theta) \geqslant a_m \sum_1^m \sin n\theta = \frac{a_m}{2\sin\frac{1}{2}\theta}\cos\tfrac{1}{2}\theta,$$

$$g_2(\theta) \geqslant a_{2m} \sum_{m+1}^{2m} \sin n\theta = -\frac{a_{2m}}{2\sin\frac{1}{2}\theta}\cos(2m+\tfrac{1}{2})\theta,$$

$$|g_3(\theta)| \leqslant a_{2m+1}\,\mathrm{Max}\left|\sum_{2m+1}^{p}\sin n\theta\right| \leqslant \frac{a_{2m}}{2\sin\frac{1}{2}\theta}\{1-\cos(2m+\tfrac{1}{2})\theta\}.$$

Hence
$$2\sin\tfrac{1}{2}\theta\, g(\theta) \geqslant a_m\cos\tfrac{1}{2}\theta - a_{2m}.$$

The left-hand side is $o(1/m)$, and we can omit the $\cos\frac{1}{2}\theta$ with error $O(1/m^2) = o(1/m)$. Hence

$$a_m - a_{2m} = o\left(\frac{1}{m}\right)$$

and

$$|a_m| \leqslant |a_m - a_{2m}| + |a_{2m} - a_{4m}| + \ldots < \frac{\epsilon}{m}(1+\tfrac{1}{2}+\tfrac{1}{4}+\ldots) = \frac{2\epsilon}{m}$$

if m is sufficiently large.

* Bromwich, **1**, 494; Haslam-Jones, **6**.

References.

1. T. J. I'A. Bromwich, *Theory of infinite series* (ed. 2, 1926).
2. T. W. Chaundy and A. E. Jolliffe, "The uniform convergence of a certain class of trigonometrical series", *Proc. London Math. Soc.* (2), 15 (1916), 214–216.
3. G. H. Hardy, "A theorem concerning trigonometrical series", *Journal London Math. Soc.*, 3 (1928), 12–13.
4. G. H. Hardy and J. E. Littlewood, "Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive", *Proc. London Math. Soc.* (2), 13 (1914), 174–191.
5. ———— "Notes on the theory of series (XIII): Some new properties of Fourier constants", *Journal London Math. Soc.*, 6 (1931), 3–9.
6. U. S. Haslam-Jones, "A note on the Fourier coefficients of unbounded functions", *Journal London Math. Soc.*, 2 (1927), 151–154.
7. E. W. Hobson, *Theory of functions of a real variable*, 2 (ed. 2, 1926).
8. A. E. Jolliffe, "On certain trigonometrical series which have a necessary and sufficient condition for uniform convergence", *Proc. Camb. Phil. Soc.*, 19 (1920), 191–195.
9. E. Landau, *Darstellung und Begrundung einiger neuerer Ergebnisse der Funktionentheorie* (ed. 1, 1916).
10. A. Zygmund, "A note on series of sines", *Quarterly Journal* (Oxford series), 1 (1930), 102–107.

ON QUESTIONS OF REALITY FOR CERTAIN GEOMETRICAL LOCI

By J. A. TODD.

[Received 30 May, 1930.—Read 19 June, 1930.]

Introduction.

The object of this paper is to examine certain familiar loci from the point of view of reality, and to obtain some classification of the various cases which arise from this point of view. The loci principally involved are the intersections of a pair of quadrics in space of four or five dimensions, and the projections of these loci, an indication of an extension to the general quadric pencil being given at the end; it will be seen, however, that the general theory presents complications which render simple results difficult to obtain. The loci under discussion are in themselves well known, but no general investigation which takes into account the distinction between real and imaginary points appears to have been made hitherto. In the course of the work the division of the cubic surfaces in space into five species is obtained by a method which seems simpler than those previously adopted.

The general result which is obtained is that, in the general quadric pencil in n dimensions, we can distinguish, from our point of view, a certain number of principal types, equal, if n is odd, to $2^{\frac{1}{2}(n+1)}$ and, if n is even, to $2^{\frac{1}{2}n+1}-1$.

It seems worth while to make some general remarks on the nature of the problem involved. The ordinary projective geometry is essentially based on an algebra which consists of symbols satisfying all the ordinary rules of arithmetic, including the law of commutability of multiplication, and which is complete, in the sense that it is assumed that the fundamental theorem of algebra applies to the symbols, any equation whose coefficients belong to the system having a root which also belongs to the system. In manipulations with the symbols, they are usually

tacitly identified with the complex numbers of ordinary arithmetic, but theoretically this identification is unnecessary. It will be convenient in what follows, however, to assume that this identification has been made.

We inquire what corresponds geometrically to the separation from the field of complex numbers of the field of real numbers. In the field of real numbers all the properties of the original algebra hold, save that the fundamental theorem is untrue; an equation with real coefficients need not have any real roots. The characteristic property of the field of real numbers which distinguishes it from that of complex numbers is the fact that it can be *ordered*, *i.e.* its elements bear a certain relation to one another, so that of two distinct elements a, b, we can say that a precedes b or that b precedes a, one of these alternatives being necessarily true; and this relation is transitive, so that if a precedes b and b precedes c we can infer that a precedes c. This idea, it should be noted, is entirely one of order, and does not introduce any ideas of measure. Thus we have not, in the general case, where the numbers involved are not in general the numbers of arithmetic, any means of determining the amount by which one symbol precedes the other; the fact that in the case of ordinary real numbers we can do so is due to certain additional conventions made about these numbers which have from our point of view no geometrical significance.

The separation of the real numbers from the complex numbers corresponds geometrically to the separation of the totality of points into two parts, one part consisting of points which we call real, and the other consisting of imaginary points. It is clear that the field of real points has the following properties.

(*a*) The line which joins two real points contains an infinite number of other real points, and the points of this line bear to one another a certain relation, namely that of order; alternatively we may say that the idea of separation attaches to points on the line, so that if A, B, C, D are four points of the line, it is significant to say that the points A, B separate C, D. This separation relation satisfies the usual relations, which need not be written down here.

(*b*) Similarly three, four, ... real points determine a plane, solid, ..., containing an infinity of real points. Such planes, etc., may be called real. If the points of reference for a coordinate system are real, a system can be chosen so that a real point has real coordinates and conversely; in that case a real line, plane, etc., are defined by real equations in the coordinates. The intersection of two real spaces is necessarily real.

Another conception of importance is that of conjugate imaginary points. We associate with every imaginary point a second point, also imaginary, such that the line joining the pair is real. These two points are called *conjugate*. A point and its conjugate will coincide if, and only if, the point is real. This association is derived from the algebra, and is to some extent arbitrary (depending on the coordinate system chosen), but it is to be noted that in all coordinate systems for which the real part of the space is invariant an imaginary point lies on one real line and one only; the position of its conjugate on this line is the element of arbitrariness in the scheme.* Any locus not real (where a real locus is understood to be one which contains imaginary points in conjugate pairs, being thus represented by real equations) is associated with another similar locus, the locus of conjugates of its points. In particular, each imaginary line, plane, ... has a conjugate line, plane, ... These need not intersect; but if they do meet the locus of their common points must be real, since any common point P is either itself real or is such that its conjugate P' also lies on both loci. Thus two conjugate lines in general will not meet; but if they do intersect the point of intersection is real, and so is the plane joining them, since it contains an infinite number of real lines joining pairs of conjugate points, one on each of the lines.

The above is not intended to be a complete logical exposition of the matter, but serves to illustrate the nature of the special assumptions involved in the consideration of reality. We now pass to more particular cases.

We shall call a locus *real* if it is represented by real equations so that it contains conjugate imaginary points in pairs. If the locus contains no real points (or only certain singular points) it will be called *invisible*. The distinction between visible and invisible is quite separate from that between real and imaginary.

1. *Quadrics.*

We now consider the classification of quadrics, in space of any number of dimensions, from our point of view. Suppose that we have a quadric Q given by a real equation. We suppose first that Q is not special. A real point P, not lying on the quadric, has a polar prime which is determinable rationally in terms of the coefficients in the equation of the quadric, and which is thus real. It is thus clear that we can get a self-polar simplex of Q whose elements are all real. By a real change of

* Compare the definition of imaginary points by means of involutions on a line.

coordinates the simplex can be taken to be the new simplex of reference, and in terms of this system the equation of the quadric assumes the form

$$a_0x_0^2+a_1x_1^2+\ldots+a_nx_n^2=0,$$

where the a_i are real numbers, different from zero, since Q is not specialised. It is known that the number of positive and negative coefficients in this expression is independent of the particular real self-polar simplex chosen. The following cases are thus to be distinguished.

If all the coefficients have the same sign the quadric cannot contain any real points, and is thus, by our definition, *invisible*.

If one of the coefficients differs in sign from the rest, the quadric contains real points in infinite number, but contains no real lines. Such a quadric may be called *convex*; an example is the ellipsoid or the two-sheeted hyperboloid. Of the $n+1$ vertices of the self-polar simplex, one behaves differently from the rest; for the lines which join it to the other vertices meet the quadric in real points, while the other edges of the simplex contain no real points of the quadric. The quadric thus divides space into two regions, one of which contains one vertex of any real self-polar simplex, and the other of which contains the remaining n vertices. The former region may be called the *interior* of the quadric, the latter the *exterior*.

If two of the coefficients differ in sign from the rest, the quadric contains real lines but no linear spaces of higher dimension that are real. By reasoning as in the previous case, it may be seen that the quadric divides space into two regions, one of which contains two vertices of every real self-polar simplex and the other of which contains the remaining vertices.

Similarly, if r of the coefficients differ in sign from the remaining $n-r+1$, the quadric will contain real linear spaces of dimension not greater than $r-1$, and will divide space into two regions containing respectively r and $n+1-r$ vertices of any self-polar simplex. The greatest value of r which arises is $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$ according as n is even or odd. In the case of n odd the last case is unique in that the two parts of space in which the two sets of vertices of a simplex lie are indistinguishable from one another.

This holds for general quadrics. If Q is specialised h times the equation, when finally reduced, has only $n+1-h$ terms, and its properties are essentially those of a quadric in $[n-h]$. It can be invisible, *i.e.* have no real points outside its double $[h-1]$, or convex, ..., just as its general section by $[n-h]$.

We are now to consider the nature of general pencils of quadrics in four and five dimensions. We shall require certain results for the three-dimensional case. These were obtained by methods of pure geometry by Du Val*.

These results are as follows. A general pencil of quadrics in [3], whose base is a non-singular twisted quartic curve, can be of one of four types (the pencil being assumed to be real), which are distinguished by the nature of the four cones which contain the curve. The first type contains four real cones, of which two, however, are invisible; the curve itself is thus invisible. The second type contains four visible cones. The curve consists of two circuits, each of which is even, *i.e.* contains an even number of points of a plane section. Of the ruled quadrics which contain the curve some are transecant, *i.e.* the intersections of a generator with the curve lie on different circuits, while some are cosecant, the intersections with a generator lying on the same circuit. Two of the cones are transecant and two cosecant. If the values of the parameter which determine the position of the quadric in the pencil are represented by the points of a line, the pencil can be represented by the following scheme, where A, B, C, D represent the vertices of the cones, and the letters show the manner in which these points are separated by the quadrics of the pencil. The cones mark the change of type of the quadric.

A		B		C		D	
Convex Quadrics	Cosecant Cone	Cosecant Quadrics	Cosecant Cone	Convex Quadrics	Transecant Cone	Transecant Quadrics	Transecant Cone	Convex Quadrics
$ABD:C$	$BD:C$	$BD:AC$	$D:AC$	$D:ACB$	$D:AB$	$DC:AB$	$C:AB$	$ABD:C$

The third type of curve lies on no real cones, and every real quadric containing it is ruled; the curve again consists of two circuits, but in this case they are odd, *i.e.* each contains an odd number of points of a plane section. The last type contains two real and two imaginary cones, the curve consists of a single circuit.

Schemes such as that given for the second type above will be of great importance in what follows. The one just given is written in full because it is required later. We notice that, as the real parameter which determines the quadric in the pencil varies continuously, the quadric can only change from one form to another by passing through the limiting position

* Du Val, *Proc. Camb. Phil. Soc.*, 24 (1928), 379–399.

between the cases, which is a cone. The real cones in the pencil thus mark the change of type of quadric, and the vertex of the cone changes from an exterior to an interior point, or *vice versa*.

2. *Pencils of quadrics in* [4].

We now consider the general pencil of quadrics in [4]. The base of this pencil is a quartic surface containing sixteen lines first investigated by Segre*. The surface lies on five cones, and the lines fall into intersecting pairs such that a pair lie in a generating plane of one of these cones. We shall use the following algorithm for the lines and the way in which they lie on the cones.

A		*B*		*C*		*D*		*E*	
0, 1	2, 12	0, 2	1, 12	0, 3	1, 13	0, 4	1, 14	0, 5	1, 15
23, 45	3, 13	13, 45	3, 23	12, 45	2, 23	12, 35	2, 24	12, 34	2, 25
24, 35	4, 14	14, 35	4, 24	14, 25	4, 34	13, 25	3, 34	13, 24	3, 35
25, 34	5, 15	15, 34	5, 25	15, 24	5, 35	15, 23	5, 45	14, 23	4, 45

Here 0, 1, ..., 5, 12, ..., 45 represent the sixteen lines, and A, ..., E the vertices of the five cones; and the table shows the distribution of the lines on each cone into eight intersecting pairs lying in generating planes of the cone, four planes of each system.

We now proceed to examine the cases which can possibly arise. We note that there are three kinds of quadric in [4], namely the invisible, convex, and ruled quadrics; we shall denote these by I, C, R respectively. Similarly there are three kinds of cone which may be similarly designated. It is clear that an I cone marks the division between two quadrics one of which is I and the other C; a C cone marks the division between quadrics C and R; while an R cone divides two R quadrics, the vertex of the cone simply passing from one of the divisions of space into the other. Of the five cones in the pencil an even number are imaginary, since the discriminant of the pencil has real coefficients. If we represent the real quadrics of the pencil by the points of a line, as explained above, then, since each invisible quadric corresponds to a point on the line lying in a segment terminated by points representing invisible cones, the number of such cones is even; similarly the number of convex cones is even; there is thus at least one ruled cone in the pencil. It follows that there can

* Segre, *Math. Ann.*, 24 (1884), 313.

only be pencils of the following seven types :

I. Two invisible, two convex, one ruled cone.

II. Four convex, one ruled cone.

III. Two convex, three ruled cones.

IV. Five ruled cones.

V. Two convex, one ruled, and two imaginary cones.

VI. Three ruled and two imaginary cones.

VII. One ruled and four imaginary cones.

The imaginary cones are, of course, conjugate in pairs.

The schemes for each of these types are given below, together with a typical equation of a pencil of each type; this shows the actual existence of the pencil. It is easy to verify that the separation between the vertices is the only one possible.

Case I.

A		*B*		*C*		*D*		*E*	
I	*I*	*C*	*C*	*R*	*R*	*R*	*C*	*C*	*I*	*I*
Quadrics	Cone	Quadrics	Cone	Quadrics	Cone	Quadrics	Cone	Quadrics	Cone	Quadrics
ABCDE	*BCDE*	*A* : *BCDE*	*A* : *CDE*	*AB* : *CDE*	*AB* : *DE*	*ABC* : *DE*	*ABC* : *E*	*ABCD* : *E*	*ABCD*	*ABCDE*

$$x_0^2+x_1^2+x_2^2+x_3^2+x_4^2=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2=0.$$

This surface must be invisible, since it contains no real points (lying on invisible quadrics).

Case II.

A		*B*		*C*		*D*		*E*	
R	*C*	*C*	*C*	*R*	*R*	*R*	*C*	*C*	*C*	*R*
Quadrics	Cone	Quadrics	Cone	Quadrics	Cone	Quadrics	Cone	Quadrics	Cone	Quadrics
AE : *BCD*	*E* : *BCD*	*E* : *ABCD*	*E* : *ACD*	*EB* : *ACD*	*EB* : *AD*	*EBC* : *AD*	*EBC* : *A*	*BCDE* : *A*	*BCD* : *A*	*BCD* : *AE*

$$x_0^2+x_1^2+x_2^2+x_3^2-x_4^2=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_3^2-a_4x_4^2=0,$$

where $$a_0<a_4<a_1<a_2<a_3.$$

The two convex cones A, E each contain the vertex of the other as an interior point. The quartic surface must therefore consist of two

distinct pieces. For through A, E can be drawn primes a, e, each of which meets the corresponding cone in no real points save the vertex A, E. These primes contain no real point of the surface. Any generator of the cone A meets the cone E in two real points lying on the surface, and these are separated by the primes a, e. It is thus impossible to draw a continuous real curve on the surface which joins these two points. The surface thus consists of two pieces.

Case III.

	A		B		C		D		E	
C Quadrics	C Cone	R Quadrics	R Cone	R Quadrics	R Cone	R Quadrics	R Cone	R Quadrics	C Cone	C Quadrics
ABDE:C	*BDE:C*	*AC:BDE*	*AC:DE*	*ABC:DE*	*AB:DE*	*AB:CDE*	*AB:CE*	*ABD:CE*	*ABD:C*	*ABDE:C*

$$x_0^2+x_1^2+x_2^2+x_3^2-x_4^2=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_3^2-a_4x_4^2=0,$$

where $$a_0<a_1<a_4<a_2<a_3.$$

Case IV.

	A		B		C		D		E	
R Quadrics	R Cone	R Quadrics	R Cone	R Quadrics	R Cone	R Quadrics	R Cone	R Quadrics	R Cone	R Quadrics
ACE:BD	*CE:BD*	*CE:ABD*	*CE:AD*	*BCE:AD*	*BE:AD*	*BE:ADC*	*BE:AC*	*BED:AC*	*BD:AC*	*BD:ACE*

$$x_0^2+x_1^2+x_2^2-x_3^2-x_4^2=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2-a_3x_3^2-a_4x_4^2=0,$$

where $$a_0<a_4<a_1<a_3<a_2.$$

Case V.

	A		B		C	
Convex Quadrics	Convex Cone	Ruled Quadrics	Ruled Cone	Ruled Quadrics	Convex Cone	Convex Quadrics

$$x_0^2+x_1^2+x_2^2-2x_3x_4=0, \quad -x_1^2+x_2^2+2x_3^2-x_4^2=0.$$

Case VI.

	A		B		C	
R Quadrics	R Cone	R Quadrics	R Cone	R Quadrics	R Cone	R Quadrics

$$-x_0^2+x_1^2+x_2^2-2x_3x_4=0, \quad -x_1^2+x_2^2+2x_3^2-x_4^2=0.$$

Case VII.

A

R Quadrics *R* Cone *R* Quadrics

$$x_0^2+2ax_1x_2+2bx_3x_4=0, \quad x_1^2-x_2^2+x_3^2-x_4^2=0.$$

3. *The reality of the lines.*

We shall now discuss in each of the above cases the number of lines lying on the surface which are real, and the number of pairs of conjugate imaginary lines which intersect. Since the plane which joins a line of the surface to the vertex of one of the cones containing the surface is a generating plane of the cone, it follows that a real line can only exist on the surface if every real cone containing the surface is ruled, *i.e.* if the pencil is of one of the three types IV, VI, VII.

Consider now the generating planes of one system of one of the ruled cones lying in the pencil. Each of these planes will meet a second quadric of the pencil in a conic, and this conic will lie on the surface. Four of the planes will touch the second quadric, meeting it in a line pair; this gives rise to the eight lines on the surface which lie in the planes of this system of the cone in question. The point of intersection of the lines of such a pair lies in the polar prime of the vertex of the cone with regard to the second quadric, *i.e.* in the prime containing the other four vertices of cones of the pencil. This prime we shall call a *principal* prime. The principal prime meets the first cone in a ruled quadric and the surface in a quartic curve lying on this quadric, four generators of each system of the quadric touch this curve, and the points of contact are the intersections of the line pairs of the system. Now the section of a pencil of quadrics by one of the principal primes is a pencil of quadric surfaces in which the four cones are the sections by the prime of the four cones of the original pencil whose vertices lie in the prime. It follows that the section of the pencil by the principal prime corresponding to a vertex A of one of the cones is a pencil of quadric surfaces for which the scheme is obtained by suppressing the letter A in the scheme of the original pencil wherever it occurs.

If the point of contact of one of the generating planes of the ruled cone with the quartic surface is real there are two lines of the surface lying in this plane and intersecting in this point. These lines are thus real or conjugate imaginary. If there is a second ruled cone in the pencil they must be real, for the plane joining one of them to the vertex of the second cone passes through a real generating line of the cone and is thus

real; if the line is imaginary its conjugate must lie in this plane, which is impossible, since it lies in a plane of the first cone through the line.

Consequently we can determine the number of pairs of conjugate intersecting lines, and the number of real lines, without difficulty. We consider the six visible cases.

Case II.

There is only one ruled cone in the pencil, C say. The section of the pencil by the prime corresponding to C, *i.e.* by the prime $ABDE$, is a pencil of quadrics of which the scheme is derived from that given above by suppressing the letter C throughout. By comparison with a result previously quoted, it is easy to see that the resulting quartic curve is of the second type, where the cones A and E are met by the prime in transecant cones and B and D in cosecant cones. The section of the cone C is an ordinary quadric represented by a point on the line in the segment BD, and hence is cosecant. The quartic curve is bipartite in two even branches and the generators of the quadric in which the prime meets C meet each circuit in an even number of points, and have their intersections with the curve on the same circuit. Consequently the four contact generators of each system are real, two of the points of contact lying on each circuit. Hence the points of intersection of the eight line pairs which lie in generating planes of the cone C are all real. Since the quartic lies on no other ruled cones it contains no real lines, so that the sixteen lines fall into eight conjugate intersecting pairs.

Case III.

Here there are three ruled cones B, C, D, of which B and D have similar properties. The section of the cone E by the principal prime corresponding to C is a cone which does not separate the vertices of A, B, D, and which is thus invisible. Hence the principal prime corresponding to C meets the surface in no real points, and there are no real generating planes of the cone C which touch the surface. The prime $ACDE$, which is the prime corresponding to B, meets the surface in a bipartite even curve, of which the generators of the section of the cone B are transecant (as follows from the scheme given above), so that none of the contact generators are real. There are thus no real lines or conjugate intersecting pairs.

Case IV.

Here all the cones are ruled and behave in the same manner, so that we need only consider one of them. The section by the prime

$ABDE$ corresponding to C is a bipartite even curve, of which the section of C is a cosecant quadric. Hence, as in II, the eight contact generators are real. In this case the sixteen lines are all real, since the curve lies on a ruled cone besides C.

Case V.

There is only one ruled cone, namely B. The section of the system by the corresponding prime is a quartic lying on two imaginary cones, *i.e.* a twisted quartic of the fourth type, consisting of a single circuit. Of the four contact generators of each system two are real and two imaginary. There are thus four real generating planes of the cone B, two of each system, which touch the surface, but the lines in each of these planes form a conjugate pair, since the curve lies on no ruled cones save B. We have thus no real lines, but four pairs of conjugate imaginary intersecting lines. These form a *double four.*

Case VI.

This is similar to the last case, there being four real planes of the cone B which touch the surface, but since the other cones in the pencil which are real are ruled it follows that the lines in these planes are real. The surface contains therefore a real set of eight lines, forming a double four.

Case VII.

Here the only real cone A is ruled, and the section of the pencil by the corresponding principal prime is a pencil of quadric surfaces, including four imaginary cones. The resulting quartic curve is thus a bipartite odd curve (the third of the four types), one system of generators being transecant and the other cosecant. The former system cannot contain any real contact generators, the other has four, two on each circuit. There are thus four real contact planes, all of the same system, each of which meets the surface in a pair of real or conjugate imaginary lines. These pairs may be denoted by 2, 12; 3, 13; 4, 14; 5, 15 after the algorithm previously given. If three of these pairs, say the first three, consisted of real lines, the line 0 which meets 2, 3, 4 would be real; this is not the case. Thus at least two of the pairs must consist of imaginary lines. If the first three pairs are imaginary, the line 0 which meets 2, 3, 4 must be conjugate to the line 1 which meets 12, 13, 14; thus the surface contains a further pair 0, 1 of conjugate intersecting lines. This is impossible. It follows that just two of the pairs are real. This

surface thus contains two pairs of real lines and two pairs of intersecting conjugates, the whole lying in the planes of one system of the unique real cone containing the surface.

The conics on the surface.

Through each point on the surface pass ten conics, lying in the generating planes of the cones which pass through the point. If one of the cones through the surface is ruled the two conics through the point lying in the planes of this cone are real; if the cone is convex they are imaginary, but the two conics are conjugate and meet in a second real point, lying on the line joining the point to the vertex of the cone. Two such conics may be called *incident*, meaning that they have two common points.

We can construct the following table showing the reality of the lines and conic systems in the several cases.

Case.	Lines.		Conic systems.	
	Real.	Meeting conjugates.	Real.	Incident with conjugates.
II	—	16	2	8
III	—	—	6	4
IV	16	—	10	—
V	—	8	2	4
VI	8	—	6	—
VII	4	4	2	—

4. *The cubic surface.*

By projecting a quartic surface of the type described from a general point of itself on to a prime we obtain a general cubic surface without singularities. Conversely every real cubic surface can be obtained as the projection of a real quartic from a real point. For the general cubic surface contains twenty-seven lines and, if it is real, imaginaries among these occur in pairs. There is thus at least one real line. If this is taken to be $z = t = 0$ the surface has an equation of the form

$$zQ + tQ' = 0,$$

Q and Q' being quadratic in x, y, z, t; and this is the projection of the real quartic

$$ut - Q = 0,\ uz + Q' = 0$$

from the point U or (0, 0, 0, 0, 1) of itself.

Thus, by projecting the six visible types of quartic, we shall obtain all the possible forms of real cubic surface without singularities. We can easily determine the number of real lines and tritangent planes on such a cubic surface. For the lines arise in three ways:

(*a*) as projections of the sixteen lines on the surface;

(*b*) as projections of the ten conics on the surface which pass through the point of projection;

(*c*) as the projection of the neighbourhood of the point of projection

Real lines thus arise from each real line and conic system on the surface, and, in addition, the line arising from (*c*) is real if the projection is real.

The forty-five triple tangent planes arise

(*a*) from the forty planes containing pairs of intersecting lines on the surface, the third line in the plane arising from a conic through the point of projection meeting both lines of a pair;

(*b*) from the neighbourhood of the point of projection and the five pairs of incident conics through the point.

The plane will be real if the elements from which it arises are real, or if one be real and the other two conjugate.

Hence the following cases arise:

Case II.

Three real lines, arising from the point of projection and two real conics. Thirteen real planes, eight from the eight pairs of conjugate intersecting lines, one from the pair of real incident conics, and four from the pairs of conjugate conics which are incident. The three real lines lie in one of the planes, the other planes pass four through each line.

Case III.

Seven real lines and five real planes. The seven lines fall into a single line and three intersecting pairs which meet it; this gives three of the planes, the other two also pass through the line.

Case IV.

All the lines and planes real.

Case V.

Three real lines and seven real planes; the three lines lie in one plane, and the other six planes pass two through each line.

Case VI.

Fifteen real lines and fifteen real planes. The fifteen lines form a configuration which may be derived from a double four by taking a line not meeting any of the lines of the double four and three pairs of intersecting lines, lying in planes through this, each line meeting two pairs of intersecting lines of the double four. (The ninth line is not arbitrary.) This configuration arises from the twenty-seven lines by suppression of a double six. It is symmetrical, three lines lying in each of the fifteen planes and three planes passing through each of the lines.

Case VII.

Seven real lines and five planes. The surface is of the same type as that obtained from a quartic of type III, the difference here being that the isolated line of the seven no longer arises as the projection of the neighbourhood of the projecting point, but comes from one of the two real conics lying on the surface which lies on the cone containing the lines and meets all the lines.

Thus we arrive by geometrical methods at the five types of cubic surface given by Schläfli*.

Cubic surface type.	Real lines.	Real planes.	Projection of
I	27	45	IV
II	15	15	VI
III	7	5	III, VII
IV	3	13	II
V	3	7	V

The Schläfli classification is sometimes obtained by considering the representation of the cubic surface on the plane by means of cubic curves through six points, and considering the various cases which may arise by letting these become conjugate imaginary in pairs, so that the system of

* Schläfli, *Phil. Trans. Royal Soc.*, 153 (1863), 193.

cubic curves remains real. This method implies the assumption that a real rational surface can be mapped on a plane so that its real points correspond to the real points of the plane. When the surface consists of two or more separate pieces this cannot be true, since the plane is a connected region and cannot be put into correspondence with an unconnected set of points. But even when the rational surface is itself connected, there is no obvious ground for the assumption that it can be mapped on a plane so that real points correspond to real points. This method of approaching the problem is thus unsatisfactory*. We may note that, since the quartic surface of type II consists of two distinct sheets, the same is true of the cubic of type IV, which is its projection.

An entirely different method of obtaining the classification is given by Klein†, who obtains all the cases by continuous deformation from the four-nodal cubic surface. He uses his method to obtain various results, and as a further example of the utility of our method we give a new demonstration of one of them.

Klein's problem relates to the case in which all the lines are real, and may be stated as follows. If a plane meets the surface in a *bipartite* cubic curve, to find the number of lines of the surface which meet the oval of this curve. Now a plane section of the cubic surface is the projection of a prime section of the quartic surface which passes through the point of projection. If the cubic is bipartite, the same is true of this prime section, which is, of course, an elliptic quartic. We have thus to consider the primes through an arbitrary point of the surface which meet the surface in bipartite curves. Two cases will arise, according as the curve is even or odd. If the curve is even, the oval in the projected curve is the projection of that branch of the quartic which does not contain the vertex of projection; if, on the other hand, the curve is odd, the branch of it which passes through the projecting point is that which gives the oval of the cubic.

Suppose, first, that we have a prime through the point of projection which meets the quartic surface in a bipartite even curve. Any ruled quadric which contains this curve is either transecant, the two points in which a generator meets the curve lying on different circuits, or cosecant, in which case they lie on the same circuit or are imaginary. In

* The plane representation is obtained essentially by drawing transversals from points on the surface to a pair of skew lines; and these transversals are real only if the two lines are real or conjugate imaginaries. We have seen that the bipartite cubic surface contains no such pair of lines.

† Klein, *Math. Annalen* 6 (1873), 551; *Ges. math. Abhandlungen*, 2, 10.

particular, the sections of the cones A, B, C, D, E are ruled quadrics, belonging to one or other of these categories. If one of these five quadrics is cosecant, the conic lying in one of the planes of the corresponding cone meets the quartic curve in two imaginary points, or else in two points lying on the same circuit; in particular, the two lines forming a degenerate conic of the system which lie on the surface must both meet the same branch of the curve. On the other hand, if the quadric is transecant, any conic meets the quartic in two points lying on different branches, and the two intersecting lines lying in the planes of such a cone meet different branches of the curve.

We therefore consider the cases in which the quadrics are all cosecant, or in which one or more are transecant. If all the quadrics are cosecant, the sixteen lines must clearly meet the same branch of the quartic curve. Suppose now that four of the quadrics—sections, for example, of the cones A, B, C, D—were cosecant, while the fifth was transecant. Then, using the algorithm previously given, the branch of the curve which meets the line 0 must also meet 1, 2, 3, 4, and 12, 13, ..., 34; but the lines 12, 34 form a pair lying in a plane of the fifth cone, and lie on different branches of the quartic if this is transecant. It follows that this case cannot arise. If the sections of A, B, C are cosecant and the other two quadrics transecant we get eight of the lines meeting each branch; the two sets of eight being

$$0, 1, 2, 3, 12, 13, 23, 45$$
$$4, 5, 14, 24, 34, 15, 25, 35,$$

this being a consistent arrangement. (Each set of eight lines forms a double four.) If two of the quadrics are cosecant and the other three transecant it is easy to see that we get a contradiction. If we have one cosecant and four transecant quadrics we have an arrangement in which eight lines meet each branch; if the cosecant quadric is the section of the cone A the arrangement is

$$0, 1, 23, 24, 25, 34, 35, 45$$
$$2, 3, 4, 5, 12, 13, 14, 15,$$

each set of eight lines consists of four intersecting pairs. If, finally, all the cones are met by the prime in transecant quadrics we get an impossibility.

Thus, if the prime meets the quartic surface in a bipartite even curve, there are three cases possible:

(*a*) The five cones are met in cosecant quadrics, and all the lines meet one branch of the curve.

(*b*) Three of the quadrics are cosecant; eight lines meet each branch.

(*c*) One of the quadrics is cosecant; eight lines meet each branch.

Now the oval of the cubic which is the projection of an even bipartite quartic is the projection of the branch of the quartic on which the point of projection does not lie. The line arising from the point of projection will not meet this branch; the line arising from a conic through the point of projection meets the oval or not, according as the cone on which the conic lies is met by the prime in a transecant or cosecant quadric; thus of the ten lines so arising, the number that meet the oval of the cubic curve is none in Case (*a*), four in Case (*b*), and eight in Case (*c*). Thus the number of lines of the cubic surface which meet the oval is none or sixteen in Case (*a*), according to the position of the projecting point, twelve in Case (*b*), sixteen in Case (*c*).

Now consider the second case, in which the quartic curve is odd. In that case, of the two systems of generators of a containing quadric, one is transecant and the other cosecant. In particular, the section of the cone A has this property. Of the eight line pairs on the cone A four have the intersections of the two lines of a pair on the same circuit, and four on different ones. It is not difficult to show, by use of the algorithm, that of the eight lines lying in cosecant planes of the cone six meet one branch and two the other; no other alternatives giving a consistent result. Thus of the sixteen lines, ten meet one branch and six the other, a typical arrangement being

$$1,\ 2,\ 3,\ 4,\ 12,\ 13,\ 14,\ 23,\ 24,\ 34$$

$$0,\ 5,\ 15,\ 25,\ 35,\ 45.$$

Of the ten conics through the point of projection five have their second point of intersection with the surface on each branch. Since the curve is odd, the oval of the cubic arises as the projection of the branch of the quartic which passes through the point of projection, and is met by the line on the cubic surface which arises from the neighbourhood of this point; thus, according to the position of the projecting point, the number of lines which meet the oval is twelve or sixteen.

In all cases, therefore, the oval meets none, twelve, or sixteen of the lines. This agrees with Klein's result.

A further application of the division of the quartic surface into species should be noted. The quartic surface is the prime section of the general intersection of two quadrics in five dimensions, and its points represent

the lines of a quadratic congruence in ordinary space, and conversely every real quadratic congruence can be so obtained. The sixteen lines on the surface correspond to sixteen flat pencils of lines belonging to the congruence; the vertices and planes of these form a configuration like that of the nodes and tropes of a Kummer surface, six points lying in each plane and six planes passing through each point. A real point will arise from a real line, and be associated with a real plane passing through the point; thus it is easy to see that, of the sixteen elements of the Kummer configuration, all, eight, four, or none are real. (This does not give an account of the possible forms of Kummer surfaces, since a Kummer surface which is real may not necessarily correspond to a real quadratic complex or congruence. See Hudson, *Kummer's quartic surface*, page 116.)

Note.—It is not a difficult matter to exhibit the equations of the various types of cubic surfaces in a form which places in evidence the nature of the lines lying thereon. To do this we recall the standard form of the equation of a cubic surface referred to a pair of Steiner trihedrals, *i.e.* a set of two pairs, each of three planes, which contain nine lines of the surface. It is known that the equation of the surface can be put in the form

$$uvw+xyz=0,$$

where u, v, w, x, y, z are six planes connected by two relations which may be taken to be

$$u+v+w+x+y+z=0,$$

$$au+bv+cw+dx+ey+fz=0.$$

Then the lines are as follows: nine of the form $u=x=0$, six of the form $u+x=v+y=w+z=0$, and twelve of the form

$$Au+Dx=Bv+Ey=Cw+Fz=0,$$

where A stands for $a+\lambda$, etc., and λ satisfies the quadratic

$$(a+\lambda)(b+\lambda)(c+\lambda)=(d+\lambda)(e+\lambda)(f+\lambda).$$

Now it is shown in the paper of Schläfli, referred to above, that for a real cubic surface we can take a trihedral pair so that the products uvw and xyz represent real cubic forms. This means that all or one of the symbols occurring on each side of the equation of the surface are real. We can then specify the lines which exist as follows.

If $u, \ldots, z$ are all real, then we evidently have the fifteen lines $u=x=0$, $u+x=v+y=0$, etc., real, and the remaining twelve lines are real or imaginary according as the values of λ determined from the quadratic equation are real or not. Thus the equation

$$uvw+xyz=0,$$

where the above syzygies hold, represents a cubic surface with twenty-seven or fifteen real lines according as

$$-4(abc-def)(a+b+c-d-e-f)+(ab+bc+ca-de-ef-fd)^2$$

is greater or less than zero.

If u, x, y, z are real, and v and w conjugate imaginary, then, putting

$$v=\tfrac{1}{2}(\tau+i\omega), \quad w=\tfrac{1}{2}(\tau-i\omega),$$

the equation of the surface takes the real form

$$u(\tau^2+\omega^2)+4xyz=0,$$

where

$$u+\tau+x+y+z=0, \quad au+\beta\tau+\gamma\omega+dx+ey+fz=0$$

$$[\beta=\tfrac{1}{2}(b+c), \quad \gamma=\tfrac{1}{2}i(b-c)],$$

in which case it is clear that the three lines

$$u=x=0, \quad u=y=0, \quad u=z=0$$

are real, and that, whatever the nature of the roots of the quadratic, no other lines are real. The two cases correspond to the two cases in which three of the lines are real, as may easily be verified.

If, finally, y, z are likewise imaginary, we put $y=\tfrac{1}{2}(\eta+i\zeta)$, $z=\tfrac{1}{2}(\eta-i\zeta)$. The surface then contains the lines

$$u=x=0, \quad \tau+\eta=0, \quad \xi=\pm\omega.$$

the equation now being

$$u(\tau^2+\omega^2)+x(\eta^2+\zeta^2)=0,$$

where

$$u+\tau+x+\eta=0, \quad au+\beta\tau+\gamma\omega+dx+\theta\eta+\phi\xi=0$$

$$[\theta=\tfrac{1}{2}(e+f), \quad \phi=\tfrac{1}{2}i(e-f)].$$

The other lines are given by

$$Au+Dx = Bv+Ey = Cw+Fz = 0, \quad \text{etc.},$$

and if the roots of the quadratic in λ are real, then the line

$$Au+Dx = 0, \quad Bv+Ey = 0, \quad Cw+Fz = 0,$$

which is the same as

$$Au+Dx = 0, \quad (\lambda+\beta-i\gamma)(\tau+i\omega)+(\lambda+\theta-i\phi)(\eta+i\xi) = 0,$$

and the conjugate equation, reduces to

$$Au+Dx = 0, \quad \lambda(\eta+\tau)+\beta\tau+\gamma\omega+\theta\eta+\phi\xi = 0,$$

$$\lambda(\xi+\omega)+(\beta\omega+\theta\xi)-(\gamma\tau+\phi\eta) = 0,$$

and is thus real; similarly there are three other real lines, making seven in all. This gives the fifth type, the case of imaginary roots repeating one previously found.

(The three equations reduce to two by virtue of the quadratic.)

5. *Pencils of quadrics in* [5].

We shall now consider in the same way the possible forms of real pencils of quadrics in [5], supposing that the pencil is general, and hence contains six cones whose vertices are distinct. In [5] there are four kinds of real quadric, namely invisible, convex, containing lines, and containing planes. We shall denote these by the letters I, C, L, P. There are three kinds of cone, corresponding to the three kinds of quadric in [4]; these will be denoted by I, C, R, as before. In the pencil it is clear that

an I cone separates an I quadric from a C quadric,

a C cone separates a C quadric from an L quadric,

an R cone separates an L quadric from a P quadric,

and in any case the number of imaginary cones in a real pencil is even; it follows also that the number of I and C cones is also even. The section of the pencil by a real prime is a real pencil of quadrics in [4], which we have seen to contain at least one R cone. It follows that the pencil necessarily contains P quadrics, these being the only quadrics of which an R cone in [4] can be a prime section.

We have therefore the following eight types of pencil, which are given with their schemes and typical equations, as in the four-dimensional case.

	Case I.		*Case* II.		*Case* III.		*Case* IV (*a*).		*Case* IV (*b*).
	I Quadrics $ABCDEF$:		L Quadrics $AF:BCDE$		L Quadrics $CF:ABDE$		L Quadrics $ACDF:BE$		L Quadrics $ABEF:CD$
$A-$	I Cone $BCDEF$:	$A-$	C Cone $F:BCDE$	$A-$	R Cone $CF:BDE$	$A-$	R Cone $CDF:BE$	$A-$	R Cone $BEF:CD$
	C Quadrics $A:BCDEF$		C Quadrics $F:ABCDE$		P Quadrics $ACF:BDE$		P Quadrics $CDF:ABE$		P Quadrics $BEF:ACD$
$B-$	C Cone $A:CDEF$	$B-$	C Cone $F:ACDE$	$B-$	R Cone $ACF:DE$	$B-$	R Cone $CDF:AE$	$B-$	R Cone $EF:ACD$
	L Quadrics $AB:CDEF$		L Quadrics $BF:ACDE$		L Quadrics $ABCF:DE$		L Quadrics $BCDF:AE$		L Quadrics $EF:ABCD$
$C-$	R Cone $AB:DEF$	$C-$	R Cone $BF:ADE$	$C-$	R Cone $ABF:DE$	$C-$	R Cone $BDF:AE$	$C-$	R Cone $EF:ABD$
	P Quadrics $ABC:DEF$		P Quadrics $BCF:ADE$		P Quadrics $ABF:CDE$		P Quadrics $BDF:ACE$		P Quadrics $CEF:ABD$
$D-$	R Cone $ABC:EF$	$D-$	R Cone $BCF:AE$	$D-$	R Cone $ABF:CE$	$D-$	R Cone $BF:ACE$	$D-$	R Cone $CEF:AB$
	L Quadrics $ABCD:EF$		L Quadrics $BCDF:AE$		L Quadrics $ABDF:CE$		L Quadrics $BF:ACDE$		L Quadrics $CDEF:AB$
$E-$	C Cone $ABCD:F$	$E-$	C Cone $BCDF:A$	$E-$	C Cone $ABDF:C$	$E-$	R Cone $BF:ACD$	$E-$	R Cone $CDF:AB$
	C Quadrics $ABCDE:F$		C Quadrics $BCDEF:A$		C Quadrics $ABDEF:C$		P Quadrics $BEF:ACD$		P Quadrics $CDF:ABE$
$F-$	I Cone $ABCDE$:	$F-$	C Cone $BCDE:A$	$F-$	C Cone $ABDE:C$	$F-$	R Cone $BE:ACD$	$F-$	R Cone $CD:ABE$
	I Quadrics $ABCDEF$:		L Quadrics $BCDE:AF$		L Quadrics $ABDE:CF$		L Quadrics $BE:ACDF$		L Quadrics $CD:ABEF$

Typical equations for these cases are :

I.

$$x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2+a_5x_5^2=0.$$

II.

$$x_0^2+x_1^2+x_2^2+x_3^2+x_4^2-x_5^2=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_3^2+a_4x_4^2-a_5x_5^2=0,$$

where $$a_0>a_5>a_1>a_2>a_3>a_4.$$

III. The same equations where

$$a_0>a_1>a_5>a_2>a_3>a_4.$$

IV (*a*).

$$x_0^2+x_1^2+x_2^2-x_3^2-x_4^2-x_5^2=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2-a_3x_3^2-a_4x_4^2-a_5x_5^2=0,$$

where $$a_0>a_3>a_1>a_4>a_2>a_5.$$

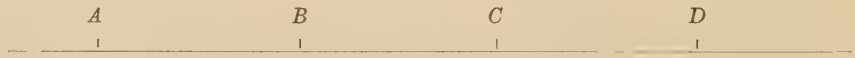
IV (*b*). The same equations where

$$a_0>a_3>a_4>a_1>a_2>a_5.$$

Case V.

$$x_0^2+x_1^2+x_2^2+x_3^2+2x_4x_5=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2+a_3x_3^2+x_4^2-x_5^2=0.$$

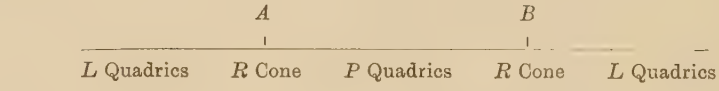
Case VI



$$x_0^2+x_1^2+x_2^2-x_3^2+2x_4x_5=0, \quad a_0x_0^2+a_1x_1^2+a_2x_2^2-a_3x_3^2+x_4^2-x_5^2=0,$$

where $$a_0>a_3>a_1>a_2.$$

Case VII.



$$x_0^2+x_1^2+2x_2x_3+2x_4x_5=0, \quad a_0x_0^2+a_1x_1^2+x_2^2-x_3^2+x_4^2-2x_5^2=0.$$

Case VIII.

$$x_0x_1+x_2x_3+x_4x_5=0, \quad x_0^2-x_1^2+x_2^2-2x_3^2+x_4^2-3x_5^2=0.$$

The notation used above is that explained previously. In Case I the locus of intersection of quadrics of the pencil is invisible, since it lies on invisible quadrics. In Case II the locus is the intersection of two convex cones A, F, each of which contains the vertex of the other as an interior point, consequently, by an argument parallel to that in the four-dimensional case, the locus of intersection consists of two distinct sheets. The

cases IV (a), IV (b) differ only in the positions of the vertices of the cones in regard to the pencil. Of the six cones in the pencil two are conjugate imaginary in Cases V and VI, four are imaginary in Case VII, and all are imaginary in Case VIII.

Through each point of the locus of intersection of the quadrics of the pencil, which is a V_3^4, pass four lines lying on the locus. In order that these should be real, it is necessary that every quadric containing the locus should be either L or P; *i.e.* the pencil must be of one of the types IV, VI, VII, or VIII. The four lines through any point of the locus lie in a solid, the intersection of the tangent primes to the quadrics of the pencil at the point. The four lines are the intersection of two quadric cones lying in the solid, and are conjugate imaginary in pairs, or else real.

In Cases I to IV, if the two quadrics whose equations are written above be denoted respectively by S and S', the general quadric of the pencil is $\lambda S - S'$, and the values of λ which give the cones are $a_0, \ldots, a_5$. It is easy to verify that, with the inequalities given above, the four convex cones in Type II correspond to the values a_0, a_1, a_4, a_5 of λ, while in Type III the two convex cones correspond to the values a_0, a_4 of λ. This remark is important later.

Corresponding to the eight types of quadric pencil in [5], we have in ordinary space eight distinct kinds of real quadratic line complexes, since all the pencils in [5] contain at least one quadric with real planes, such as represents the identical relation between the usual line coordinates. These do not, however, correspond to the eight distinct types of real Kummer surfaces enumerated by Hudson (*op. cit. ante*), since it is possible, as is there remarked, for an imaginary quadratic complex to have a real singular surface.

There is a particular case of the quadric pencil which is more interesting than the general one, in that it has more applications. This is the case when the pencil of quadrics, instead of containing six point-cones, contains three line-cones. The corresponding complex of lines is the tetrahedral complex. Analytically, the discriminant of the pencil has three pairs of equal roots instead of six distinct ones, these roots also being roots of the first minors of the discriminant when this is written in the usual determinant form. Of the three line-cones all or one are real. A line-cone in [5] has the properties of an ordinary quadric in [3], and there are three species, which may be denoted by I, C, R as usual.

In enumerating the various types of pencil of this form two methods are open to us: we can proceed directly or else derive the types as limiting cases of the general types when the six cones coincide in pairs. If the former course is adopted we note that of the three line-cones all or one

must be real, and that they separate quadrics in the pencil as follows :

an I line-cone separates quadrics which are I and L or C and C,

a C line-cone separates quadrics which are C and P or L and L,

an R line-cone separates quadrics which are L and L or P and P.

In the second method, we consider the schemes of the general pencils and let consecutive cones approach coincidence. The nature of the line-cones is then determined by the original cones as follows :

(a) If the two point-cones were I, so is the line-cone.

(b) If one was I and one was C, the line-cone is I.

(c) If both the point-cones were C, the line-cone is I if the two point-cones were separated by L quadrics, and is C if they were separated by C quadrics.

(d) If the cones were C and R, the line-cone is C.

(e) If the two cones were R and were separated by the quadrics of Type L, the line-cone is R. If the cones were separated by P quadrics, the line-cone is R or C; to determine which, consider the two sets of L quadrics which are separated by the two cones. If the two vertices of the self-polar tetrad which were outside one of the quadrics were inside the other the line-cone is ruled; if only one vertex was inside the second quadric, the line-cone is C.

These results are easily verified directly.

We find eight types of pencil, as follows.

I.

I Quadrics	I Cone	L Quadrics	R Cone	L Quadrics	I Cone	I Quadrics

$$x_0^2+x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0, \quad a(x_0^2+x_1^2)+b(x_2^2+x_3^2)+c(x_4^2+x_5^2)=0.$$

II.

C Quadrics	I Cone	C Quadrics	C Cone	P Quadrics	C Cone	C Quadrics

$$x_0^2-x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0, \quad a(x_0^2-x_1^2)+b(x_2^2+x_3^2)+c(x_4^2+x_5^2)=0$$

$$(a>b>c).$$

III.

C Quadrics	C Cone	P Quadrics	R Cone	P Quadrics	C Cone	C Quadrics

$$x_0^2-x_1^2+x_2^2+x_3^2+x_4^2+x_5^2=0, \quad a(x_0^2-x_1^2)+b(x_2^2+x_3^2)+c(x_4^2+x_5^2)=0$$

$$(b>a>c).$$

IV.

L Quadrics	C Cone	L Quadrics	R Cone	L Quadrics	C Cone	L Quadrics

$$x_0^2-x_1^2+x_2^2-x_3^2+x_4^2+x_5^2=0, \quad a(x_0^2-x_1^2)+b(x_2^2-x_3^2)+c(x_4^2+x_5^2)=0$$

$$(a>b>c).$$

V.

P Quadrics	R Cone	P Quadrics	R Cone	P Quadrics	R Cone	P Quadrics

$$x_0^2-x_1^2+x_2^2-x_3^2+x_4^2-x_5^2=0, \quad a(x_0^2-x_1^2)+b(x_2^2-x_3^2)+c(x_4^2-x_5^2)=0.$$

VI.

L Quadrics	R Cone	L Quadrics	R Cone	L Quadrics	R Cone	L Quadrics

$$x_0^2+x_1^2+x_2^2+x_3^2-x_4^2-x_5^2=0, \quad a(x_0^2+x_1^2)+b(x_2^2+x_3^2)-c(x_4^2+x_5^2)=0$$

$$(a>c>b).$$

VII.

L Quadrics	R Cone	L Quadrics

$$x_0^2+x_1^2+2x_2x_3+2x_4x_5=0, \quad a(x_0^2+x_1^2)+x_2^2-x_3^2+x_4^2-x_5^2=0.$$

VIII.

P Quadrics	R Cone	P Quadrics

$$x_0^2-x_1^2+2x_2x_3+2x_4x_5=0, \quad a(x_0^2-x_1^2)+(x_2^2-x_3^2)+(x_4^2-x_5^2)=0.$$

To see how these arise from the general types, we note that only those general types in which all or two of the cones are real can be specialized so as to give three line-cones in the pencil, and that each of these types will admit of two ways of specialization according, in the first case, to the two ways of pairing the six vertices in three consecutive pairs (namely AB, CD, EF, and BC, DE, FA), and, in the second case, to the two segments of quadrics, one of which is to be suppressed. The following table shows the resulting cases.

Type of general quadric pencil.	Type of special pencil.	
I	I (*AB CD EF*)	II (*BC DE FA*)
II	IV (*AB CD EF*)	II (*BC DE FA*)
III	IV (*AB CD EF*)	III (*BC DE FA*)
IV (*a*)	IV (*AB CD EF*)	V (*BC DE FA*)
IV (*b*)	VI (*AB CD EF*)	V (*BC DE FA*)
VII	VII	VIII

The V_3^4, which is the base of a pencil of quadrics of this type, is distinguished projectively from the general V_3^4 by the following properties. Each line vertex of one of the containing cones meets the locus in the two points in which it meets any other quadric of the pencil. These points are nodes of V_3^4. If the three pairs of nodes so obtained be denoted by XX', YY', ZZ' the plane XYZ and the seven other planes obtained by replacing one or more of $X\,Y\,Z$ by the corresponding dashed letters all lie on the locus. These eight planes form a double four, thus :

$$\begin{array}{cccc} XYZ & XY'Z' & X'YZ' & X'Y'Z \\ X'Y'Z' & X'YZ & XY'Z & XYZ', \end{array}$$

in which two planes do not meet if they are in the same column of the double four, meet in a point if they lie in the same row, and meet in a line if they lie neither in the same row nor column. The four lines of the locus, which pass through each general point, now fall into four distinct systems, and are the transversals drawn from the point to the four pairs of skew planes which lie on the locus.

We shall now investigate the reality of the nodes, lines, and planes in the various cases which arise, *i.e.* in the six visible cases III to VIII, observing here that in Case II there are just two real isolated nodes of the locus and no other real points.

The reality is completely determined by that of the nodes. In the equations for the types which we have given the three line vertices, when these are real, are given by

$$x_2 = x_3 = x_4 = x_5 = 0,$$

$$x_0 = x_1 = x_4 = x_5 = 0,$$

$$x_0 = x_1 = x_2 = x_3 = 0,$$

and when only one is real it is the line

$$x_2 = x_3 = x_4 = x_5 = 0.$$

We shall suppose the points XX', YY', ZZ' to lie on the three lines in the order given, so that the pair XX' always lie on a real line. A plane is real if the three nodes lying in it are all real or one real and two conjugate imaginary. Since the three nodes in a plane lie on different line vertices, it follows that the second case will occur only if two of the lines are conjugate imaginary, for if an imaginary point lies on a real line its conjugate lies on the same line. A line system is real if the two skew planes which determine it are real or conjugate imaginary; the latter case can only happen if three line vertices are real and the pairs of nodes on each are conjugate imaginary.

Now consider the cases separately.

III. The points XX' are real, while YY', ZZ' are pairs of conjugate imaginary points. There are thus two real nodes $X X'$ and no real planes; further, since a pair of conjugate planes must contain the same real node, there are no real lines.

IV. The points XX', YY' are real, while ZZ' are conjugate. Thus there are four real nodes, no real planes, and no real lines.

V. All the nodes are real. Thus we have six real nodes, eight real planes, and four real systems of lines.

VI. There are no real nodes, the pairs XX', YY', ZZ' being conjugate. It follows that the planes are all imaginary, two conjugate planes containing between them all the nodes. Thus the lines of the locus, being the transversals of these pairs of planes, are real, and we have no real nodes or planes, but four real systems of lines.

VII. The points XX' are conjugate imaginary; the lines YY', ZZ' are not real, so that the conjugate of one of the points on one of these lines lies on the other. Thus the pairs YZ, $Y'Z'$ may be taken to consist of conjugate points. There are no real nodes or planes. The pairs of conjugate imaginary planes are $(XYZ,\ X'YZ)$, $(XY'Z',\ X'Y'Z')$, $(XYZ',\ X'ZY')$, $(XZY',\ X'YZ')$, of which the last two are skew, so that two of the four line systems are real.

VIII. As before, the pairs YZ, $Y'Z'$ consist of conjugate imaginary points, but now the points $X X'$ are real. There are thus four real planes,

namely XYZ, $X'YZ$, $XY'Z'$, $X'Y'Z'$, which form two skew pairs, and give rise to two real systems of lines. The other pairs of conjugate imaginary planes have a point X or X' in common; the other line systems are thus not real.

Summarizing we have

Type.	Nodes real.	Planes real.	Line systems real.
III	2	0	0
IV	4	0	0
V	6	8	4
VI	0	0	4
VII	0	0	2
VIII	2	4	2

6. *The Segre cubic primal.*

The projection of the V_3^4 just considered from a point of itself is a Segre cubic primal. This contains ten nodes, arising from the six nodes of V_3^4 and the four lines which pass through the point of projection. It contains fifteen planes. Eight of these are projections of the planes of V_3^4, one arises from the neighbourhood of the projecting point, and the other six arise from quadric surfaces through the projecting point which are the intersections of V_3^4 with the generating solids of the line-cones which pass through the point. There being three line-cones, each with two systems of generating solids, we get in this way six quadric surfaces. Through any chord of V_3^4 passes just one quadric of the pencil, and the two generating planes of the quadric which pass through this chord meet the V_3^4 in conics passing through the two points in which the chord meets V_3^4. Thus the V_3^4 contains two conics through each pair of points. The projected cubic primal in [4] thus contains six systems of lines, four arising from the lines of V_3^4 and two from the conics of V_3^4 passing through the point of projection.

Now if the cubic primal has a real equation, so that imaginary elements occur in conjugate pairs, it is clear that, of its fifteen planes, an odd number are real. Taking one of them to be $u=v=0$, the equation of the primal is capable of the form

$$uQ+vQ'=0$$

where Q and Q' are quadratic, and it follows, just as for the cubic surface in [3], that the primal can be obtained by projecting a real V_3^4 of [5]. Thus every cubic primal with ten nodes is so obtainable.

Now the plane of the cubic primal which arises from the neighbourhood of the projecting point is real; the planes arising from quadric surfaces through the point are real if these surfaces are real; this requires that the cones should have real generating solids; they are thus R cones, and two real quadric surfaces through the point of projection will arise for every R cone in the pencil. The two conic systems of V_3^4 are real if, and only if, the pencil contains P quadrics distinct from the cones.

The various cases are thus as follows.

III. Two real nodes, arising from the nodes on the V_3^4. Three real planes, two coming from real quadric surfaces, there being one ruled cone in the pencil, and one from the point of projection. Two real line systems, coming from the conic systems of V_3^4.

IV. Four real nodes, the projection of nodes of V_3^4. Three real planes derived in the same way as those in Case III. No real lines, since there are no proper P quadrics. The V_3^4 which this is derived from is a limiting case of the second general type, which is bipartite; the same is true of the cubic primal.

V. All the lines, planes, and nodes real.

VI. Four real nodes, from lines on V_3^4 through the projecting point. Seven real planes, one from the projecting point and two from each of the three ruled cones. Four real line systems, from those of V_3^4.

VII. Two real nodes, from lines of V_3^4. Three real planes, one from the point of projection and two from the ruled cone in the pencil. Two real systems of lines, from the lines on V_3^4.

VIII. Four real nodes, from the nodes and lines through the point of projection. Seven real planes, four from planes of V_3^4, two from the ruled cone, and one from the point of projection. Four systems of lines real; two arising from lines on V_3^4 and the other two from the conic systems.

Cases III and VII, VI and VIII, give rise to the same cubic primal; this is because the configuration of the real planes on such a primal is unsymmetrical, as is easy to verify. (Compare the similar occurrence

in the case of the cubic surface.) There are thus four types of real Segre cubic primals :

(*a*) Two nodes, three planes, two line systems real.

(*b*) Four nodes, three planes, no line systems real.

(*c*) Four nodes, seven planes, four line systems real.

(*d*) Ten nodes, fifteen planes, six line systems real.

The case (*b*) is the bipartite one. If we consider the representation of the primal on [3] by quadrics through five points, the three unipartite cases arise by letting one, three, or five points be real; but the bipartite case does not arise, and, indeed, cannot be represented on space in a real way.

7. *The Kummer surface.*

The apparent contour of the primal, viewed from a point of itself, is a Kummer surface. Conversely, any real Kummer surface which contains a real node and a real trope not incident with it can be obtained as the projection of a real Segre cubic primal. For, if T be the node and the plane $t=0$ the trope, touching the surface along the conic $t=\psi=0$, the equation of the surface is of the form

$$t^2\phi_2+2t\phi_3+\psi^2=0,$$

showing that it is the apparent contour of the primal

$$u^2t-2u\psi-(2\phi_3+t\phi_2)=0$$

viewed from the point U.

Thus, by projecting the cubic primals of the four types given above, we obtain the four types of real Kummer surface which contain a real trope; they contain respectively four, four, sixteen, and eight real tropes, and a similar number of real nodes; the second type consists of two sheets. These types are, in Hudson's classification*, described as III, IVa, Ia, IIa respectively; Hudson obtains four other types, one of which is invisible, another of which contains four nodes and no other real points, and the other two of which contain no real nodes or tropes.

8. Reverting to the general quadric pencil in [5], it is known that of the generating planes of each system of one of the quadrics of the

* *Op. cit. ante*, ch. 11.

pencil sixteen touch any other quadric (and hence V_3^4) along lines. These may be derived from one of them as follows. If a plane α touches the V_3^4 along a line 0, the plane joining 0 to the vertex of one of the cones of the pencil meets V_3^4 in another line, 1 say. There is a [3] joining the line 1 to the plane α, and this [3] meets V_3^4 in the lines 0 and 1, each counted twice. This [3] contains the plane α, which touches the cone along 0, and meets the quadric of the pencil under consideration in a further plane π, of the opposite system to α, which meets α in a line and contains the line 1; the plane π touches V_3^4 along 1, since it lies in a tangent solid to the cone, and this cannot contain any further points of V_3^4. Thus from the six cones of the pencil we get six lines 1, 2, ..., 6 along which the V_3^4 is touched by planes of the quadric of the opposite system to α, meeting α in lines. Each of these lines meets 0. From these lines, by similar processes, we derive new lines 12, ..., 56; the line 12 arising from the line 1 and the second cone, or from the line 2 and the first cone, and from these again the lines 123, ... arise; here the lines 123 and 456 are identical. The planes containing 0 and ij are of one system, those containing i and ijk of the other; each plane is incident with six planes of the other system. This is, of course, the ordinary notation for the nodes and tropes of the Kummer surface, with which the two sets of planes are in correspondence.

It is immediately clear from the above that the configuration can only be real if all the real cones in the pencil contain real generating planes. If there are six such real cones, then the whole configuration is real; if four of the cones are real, eight planes of each system are real (*e.g.*, 0, 1, ..., 4; 12, ..., 34; 123, ..., 234; and $56 = 1234$). If two cones are real there are, in fact, four real planes of each system, *e.g.*, 0, 1, 2, 12; 34, 56, 234, 134; and these give the various kinds of Kummer configuration with real elements*.

The thirty-two planes of the quadric so obtained lie in pairs in sixteen tangent solids of one of the cones of the pencil; thus the sixteen planes of one system are in correspondence with the lines on the Segre quartic surface, for the sixteen lines in which the plane pairs in these solids intersect project from the vertex of the cone into the sixteen lines on the surface which is the section of V_3^4 by the prime containing the other five vertices of the cones in the pencil. This remark leads at once to the various possible cases above.

* The case of six imaginary cones is noted below, in the general case. Four planes of each system are real, *e.g.* 0, 12, 34, 56; 135, 136, 245, 246.

9. *The general quadric pencil**.

We can obtain the various cases possible for the general pencil of quadrics, in any number of dimensions, by analogous methods; this is done below. We make a distinction between spaces of odd and even dimension.

In all cases a cone with real generating $[k]$ marks the division in the pencil between quadrics containing real $[k]$ and quadrics containing real $[k-1]$, except when n is even and $k=\frac{1}{2}n$, when both sets of quadrics contain real $[k-1]$. This includes, when $k=0$, the result that an invisible cone divides invisible quadrics from convex quadrics, if we make the convention that an invisible cone may, in this sense, be said to have real points (its vertex) but no real lines. It follows that a segment of line, in the scheme for the quadrics of the pencil, which represents quadrics with real generating $[k]$, is bounded by points representing cones with real $[k]$ or real $[k+1]$.

If n denote the dimension of the space considered, the pencil contains $n+1$ cones, of which a certain number, say $2s$, may be conjugate imaginary in pairs. In that case, of the vertices of the self-polar simplex, $n+1-2s$ are real, and the rest are conjugate in pairs, lying on s real lines. We can choose a simplex of reference to consist of the real vertices of the simplex and $2s$ suitably chosen points on these s lines, so that the equations of two of the quadrics of the pencil take the form

$$a_0x_0^2+a_1x_1^2+\ldots+a_{n-2s}x_{n-2s}^2+2\lambda_1 x_{n-2s+1}x_{n-2s+2}+\ldots+2\lambda_s x_{n-1}x_n=0,$$

$$b_0x_0^2+b_1x_1^2+\ldots+b_{n-2s}x_{n-2s}^2+(x_{n-2s+1}^2-x_{n-2s+2}^2)+\ldots+(x_{n-1}^2-x_n^2)=0.$$

To see this we observe that, since the points $A_0, \ldots, A_{n-2s}$ of the common self-polar simplex are real, and are vertices of the fundamental simplex, and since, further, the other vertices lie on the real edges of the self-polar simplex, the terms in the equation which involve $x_0, x_1, \ldots, x_{n-2s}$ can only be the square terms. For a similar reason, the s lines being conjugate, there can be no terms involving products of coordinates corresponding to points on different lines. Thus the only terms which can

* Since this paper was written my attention has been called by Prof. Baker to a paper by Togliatti, *Ann. di Mat.* (3), 30 (1921), 75–117. This deals exclusively with the case of the general quadric pencil in $[n]$, and the chief difference between his investigation and the present one lies in the fact that in the present paper emphasis is laid on the nature of the cones in the pencil, while Togliatti stresses the nature of the undegenerate quadrics. Many of the results of § 9 of the present paper are also to be found in this memoir. The use of the algorithm to obtain the system of real spaces on the loci, however, seems new. The consideration of the projected loci is not found in Togliatti's work.

arise are of the form given, and the two points on each of the s lines can be chosen so that the terms involving the two coordinates corresponding to the vertices of the simplex of reference lying on the line take the forms given above for two of the quadrics.

The general quadric of the pencil is got by taking linear multiples of these. The terms involving the last two coordinates are then

$$\lambda(x_{n-1}^2-x_n^2)+2\mu x_{n-1}x_n,$$

which can be put in the form

$$\lambda\left[\left(x_{n-1}+\frac{\mu}{\lambda}x_n\right)^2-x_n^2\left(1+\frac{\mu^2}{\lambda^2}\right)\right],$$

the difference of two real squares. Thus the nature of the quadric in the pencil, which depends on the number of positive and negative terms in its equation, is entirely dependent on the a_i and b_i. The quadric always contains at least s pairs of positive and negative terms, arising from the last $2s$ coordinates, and hence must contain real $[s-1]$ at least. Hence it follows that, if a pencil contains invisible quadrics, all the cones are real; if it contains convex quadrics, at most two of the cones are unreal; and so on.

Suppose first that all the cones are real, so that the equations of both quadrics may be simultaneously reduced to the sum or difference of square terms.

If the pencil contains invisible quadrics, the equation of one of these may be taken to be

$$x_0^2+x_1^2+\ldots+x_n^2=0,$$

and any other quadric of the pencil has for its equation

$$a_0x_0^2+a_1x_1^2+\ldots+a_nx_n^2=0,$$

where we may suppose the a_i to be in descending order of magnitude. The general quadric of the pencil has then the equation

$$(\lambda-a_0)x_0^2+(\lambda-a_1)x_1^2+\ldots+(\lambda-a_n)x_n^2=0,$$

and is a cone when λ takes the values $a_0, \ldots, a_n$. Now it is clear that of the $n+1$ cones so obtained, two and only two are invisible, namely those which correspond to the values a_0, a_n of λ; two more are convex, arising when λ is a_1 or a_{n-1}, and generally there are two cones of the pencil containing real $[s-1]$, given by $\lambda=a_s$, a_{n-s}, the only exception

occurring when n is even and $s=\frac{1}{2}n$, in which case there is only one cone of the highest type. The pencil has thus only one form, which may be represented by the scheme

two cones of each possible type arising if n is odd, but only one of the highest type if n is even.

Now suppose the pencil to contain convex quadrics but no invisible quadrics. The quadrics

$$x_0^2+x_1^2+\ldots+x_{n-1}^2-x_n^2=0$$

$$a_0x_0^2+a_1x_1^2+\ldots+a_{n-1}x_{n-1}^2-a_nx_n^2=0,$$

where $a_0, \ldots, a_{n-1}$ are in descending order of magnitude, determine the pencil of which the general quadric has the equation

$$(\lambda-a_0)x_0^2+(\lambda-a_1)x_1^2+\ldots+(\lambda-a_{n-1})x_{n-1}^2-(\lambda-a_n)x_n^2=0,$$

and, if the pencil contains no invisible quadrics, a_n must lie between a_0 and a_{n-1}. The cones, as before, arise when λ is equal to one of the a_i. It is clear that the values a_0, a_{n-1} of λ correspond to convex cones in the pencil. If p is any number such that $p<n-1-p$ and $a_p>a_n$, then the cones given by $\lambda=a_p$, $\lambda=a_{n-1-p}$ are similar, and contain real generating $[p]$. Now suppose that a_n lies between a_r and a_{r+1}. If $r<n-1-r$, the cones given by $\lambda=a_r$, $\lambda=a_{n-1-r}$ are of the same type, and contain real generating $[r]$; the cones given by a_n and a_{r+1} are also of this type. If, on the other hand, $r>n-1-r$, then the cones a_{r+1} and a_{n-r-2} are of the same type, containing real $[n-r-2]$, and the cones a_r, a_n are of the same type. It follows readily that, if n is odd, there are two cones of each type (save the invisible one) except one, of which there are four; if n is even the same holds with the modification that there is either only one of the highest type or three of this type and two of the rest, the total number of cones in the pencil being odd.

Generally, if n is odd, every type of cone is represented by an even number in the pencil; if n is even, the highest type only must occur an odd number of times (and hence once at least). Each additional pair of cones of any given type in the pencil reduces by two the number of possible cones of lower types which can exist in the pencil. The process started above may thus be continued. In brief, the result is that there cannot

be more than two invisible cones, more than four convex or invisible, more than six which contain no real [3] . . . , more than $2s$ which contain no real $[s]$. In this way we get the various types of pencil containing all real cones; if n is odd, $2m+1$ say, the number of cases is clearly

$$1+2+4+\ldots+2^{m-1}+1 = 2^m$$

(for there is the case in which all the cones are of the highest type).

If all the cones are not real, it is clear that the real cones can be considered in the same way; if there are s pairs of conjugate imaginary cones, the number of types obtained is the same as the number of types of the pencil in $[n-2s]$ with all the cones real, or 2^{m-s}. There is, in addition, the pencil containing no real cones, so that the total number of types is

$$2^m+2^{m-1}+\ldots+1+1 = 2^{m+1}.$$

It is to be noted that the specification of the cones in the pencil, given above, does not completely determine the pencil: an example of the incomplete determination is when $n=5$, where all the cones are real, and we have seen that there are two types, IV(a) and IV(b), in which all the cones contain real generating planes, these types differing in the arrangement of vertices with regard to the cones. These cases can be obtained in any given instance by examining all possible separation relations between the vertices, but there does not appear to be any simple general result. It will be seen below, however, that this subdivision is of secondary importance.

When n is even, $2m$ say, there is an odd number of cones in the pencil, one of which at least must be real. The argument which limits the number of cones containing real spaces of dimension not greater than s to $2s+2$ applies to this case just as to the case of n odd; it therefore follows that there is a case of this type corresponding to each case of the other type, except the case when all the cones are imaginary. Thus, if $n=2m$, the number of types of pencil is $2^{m+1}-1$. When n exceeds four there will be subdivisions of some of these types as described above. These general formulae verify the cases $n=4$, 5, already discussed.

10. *The base locus of the pencil.*

The base locus of the pencil is a V^4_{n-2}. When n is odd the general quadric of the pencil has two systems of generating $[\frac{1}{2}(n-1)]$, and these meet the locus in general in quadric $V^2_{\frac{1}{2}(n-3)}$, but for a finite number of the spaces of each system this will degenerate into a repeated $[\frac{1}{2}(n-3)]$. The

number of these spaces will be shown below to be 2^{n-1} of each system When n is even the locus contains 2^n $[\frac{1}{2}n-1]$, as is known*; a demonstration of this is given below, and it gives an algorithm for the spaces necessary when questions of reality are considered.

The two problems are intimately connected, and we may proceed to them by induction. Suppose first that n is even. Then each of the $n+1$ cones in the system contains two systems of generating $[\frac{1}{2}n]$, which in general meet the V^4_{n-2} in a series of quadrics $V^2_{\frac{1}{2}n-1}$. These $[\frac{1}{2}n]$ are met by the prime containing the vertices of the other n vertices of cones of the pencil in $[\frac{1}{2}n-1]$ lying on the quadric V^2_{n-2} in which the cone is met by the prime. This quadric contains two systems of generating $[\frac{1}{2}n-1]$, and meets V^4_{n-2} in a V^4_{n-3}. By the result assumed for the case $n-1$, there are 2^{n-2} of these $[\frac{1}{2}n-1]$ of each system, which touch V^4_{n-2} along $[\frac{1}{2}n-2]$. The generating $[\frac{1}{2}n]$ of the cone passing through such a $[\frac{1}{2}n-1]$ thus meets V^4_{n-2} in a $V^2_{\frac{1}{2}n-1}$ containing a double $[\frac{1}{2}n-2]$, that is, in a pair of $[\frac{1}{2}n-1]$. Thus a pair of $[\frac{1}{2}n-1]$ of the locus arises from each contact $[\frac{1}{2}n-2]$ of the quadric in the prime, and hence the total number is 2^n.

If n is odd, each quadric contains two systems of generating $[\frac{1}{2}(n-1)]$. If one of the generating spaces touches the V^4_{n-2} along a $[\frac{1}{2}(n-3)]$, the join of this generating space to the vertex of a cone of the pencil meets the cone in two of its generating $[\frac{1}{2}(n-1)]$, one of which contains the $[\frac{1}{2}(n-3)]$ of contact; this $[\frac{1}{2}(n+1)]$ touches the cone along the intersection of its generating spaces which lie in the space, and accordingly the second generating space of the cone meets V^4_{n-2} in a repeated $[\frac{1}{2}(n-3)]$ like the first. The generating space of the original quadric which passes through this must touch the V^4_{n-2} along this space, and it is of the opposite system to the one first chosen, meeting it in a $[\frac{1}{2}(n-3)]$. The two $[\frac{1}{2}(n-3)]$ of contact meet in a $[\frac{1}{2}(n-5)]$, which clearly lies in the prime containing the vertices of the cones of the pencil, other than the one considered, since the space of intersection is the space of contact of tangents from the vertex of this cone to V^4_{n-2}. It thus follows that, given any cone of the pencil, the contact spaces fall into pairs, of opposite systems on the quadric, lying in tangent $[\frac{1}{2}(n+1)]$ of the cone, and that these tangent spaces are in correspondence with the $[\frac{1}{2}(n-3)]$ lying on the section of V^4_{n-2} by the polar prime of the vertex of the cone with regard to the pencil. Thus the number of such contact spaces is 2^n, half belonging to each system.

This proves the result inductively, since it is true when $n = 4, 5$, as has been seen.

---

* Rosati, *Rend. Ist Lomb.*, 32 (1899), 1267; the above proof is similar to his.

It is then easy to verify that the following algorithm defines the properties of the system of spaces.

If n is *even* we can denote the $n+1$ cones in the pencil by $1, 2, \ldots, n+1$. Take an arbitrary space of the system and call it 0. The spaces which lie with this in the generating spaces of the cones $1, 2, \ldots$ may be denoted by $(1), (2), \ldots$. The generating space of the cone j which contains (i) contains a second space (ij), which is easily seen to lie with (j) in a space of the cone i. Similarly we obtain spaces (ij), (ijk), $\ldots$, $(ijk \ldots p)$ where the number of letters is $\frac{1}{2}n$. If q is another of the symbols it is easy to verify that the space which would be called $(ij \ldots pq)$ is the same as that represented by the remaining $\frac{1}{2}n$ symbols. This gives in all 2^n symbols, as may easily be verified. The incidences are as follows.

The pairs of spaces which intersect in $[\frac{1}{2}n-2]$ are

$$\begin{array}{ccc} 0 & \text{and} & (i) \\ (j) & \text{and} & (ij) \\ (jk) & \text{and} & (ijk) \\ \ldots & \ldots & \ldots \\ (jk \ldots l) & \text{and} & (ijk \ldots l) \\ (jk \ldots p) & \text{and} & (qr \ldots s), \end{array}$$

where in the last line all the symbols save i occur, each once. All the other incidences are consequences of these; *e.g.*, 0 and (ij) meet in $[\frac{1}{2}n-4]$, and so on.

If we regard the addition of (i) to a symbol as symbolic multiplication by (i), then the pairs of spaces lying in generating spaces of the cone i are simply $[X, iX]$, where it is understood that if i occurs twice in a term it is dropped, and if a term contains more than $\frac{1}{2}n$ of the $n+1$ symbols it is replaced by the term containing the others.

If n is *odd* we proceed in a similar way, obtaining from 0 and $n+1$ symbols (i) the symbols (ij), (ijk), $\ldots$ as before; the process now stops when each symbol contains $\frac{1}{2}(n+1)$ terms, and is then to be regarded as the same as the symbol containing the other $\frac{1}{2}(n+1)$ terms. The incidence laws are the same as in the previous case; the condition for two spaces to lie in the generating spaces of a cone of the pencil is now replaced by the condition that they should lie in one of the tangent spaces to such a cone, and this condition is formally the same as in the previous case for the corresponding problem. The space 0 and those whose symbols contain an even number of letters (ij), $(ijkl)$, $\ldots$ belong to one system on the

quadric, those involving an odd number of letters, such as (i), (ijk), ... belong to the opposite system.

With this algorithm it is easy to obtain the various possible cases of reality of the linear spaces. It is clear that, if V^4_{n-2} contains real spaces of the highest possible dimension, all the cones must be either imaginary or of the highest possible type (n being even), and, when n is odd, it is similarly true that the contact spaces can be real only if the same condition holds. If there is a real cone in the pencil the case depends on that of the previous value of n, each real space associated with the section of the locus by the polar prime of the vertex of the cone giving rise to a pair of spaces on V^4_{n-2} which are real or conjugate imaginary, the latter case only occurring if there is no other real cone on the pencil.

The possible cases can be enumerated simply by use of the algorithm. For consider first the case in which n is even, $2m$ say. Let the cones $1, 2, \ldots, 2h-1$ be real, and let the pairs

$$(2h,\ 2h+1),\ (2h+2,\ 2h+3),\ \ldots,\ (2m,\ 2m+1)$$

be conjugate imaginary pairs. If 0 is a real space, then so are the spaces $(1), \ldots, (2h-1)$, and any combination of these symbols also gives a real space. The space $(2h)$ is imaginary and is conjugate to $(2h+1)$; the space $(2h,\ 2h+1)$ is self-conjugate and hence real. Thus the real spaces are those whose symbols contain pairs $(2h,\ 2h+1)$, etc., combined with the symbols $1, \ldots, 2h-1$. It follows that the number of such real spaces is

$$\tfrac{1}{2} \cdot 2^{2h-1} \cdot 2^{m-h+1} = 2^{m+h-1}.$$

In particular, when $h = m+1$, and all the cones are real, there are 2^{2m} real spaces, so that all the spaces are real; if $h = 1$, so that there is only one real cone, the number of real spaces is 2^m. This is in agreement with Togliatti's result*.

If n is odd, $2m+1$ say, we suppose that the cones $1, \ldots, 2h$ are real and that $(2h+1,\ 2h+2), \ldots, 2m+1,\ 2m+2)$ are conjugate pairs. If $h > 0$, so that there are real cones in the pencil, the real contact spaces are represented by symbols containing the numbers $1, \ldots, 2h$ and pairs out of the set $(2h+1,\ 2h+2)$, etc., so that the number of real spaces, including the primitive one 0, is 2^{m+h} as before. When $h = 0$, so that all the cones are imaginary, any space represented by a number of these pairs is real. The space $(135 \ldots 2m+1)$ has for conjugate $(246 \ldots 2m+2)$, which is the same space; consequently this is real also. If m is odd, such spaces

* *Loc. cit. ante*, 95.

involve an even number of symbols, and so belong to the same system on the quadric as the ones previously mentioned; if m is even, they belong to the opposite system.

Considering the two cases separately, if m is odd, n is of the form $4p+3$. We then get, when $h=0$, real spaces whose symbols either consist of a selection of q of the $m+1$ pairs $(2k,\ 2k+1)$, or else consist of $m+1$ symbols taken one from each of these pairs. In all, then, there are $\frac{1}{2}(2^{m+1}+2^{m+1})$ or 2^{m+1} of these spaces. In this case, as we remarked above, they must all belong to the same system on the quadric in question. In the case when m is even, so that $n=4p+1$, there are the same number of spaces, but half of them belong to each system on the quadric.

We may therefore state the results of this section in the following form.

Consider a general pencil of quadrics in $[n]$, such that of the $n+1$ cones in the pencil $2k$ are imaginary, and the remaining ones are of the highest type possible. Then

(i) if n is even the number of real spaces on the intersection of the quadric pencil is 2^{n-k};

(ii) if n is odd the number of generating spaces of any of the quadrics in the pencil which touch the V^4_{n-2} along $[\frac{1}{2}(n-3)]$ which are real is

$$2^{n-k} \text{ if } k \neq \tfrac{1}{2}(n+1) \quad \text{and} \quad 2^{\frac{1}{2}(n+1)} \text{ if } k=\tfrac{1}{2}(n+1);$$

these spaces all belong to the same system if n is of the form $4p+3$, and belong half to each system if n is of the form $4p+1$.

THE ZEROS OF DIRICHLET'S L-FUNCTIONS

By E. C. TITCHMARSH.

[Received June 11, 1930.—Read June 19, 1930.]

1. It is well known that the L-function*

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad (s = \sigma + it) \tag{1.1}$$

has in general three classes of zeros:—

(i) zeros on the negative real axis,

(ii) zeros on the imaginary axis,

(iii) other zeros, all of which lie in the strip $0 < \sigma < 1$.

The distribution of the zeros of the first two classes is perfectly regular, and no particular interest attaches to them. The zeros of the third class are the interesting ones. Their distribution is somewhat irregular. The number $N(T, \chi)$ of those of them which lie between $t = 0$ and $t = T$ is given approximately by the formula†

$$N(T, \chi) = \frac{1}{2\pi} T \log T - \frac{1 + \log(2\pi/K)}{2\pi} T + O(\log T). \tag{1.2}$$

Here K is a certain factor of the modulus k of the character χ with which the function $L(s, \chi)$ is formed, and k is, of course, supposed to be fixed.

If k also is variable, the formula requires modification; but there is not much difficulty in seeing what the modified result is. If $\phi(k, T)$ is a positive function of k and T which is monotonic in either variable, we denote by $O\{\phi(k, T)\}$ a function $f(k, T)$ such that

$$|f(k, T)| < A\,\phi(k, T),$$

* For the theory of L-functions see Landau's *Handbuch der Verteilung der Primzahlen*, 1 (1909), 391–535.

† *Handbuch*, 535.

where A denotes an absolute constant, $T \geqslant 0$, and $k \geqslant 2$. Then

$$(1.3) \quad N(T, \chi) = \frac{1}{2\pi} T \log T - \frac{1+\log(2\pi/K)}{2\pi} T + O[\log\{k(T+1)\}].$$

To prove this we need merely follow out Landau's proof of (1.2), given for fixed k, and observe that*

$$(1.4) \qquad N(T+1, \chi) - N(T, \chi) = O[\log\{k(T+1)\}].$$

2. Consider now the aggregate of all the zeros of L-functions to modulus k, and let $N_k(T)$ be the number of those of them with ordinates between 0 and T. Then

$$(2.1) \qquad N_k(T) = \sum_{\chi} N(T, \chi).$$

This number $N_k(T)$ is also the number of zeros of the third class of a certain type of zeta-function, viz. the function $\zeta_k(s)$ associated with the corpus of the k-th roots of unity; for†

$$\zeta_k(s) = \prod_{\chi} L(s, \chi) \prod_{p \mid k} (1-p^{-fs})^q,$$

where f and q are constants depending on k and p.

Let us consider for simplicity the case where k is a prime, so that all the characters are primitive. Then in (1.3) $K = k$ for the $k-2$ non-principal characters, and $K = 1$ for the principal character; and it follows from (1.3) and (2.1) that

$$(2.2) \quad N_k(T) = \frac{k-1}{2\pi} T \log T + \frac{1}{2\pi}\{(k-2)\log k - (k-1)(1+\log 2\pi)\} T + O[k \log\{k(T+1)\}].$$

We now assume the extended Riemann hypothesis, that all the zeros of the third class of all L-functions lie on the line $\sigma = \frac{1}{2}$. On this hypothesis we can, by an argument‡ similar to one used for the ordinary zeta-function, make an improvement in the error-term in (1.3), considered as a function of T; but the argument does not seem to affect it as a function of k. On the other hand we can improve the error-term in (2.2), both as a function

* E. C. Titchmarsh, *Rend. di Palermo*, 54 (1930), 414–429.

† See H. Bohr and H. Cramér, "Die neuere Entwicklung der analytischen Zahlentheorie", *Encyk. Math. Wiss.* II 3 (1922), 722–849 (843).

‡ J. E. Littlewood, *Proc. Camb. Phil. Soc.*, 22 (1924), 295–318, and E. C. Titchmarsh, *Journal London Math. Soc.*, 2 (1927), 247–254.

of T and as a function of k. We shall prove that, on the extended Riemann hypothesis,

$$(2.3)\quad N_k(T) = \frac{k-1}{2\pi} T \log T + \frac{1}{2\pi}\{(k-2)\log k - (k-1)(1+\log 2\pi)\} T + O\left[\frac{k \log\{k(T+1)\}}{\log\{k \log(T+2)\}}\right].$$

We thus obtain a considerably smaller value for the error-term by considering all the L-functions to modulus k together than we could do by considering them separately and adding the results.

In particular, if T_1 and T_2 are fixed,

$$(2.4)\quad N_k(T_2) - N_k(T_1) = \frac{T_2 - T_1}{2\pi} k \log k + O(k).$$

It follows from this that, however small $T_2 - T_1$ may be, $N_k(T_2) - N_k(T_1) > 0$ if k is sufficiently large; and so that the aggregate of all zeros of the third class of *all* L-functions is everywhere dense on the line $\sigma = \frac{1}{2}$.

The proof of (2.3) is similar in principle to the proofs just referred to of Littlewood's result for the zeta-function, but it follows a different line in detail. Incidentally the method, applied to the zeta-function, could be used to give still another proof of Littlewood's theorem.

3. It was proved by Backlund* that, if $N(T)$ denotes the number of zeros of Riemann's zeta-function for which $0 < t < T$, then

$$(3.1)\quad N(T) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + \frac{7}{8} + \frac{1}{\pi} \operatorname{am} \xi(\tfrac{1}{2}+it) + O\left(\frac{1}{T}\right),$$

the amplitude of $\xi(\frac{1}{2}+it)$ being found by continuous variation along the straight lines joining 2, $2+iT$, $\frac{1}{2}+iT$. We begin by obtaining a similar formula for the zeros of $\Pi L(s, \chi)$.

In the first place, if χ_0 is the principal character,

$$L(s, \chi_0) = (1-k^{-s})\,\xi(s).$$

The zeros of the third class of $L(s, \chi_0)$ are therefore the zeros of the third class of $\xi(s)$. Observing that

$$\operatorname{am}(1-k^{-\frac{1}{2}-it}) = \arctan\left\{\frac{\sin(t \log k)}{\sqrt{k} - \cos(t \log k)}\right\} = O\left(\frac{1}{\sqrt{k}}\right),$$

* R. J. Backlund, *Comptes rendus*, 158 (1914), 1979–1981.

we obtain

$$(3.2)\quad N(T, \chi_0) = \frac{1}{2\pi} T \log T - \frac{1+\log 2\pi}{2\pi} T + \tfrac{7}{8} + \frac{1}{\pi} \operatorname{am} L(\tfrac{1}{2}+iT, \chi_0) + O\left(\frac{1}{\sqrt{k}}\right) + O\left(\frac{1}{T+1}\right).$$

If χ is a non-principal character, we have

$$L(s, \chi) = \left(\frac{\pi}{k}\right)^{\frac{1}{2}(s+a)} \frac{\xi(s, \chi)}{\Gamma\{\frac{1}{2}(s+a)\}},$$

where $a = 0$ if $\chi(-1) = 1$, and $a = 1$ if $\chi(-1) = -1$, and where

$$\xi(s, \chi) = \epsilon(\chi)\, \xi(1-s, \bar{\chi}), \quad |\epsilon(\chi)| = 1.$$

Since $\sum_{\chi} \chi(-1) = 0$, and $\chi_0(-1) = 1$, we have $a = 0$ for $\frac{1}{2}(k-3)$ non-principal characters, and $a = 1$ for $\frac{1}{2}(k-1)$ non-principal characters. Hence, if Π' denotes a product over non-principal characters,

$$(3.3)\quad \Pi' L(s, \chi) = \left(\frac{\pi}{k}\right)^{\frac{1}{2}s(k-2)+\frac{1}{4}(k-1)} \frac{\Pi' \xi(s, \chi)}{\{\Gamma(\frac{1}{2}s)\}^{\frac{1}{2}(k-3)} \{\Gamma(\frac{1}{2}s+\frac{1}{2})\}^{\frac{1}{2}(k-1)}}.$$

Now

$$\Pi' \xi(s, \chi) = \Pi' \xi(1-s, \chi),$$

and hence, as in the case of $\xi(s)$, the number of zeros of $\Pi' \xi(s, \chi)$ is equal to

$$\pi \operatorname{am}\{\Pi' \xi(\tfrac{1}{2}+iT, \chi)\},$$

the amplitude being found by continuous variation along the straight lines joining 2, $2+iT$, $\frac{1}{2}+iT$. Now, by (3.3),

$$\operatorname{am}\{\Pi' \xi(s, \chi)\} = \tfrac{1}{2}(k-3) \operatorname{am} \Gamma(\tfrac{1}{2}s) + \tfrac{1}{2}(k-1) \operatorname{am} \Gamma(\tfrac{1}{2}s+\tfrac{1}{2}) - \operatorname{am}\left(\frac{\pi}{k}\right)^{\frac{1}{2}s(k-2)+\frac{1}{4}(k-1)} + \operatorname{am} \Pi' L(s, \chi),$$

and so

$$\operatorname{am}\{\Pi' \xi(\tfrac{1}{2}+iT, \chi)\} = \tfrac{1}{2}(k-3)\, \mathfrak{I} \log \Gamma(\tfrac{1}{4}+\tfrac{1}{2}iT) + \tfrac{1}{2}(k-1)\, \mathfrak{I} \log \Gamma(\tfrac{3}{4}+\tfrac{1}{2}iT) - \tfrac{1}{2}T(k-2) \log \frac{\pi}{k} + \operatorname{am}\{\Pi' L(\tfrac{1}{2}+iT, \chi)\}.$$

Now

$$\mathfrak{I} \log \Gamma(\tfrac{1}{4}+\tfrac{1}{2}iT) = \tfrac{1}{2}T \log \tfrac{1}{2}T - \tfrac{1}{2}T - \tfrac{1}{8}\pi + O\left(\frac{1}{T+1}\right),$$

$$\mathfrak{I} \log \Gamma(\tfrac{3}{4}+\tfrac{1}{2}iT) = \tfrac{1}{2}T \log \tfrac{1}{2}T - \tfrac{1}{2}T + \tfrac{1}{8}\pi + O\left(\frac{1}{T+1}\right).$$

Hence

$$(3.4)\quad \Sigma' N(T,\chi)=\frac{k-2}{2\pi}(T\log\tfrac{1}{2}T-T)-\frac{k-2}{2\pi}\log\frac{\pi}{k}\,T$$
$$+\tfrac{1}{8}+\operatorname{am}\Pi' L(\tfrac{1}{2}+iT,\chi)+O\Big(\frac{k}{T+1}\Big).$$

Finally, combining this formula with that already obtained for $N(T,\chi_0)$, we have

$$(3.5)\quad N_k(T)=\frac{k-1}{2\pi}\,T\log T+\frac{1}{2\pi}\{(k-2)\log k-(k-1)(1+\log 2\pi)\}\,T$$
$$+1+\operatorname{am}\prod_{\chi} L(\tfrac{1}{2}+iT,\chi)+O\Big(\frac{1}{\sqrt{k}}\Big)+O\Big(\frac{k}{T+1}\Big).$$

It is, of course, possible to use this formula to obtain an alternative proof of (2.2).

4. We write

$$S(T)=\frac{1}{\pi}\operatorname{am}\{\prod_{\chi} L(\tfrac{1}{2}+iT,\chi)\},$$

and

$$S_1(T)=\int_0^T S(t)\,dt.$$

Then*, as in the case of $\zeta(s)$,

$$(4.1)\qquad \pi S_1(T)=\int_{\frac{1}{2}}^{\infty}\log|F(\sigma+iT)|\,d\sigma-\int_{\frac{1}{2}}^{\infty}\log|F(\sigma)|\,d\sigma,$$

where

$$F(s)=\prod_{\chi} L(s,\chi).$$

Now, for $\sigma>1$,

$$\log|F(s)|=\Re\sum_{\chi}\log L(s,\chi)=\Re\sum_{\chi}\sum_{n=2}^{\infty}\frac{\Lambda_1(n)\chi(n)}{n^s},$$

where $\Lambda_1(n)=1$ if n is prime, and otherwise $\Lambda_1(n)=0$. Also

$$\sum_{\chi}\chi(n)=k-1$$

if $n\equiv 1 \pmod k$, and otherwise this sum is zero. Hence we have

$$\log|F(s)|=(k-1)\,\Re\sum_{n\equiv 1(\mathrm{mod}\ k)}\frac{\Lambda_1(n)}{n^s}$$
$$=O[k\{(k+1)^{-\sigma}+(2k+1)^{-\sigma}+\dots\}]=O(k^{1-\sigma}).$$

---

* J. E. Littlewood, *ibid.*, Theorem 3.

Hence

$$\int_{\frac{3}{8}}^{\infty} \log|F(\sigma+it)|\,d\sigma = O\Big(\int_{\frac{3}{8}}^{\infty} k^{1-\sigma}\,d\sigma\Big) = O\Big(\frac{1}{k^{\frac{1}{8}}\log k}\Big),$$

and (4.1) takes the form

$$\pi S_1(T) = \int_{\frac{3}{8}}^{2} \log|F(\sigma+iT)|\,d\sigma - \int_{\frac{1}{8}}^{2} \log|F(\sigma)|\,d\sigma + O(k^{-\frac{1}{8}}). \tag{4.2}$$

The upper limit $\frac{3}{8}$ is (as in Littlewood's work) taken as a convenient constant just greater than 1.

5. The proof now depends on the following two lemmas, each of which is an adaptation of Littlewood's work to the present case.

LEMMA A. *If* $\frac{1}{2} < \sigma \leqslant \frac{3}{8}$, $t \geqslant 0$, *and* $0 < y \leqslant 1$, *then*

$$-\frac{F'(s)}{F(s)} = (k-1)\sum_{n\equiv 1 \pmod k} \frac{\Lambda(n)}{n^s} e^{-ny} + \sum_{\rho} y^{s-\rho}\Gamma(\rho-s) - y^{s-1}\Gamma(1-s) + O\left[ky^{\sigma-\frac{1}{2}}\log\{k(t+1)\}\right],$$

where ρ *runs through all the zeros on* $\sigma = \frac{1}{2}$ *of all* L*-functions to modulus* k.

This is a straightforward extension of a theorem of Littlewood* on the ξ-function. We have

$$\sum \frac{\Lambda(n)\chi(n)}{n^s} e^{-ny} = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{L'(s+z,\chi)}{L(s+z,\chi)}\, y^{-z}\Gamma(z)\,dz. \tag{5.1}$$

We move the path of integration to $\Re(z) = \frac{1}{4}-\sigma$, passing over poles at the following points:

(i) $z = 0$, with residue $L'(s,\chi)/L(s,\chi)$,

(ii) the zeros $\rho - s$ of $L(s+z,\chi)$, with residues $y^{s-\rho}\Gamma(\rho-s)$,

(iii) in the case of the principal character, $z = 1-s$, the residue being $-y^{s-1}\Gamma(1-s)$.

Hence

$$\sum \frac{\Lambda(n)\chi(n)}{n^s} e^{-ny} = -\frac{L'(s,\chi)}{L(s,\chi)} - \sum y^{s-\rho}\Gamma(\rho-s) + y^{s-1}\Gamma(1-s)E(\chi) - \frac{1}{2\pi i}\int_{\frac{1}{4}-\sigma-i\infty}^{\frac{1}{4}-\sigma+i\infty} \frac{L'(s+z,\chi)}{L(s+z,\chi)}\, y^{-z}\Gamma(z)\,dz, \tag{5.2}$$

* J. E. Littlewood, *Proc. London Math. Soc.* (2), 24 (1925), 175-201.

where $E(\chi)=1$ for the principal character, and otherwise $E(\chi)=0$. Now*, if $z=\frac{1}{4}-\sigma+i\eta$,

$$\frac{L'(s+z,\chi)}{L(s+z,\chi)}=O[\log\{k(t+|\eta|+1)\}].$$

Hence the integral in (5 . 2) is

$$O\Big[y^{\sigma-\frac{1}{2}}\int_{-\infty}^{\infty}\log\{k(t+|\eta|+1)\}e^{-A|\eta|}d\eta\Big]=O[y^{\sigma-\frac{1}{2}}\log\{k(t+1)\}].$$

The result stated now follows on summing over the $k-1$ characters (mod k).

6. LEMMA B. *If $\phi(k, T)$ is a steadily increasing function of k and of T, such that $\phi(k, 2T)<k\log\{k(T+1)\}$, and if*

$$S_1(T)=O\{\phi(k, T)\},$$

then

$$S(T)=O\Big([k\log\{k(T+1)\}\phi(k, 2T)]^{\frac{1}{2}}\Big)+O\Big(\frac{k}{T+1}\Big).$$

Let

$$N_k(T)=\frac{k-1}{2\pi}T\log T+\frac{1}{2\pi}\{(k-2)\log k-(k-1)(1+\log 2\pi)\}T+1+R(T),$$

so that, by (3 . 5),

$$R(T)=S(T)+O\Big(\frac{1}{\sqrt{k}}\Big)+O\Big(\frac{k}{T+1}\Big).$$

Now $$N_k(T+x)-N_k(T)\geqslant 0 \quad (x>0),$$

and hence, if $T>1$, $0<x<\frac{1}{2}$,

$$\begin{aligned}R(T+x)-R(T)&\geqslant-\frac{k-1}{2\pi}\{(T+x)\log(T+x)-T\log T\}\\&\quad-\frac{1}{2\pi}\{(k-2)\log k-(k-1)(1+\log 2\pi)\}x\\&>-Akx\log(T+1)-Akx\log k=-Akx\log\{k(T+1)\}.\end{aligned}$$

Hence

$$\begin{aligned}\int_T^{T+x}R(t)dt&=xR(T)+\int_0^x\{R(T+u)-R(T)\}du\\&>xR(T)-\int_0^x Aku\log\{k(T+1)\}du\\&>xR(T)-Akx^2\log\{k(T+1)\}.\end{aligned}$$

* E. C. Titchmarsh, *Rend. di Palermo*, 54 (1930), 414–429, Lemma 14; or J. E. Littlewood, *Proc. London Math. Soc.* (2), 27 (1928), 358–372, Lemma 5, and the functional equation.

But

$$\int_T^{T+x} R(t)dt = S_1(T+x)-S_1(T)+O\left(\frac{x}{\sqrt{k}}\right)+O\left(\frac{xk}{T+1}\right)$$

$$= O\{\phi(k, 2T)\}+O\left(\frac{x}{\sqrt{k}}\right)+O\left(\frac{xk}{T+1}\right).$$

Hence

$$R(T) < Akx \log\{k(T+1)\}+O\{\phi(k, 2T)\}+O\left(\frac{x}{\sqrt{k}}\right)+O\left(\frac{xk}{T+1}\right),$$

and, putting $$x = \left[\frac{\phi(k, 2T)}{2k \log\{k(T+1)\}}\right]^{\frac{1}{2}}$$

(so that $x < \frac{1}{2}$), we have

$$(6.1) \quad R(T) < A[k \log\{k(T+1)\}\phi(k, 2T)]^{\frac{1}{2}}+O\left(\frac{1}{\sqrt{k}}\right)+O\left(\frac{k}{T+1}\right).$$

If $T \leqslant 1$, we have similarly

$$R(T+x)-R(T) > -Akx \log(1/x)-Akx \log k,$$

and hence

$$\int_T^{T+x} R(t)dt > xR(T)-\int_0^x \left(Aku \log\frac{1}{u}+Aku \log k\right)du$$

$$> xR(T)-Akx^2 \log(1/x)-Akx^2 \log k.$$

Hence, as before,

$$R(T) < Akx \log\frac{k}{x}+O\left\{\frac{\phi(k, 2)}{x}\right\}+O(k),$$

and putting $$x = \left\{\frac{\phi(k, 2)}{k \log k}\right\}^{\frac{1}{2}}$$

we have

$$(6.2) \quad R(T) < A\{k \log k \phi(k, 2)\}^{\frac{1}{2}}+O(k).$$

The required upper bound for $S(T)$ thus follows in either case. Similarly the lower bound may be obtained by considering integrals over $(T-x, T)$.

7. *Proof of the main theorem.* We take the result of Lemma A, and integrate from $\sigma+it$ to $\frac{9}{8}+it$, where $\frac{1}{2} < \sigma \leqslant \frac{9}{8}$, $t \geqslant 0$. If $\sigma > 1$, or if $\sigma \leqslant 1$ and $t \geqslant \frac{1}{8}(1-\sigma)$, the integral of each term is taken along a straight line. If $\sigma < 1$, $t < \frac{1}{8}(1-\sigma)$, the path of integration of $y^{s-1}\Gamma(1-s)$ is deformed so as to pass round an arc of the circle with

centre 1 and radius $\frac{1}{8}(1-\sigma)$ It is easily seen that in any case

$$\int y^{s-1}\Gamma(1-s)ds = O(y^{\sigma-1}\log|1-\sigma|).$$

Hence

$$(7.1)\quad \log F(\tfrac{9}{8}+it)-\log F(\sigma+it) = (k-1)\sum_{n\equiv 1 \pmod k}\Lambda_1(n)e^{-ny}\left(\frac{1}{n^s}-\frac{1}{n^{\frac{9}{8}+it}}\right)$$
$$+\int_{\sigma+it}^{\frac{9}{8}+it}\left\{\sum_\rho y^{s-\rho}\Gamma(\rho-s)\right\}ds+O(y^{\sigma-1}\log|1-\sigma|)+O[ky^{\sigma-\frac{1}{4}}\log\{k(t+1)\}].$$

As in § 4, the term on each side arising from the upper limit of the integral is $O(k^{-\frac{1}{8}})$. Hence, integrating again and taking the real part,

$$(7.2)\quad \int_{\frac{1}{2}}^{\frac{9}{8}}\log|F(\sigma+it)|d\sigma = -(k-1)\Re\sum_{n\equiv 1 \pmod k}\Lambda_2(n)e^{-ny}\left(\frac{1}{n^{\frac{1}{2}+it}}-\frac{1}{n^{\frac{9}{8}+it}}\right)$$
$$+\Re\int_{\frac{1}{2}}^{\frac{9}{8}}d\sigma\int_{\sigma+it}^{\frac{9}{8}+it}\left\{\sum_\rho y^{s-\rho}\Gamma(\rho-s)\right\}ds+O(k^{-\frac{1}{8}})+O(y^{-\frac{1}{2}})+O[ky^{\frac{1}{4}}\log\{k(t+1)\}],$$

where $\Lambda_2(n)=\Lambda_1(n)/\log n$. The first term on the right is

$$O\left(k\sum_{n\equiv 1 \pmod k}\frac{e^{-ny}}{\sqrt{n}}\right) = O\left[k\left\{\frac{e^{-(k+1)y}}{\sqrt{(k+1)}}+\frac{e^{-(2k+1)y}}{\sqrt{(2k+1)}}+\dots\right\}\right]$$
$$= O\left\{\sqrt{k}\left(\frac{e^{-ky}}{\sqrt{1}}+\frac{e^{-2ky}}{\sqrt{2}}+\dots\right)\right\} = O\left(\sqrt{k}\int_1^\infty\frac{e^{-uky}}{\sqrt{u}}\,du\right)$$
$$= O\left(\frac{1}{\sqrt{y}}\right).$$

Now consider the second term. From the asymptotic formula for the Γ-function,

$$|\Gamma(\rho-s)| = \left|\frac{\Gamma(\rho-s+1)}{\rho-s}\right| < A\,\frac{e^{-A|\gamma-t|}}{\sqrt{\{(\sigma-\frac{1}{2})^2+(\gamma-t)^2\}}}.$$

Suppose now that it has been proved that

$$(7.3)\quad |S(T)| \leqslant M\frac{k\log\{k(T+1)\}}{[\log\{k\log(T+2)\}]^\alpha} = M\psi(\alpha),$$

say, where $M\geqslant 1$, $0\leqslant\alpha\leqslant 1$. By (2.2) this is certainly true with $\alpha=0$. Let

$$\delta = [\log\{k\log(t+2)\}]^{-\alpha}.$$

Then

$$\sum_\gamma\frac{e^{-A|\gamma-t|}}{\sqrt{\{(\sigma-\frac{1}{2})^2+(\gamma-t)^2\}}} < \sum_{|\gamma-t|<\delta}\frac{1}{\sigma-\frac{1}{2}}+\sum_{\delta\leqslant|\gamma-t|<1}\frac{1}{|\gamma-t|}+\sum_{|\gamma-t|\geqslant 1}e^{-A|\gamma-t|}.$$

If $t > 2$, the number of terms in the first sum on the right is, by (3 . 5),

$$O(k\delta \log t)+O(k\delta \log k)+O\{M\psi(a)\} < AM\psi(a),$$

the constants being independent of M; while if $t \leqslant 2$ it is

$$O(k)+O(k\delta \log k)+O[M\psi(a)] < AM\psi(a).$$

Hence
$$\sum_{|\gamma-t|<\delta} \frac{1}{\sigma-\frac{1}{2}} < \frac{AM\psi(a)}{\sigma-\frac{1}{2}}.$$

Again,
$$\sum_{\delta\leqslant|\gamma-t|<1} \frac{1}{|\gamma-t|} \leqslant \sum_{m=1}^{1/\delta} \sum_{m\delta\leqslant|\gamma-t|<(m+1)\delta} \frac{1}{|\gamma-t|},$$

and the number of terms in each partial sum is less than $AM\psi(a)$. Hence
$$\sum_{\delta\leqslant|\gamma-t|<1} \frac{1}{|\gamma-t|} < \sum_{m=1}^{1/\delta} \frac{AM\psi(a)}{m\delta} < \frac{AM\psi(a)\log(1/\delta)}{\delta}.$$

Finally
$$\begin{aligned}\sum_{|\gamma-t|\geqslant 1} e^{-A|\gamma-t|} &= \sum_{n=1}^{\infty} \sum_{n\leqslant|\gamma-t|<n+1} e^{-A|\gamma-t|} \\ &= \sum_{n=1}^{\infty} O[k\log\{k(t+n+1)\}]e^{-An} \\ &= O[k\log\{k(t+1)\}].\end{aligned}$$

Since
$$\int_{\frac{1}{2}}^{\beta} d\sigma \int_{\sigma'}^{\beta} \frac{y^{\sigma'-\frac{1}{2}}}{\sigma'-\frac{1}{2}}\, d\sigma' = \int_{\frac{1}{2}}^{\beta} y^{\sigma'-\frac{1}{2}} d\sigma' = O\left\{\frac{1}{\log(1/y)}\right\},$$

$$\int_{\frac{1}{2}}^{\beta} d\sigma \int_{\sigma'}^{\beta} y^{\sigma'-\frac{1}{2}} d\sigma' = \int_{\frac{1}{2}}^{\beta} y^{\sigma'-\frac{1}{2}}(\sigma'-\tfrac{1}{2})\, d\sigma' = O\left\{\frac{1}{\log^2(1/y)}\right\},$$

we conclude that

$$\begin{aligned}(7\,.\,4)\quad &\int_{\frac{1}{2}}^{\beta} d\sigma \int_{\sigma+it}^{\beta+it} \left\{\sum_{\rho} y^{s-\rho}\Gamma(\rho-s)\right\} ds \\ &= O\left\{\frac{M\psi(a)}{\log(1/y)}\right\} + O\left\{\frac{M\psi(a)\log(1/\delta)}{\delta\log^2(1/y)}\right\} + O\left\{\frac{k\log\{k(t+1)\}}{\log^2(1/y)}\right\}.\end{aligned}$$

We insert this result in (7 . 2), and take

$$y = \frac{1}{k\log(t+1)},$$

and obtain a result of the form

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \log|F(\sigma+it)|\,d\sigma = O\{M\psi(a+1)\}.$$

Hence (4.2) gives

$$S_1(T) = O\{M\psi(a+1)\}, \tag{7.5}$$

and so Lemma B gives, say,

$$|S(T)| < A_1\sqrt{M\psi(\tfrac{1}{2}a+\tfrac{1}{2})}. \tag{7.6}$$

Since we actually start with $a = 0$, the first application of the passage from (7.3) to (7.6) gives

$$|S(T)| < A_1\sqrt{M\psi(\tfrac{1}{2})}.$$

Repeating the process, we obtain

$$|S(T)| < A_1\sqrt{(A_1\sqrt{M})\psi(\tfrac{1}{2}+\tfrac{1}{4})},$$

and so generally

$$|S(T)| < A_1^{1+\frac{1}{2}+\ldots+1/2^{n-1}} M^{1/2^n}\psi\left(\tfrac{1}{2}+\tfrac{1}{4}+\ldots+\frac{1}{2^n}\right).$$

Making $n\to\infty$,

$$|S(T)| < A_1^2\psi(1),$$

the required result.

8. As an appendix to the above analysis, we observe that the logarithmic derivative of the function

$$F(s) = \prod_{\chi} L(s,\chi)$$

of the above proof satisfies the following inequality:

On the extended Riemann hypothesis

$$\frac{F'(s)}{F(s)} = O\Big(\left[k\log\{k(t+1)\}\right]^{2-2\sigma}\Big) \tag{8.1}$$

uniformly for $\frac{1}{2}+\delta \leqslant \sigma \leqslant 1-\delta$, *where* $0 < \delta < \frac{1}{4}$.

This is analogous to Littlewood's theorem*, that

$$\frac{\zeta'(s)}{\zeta(s)} = O\{(\log t)^{2-2\sigma}\}. \tag{8.2}$$

---

* J. E. Littlewood, *Proc. London Math. Soc.* (2), 24 (1925), 175–201, Theorem 2.

It does not, however, seem possible to prove (8.1) in quite the same way as (8.2). A new device is required, and fortunately a suitable one is available.

We start from Lemma A. Since (as in §7)

$$\sum_{\rho} y^{s-\rho}\Gamma(\rho-s) = O\Big(y^{\sigma-\frac{1}{2}}\sum_{\gamma} e^{-A|\gamma-t|}\Big)$$
$$= O[y^{\sigma-\frac{1}{2}}k\log\{k(t+1)\}],$$

we have

$$(8.3)\qquad \frac{F'(s)}{F(s)} = O\Big(k\sum_{n\equiv 1 \pmod k}\frac{\Lambda(n)}{n^{\sigma}}e^{-ny}\Big)+O[y^{\sigma-\frac{1}{2}}k\log\{k(t+1)\}]+O(y^{\sigma-1}).$$

Let $\pi(x;\ k,\ l)$ denote, as usual, the number of primes, less than or equal to x, which are congruent to l (mod k). Then

$$\sum_{n\equiv 1 \pmod k}\frac{\Lambda(n)}{n^{\sigma}}e^{-ny} = \sum_{n=2}^{\infty}\frac{\log n}{n^{\sigma}}\{\pi(n;\ k,\ 1)-\pi(n-1;\ k,\ 1)\}e^{-ny}$$
$$= \sum_{n=2}^{\infty}\pi(n;\ k,\ 1)\Big\{\frac{\log n}{n^{\sigma}}e^{-ny}-\frac{\log(n+1)}{(n+1)^{\sigma}}e^{-(n+1)y}\Big\}.$$

We require an inequality for $\pi(n;\ k,\ 1)$ which holds uniformly with respect to k. A result of this type is provided by the work of Brun*. It is easily deduced from Brun's work† that, if $(k,\ l)=1$,

$$\pi(x;\ k,\ l) = O\Big\{\frac{x}{\phi(k)\log x}\Big\}$$

uniformly for $k\leqslant x^{a}$, where $0<a<1$. Thus, taking $a=\frac{2}{3}$, we have, since $\phi(k)$ is here $k-1$,

$$\pi(n;\ k,\ 1) = O\Big(\frac{n}{k\log n}\Big)\quad (n\geqslant k^{\frac{3}{2}}).$$

Also $\pi(n;\ k,\ 1)\leqslant n/k$ for *all* values of n. Hence

$$(8.4)\qquad \sum_{n\equiv 1 \pmod k}\frac{\Lambda(n)}{n^{\sigma}}e^{-ny}$$
$$= \sum_{n<k^{\frac{3}{2}}}O\Big(\frac{n}{k}\Big)\Big\{\frac{\log n}{n^{\sigma}}e^{-ny}-\frac{\log(n+1)}{(n+1)^{\sigma}}e^{-(n+1)y}\Big\}$$
$$+\sum_{n\geqslant k^{\frac{3}{2}}}O\Big(\frac{n}{k\log n}\Big)\Big\{\frac{\log n}{n^{\sigma}}e^{-ny}-\frac{\log(n+1)}{(n+1)^{\sigma}}e^{-(n+1)y}\Big\}.$$

---

* V. Brun, *Videnskapsselskapets Skrifter* 1, *Mat. Naturv. Klasse* (Kristiania, 1920), No. 3.

† See E. C. Titchmarsh, *Rend. di Palermo*, 54 (1930), 416.

Now

$$\frac{\log n}{n^\sigma}\, e^{-ny} - \frac{\log (n+1)}{(n+1)^\sigma}\, e^{-(n+1)y}$$

$$= \frac{\log n}{n^\sigma}\, e^{-ny}(1-e^{-y}) + \left\{\frac{\log n}{n^\sigma} - \frac{\log (n+1)}{(n+1)^\sigma}\right\} e^{-(n+1)y}$$

$$= O\left(y\, \frac{\log n}{n^\sigma}\, e^{-ny}\right) + O\left(\frac{\log n}{n^{\sigma+1}}\, e^{-ny}\right).$$

Hence the first sum on the right of (8 . 4) is

$$O\left(\frac{y}{k} \sum_{n < k^{\frac{3}{2}}} \frac{\log n}{n^{\sigma-1}}\right) + O\left(\frac{1}{k} \sum_{n < k^{\frac{3}{2}}} \frac{\log n}{n^\sigma}\right)$$

$$= O(yk^{\frac{3}{2}(2-\sigma)-1} \log k) + O(k^{\frac{3}{2}(1-\sigma)-1} \log k),$$

and the second is

$$O\left(\frac{y}{k} \sum_{n \geqslant k^{\frac{3}{2}}} \frac{e^{-ny}}{n^{\sigma-1}}\right) + O\left(\frac{1}{k} \sum_{n \geqslant k^{\frac{3}{2}}} \frac{e^{-ny}}{n^\sigma}\right) = O\left(\frac{y}{k} \int_0^\infty \frac{e^{-uy}}{u^{\sigma-1}}\, du\right) + O\left(\frac{1}{k} \int_0^\infty \frac{e^{-uy}}{u^\sigma}\, du\right)$$

$$= O\left(\frac{y^{\sigma-1}}{k}\right).$$

Thus (8 . 3) gives

$$\frac{F'(s)}{F(s)} = O(yk^{\frac{3}{2}(2-\sigma)} \log k) + O(k^{\frac{3}{2}(1-\sigma)} \log k) + O(y^{\sigma-1}) + O[y^{\sigma-\frac{1}{2}} k \log \{k(t+1)\}].$$

Taking $$y = \frac{1}{k^2 \log^2 \{k(t+1)\}}$$

the result follows.

ON HECKE'S MODULAR FUNCTIONS, ZETA FUNCTIONS, AND SOME OTHER ANALYTIC FUNCTIONS IN THE THEORY OF NUMBERS

By L. J. MORDELL.

[Received 13 March, 1930.—Read 12 December, 1929.]

Introduction.

1. For about the last fifteen years, Hecke's researches upon the subject forming the title of this paper have been developed in a series of most interesting papers leading to new and important developments in both the theory of numbers and the theory of functions. They commenced* with a proof that the Dedekind zeta function in the theory of algebraic numbers, defined as a function of a complex variable s for $R(s) > 1$, could be continued throughout the whole s plane and satisfied a functional equation of a familiar type. They were extended† immediately to the *zeta* functions formed with characters. Applications‡ were made to Gauss' sums in algebraic fields and to laws of reciprocity. He also investigated other types of analytic functions§, and in particular he has shown‖ that modular functions can be constructed from a series of exponentials of quadratic *indefinite* forms in the two variables of summation. He has also discussed¶ the effect of modular substitutions upon an allied function.

Two principles were used by Hecke. The first was the transformation of a series by the use of a Fourier series**, really an application of Poisson's summation formula in several variables. One of the most important results of this was the association of a theta function with an ideal in an algebraic field, and was the first stage in the proof of the new properties of the Dedekind zeta function. The second stage was

* Hecke, **1**.
† Hecke, **2**, **6**.
‡ Hecke, **3**.
§ Hecke, **7**, **8**.
‖ Hecke, **5**, **11**.
¶ Hecke, **12**.
** See also Hecke, **9**, **10**.

tantamount to the introduction of the second principle by which a sum taken over the non-associated numbers of an ideal could be sometimes transformed into one taken over *all* the numbers of the ideal. This was done by expressing the terms of the first series as integrals taken between limits 0 and ∞; the sum could then be extended to include all the numbers of the ideal by shortening the range of integration.

The summation formula used by Hecke is, of course, well known, and, if we confine ourselves to two variables, states that

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} f(m, n) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(mx+ny)} f(x, y)\, dx\, dy. \tag{1}$$

Hecke's use of (1) is sometimes of a formal nature, but this does not detract from the brilliance of his discovery, since his results are usually of such a character as to make the justification easy and comparatively unimportant. But by using it thus or as a particular case of Fourier series, without giving* some general conditions for $f(x, y)$ for which (1) is valid, some important applications are missed. Further, several of his results, as well as some due to Siegel† and other writers, appear unnecessarily isolated from the general theory in which they occur as special cases.

I have recently given a number‡ of simple applications of the summation formula to the proofs of important formulae, some of which, *e.g.* Kronecker's fundamental limit formula, can only have been overlooked because of a lack of enunciations of simple conditions sufficient for the validity of the formula. It was shown by integration by parts§, without using the theory of Fourier series, that (1) is valid if

(1A) *For all real values of* x, y,

$$\frac{\partial^{\alpha+\beta} f(x, y)}{\partial x^{\alpha}\partial y^{\beta}} \quad (\alpha, \beta = 0, 1, 2)$$

are continuous functions|| *of* x *and* y *and, excluding* $\alpha = \beta = 2$, *tend to zero when* $x \to \infty$ *or* $y \to \infty$; *and if also*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(|f| + \left|\frac{\partial^2 f}{\partial x^2}\right| + \left|\frac{\partial^2 f}{\partial y^2}\right| + \left|\frac{\partial^4 f}{\partial x^2\partial y^2}\right|\right) dx\, dy$$

converges.

---

* Hecke in **8**, 110, gives a *formal* derivation of (1).

† Siegel, **1**, **5**.

‡ Mordell, **2**–**6**.

§ See also Mordell, **7**.

|| The continuity of $\partial^4 f/\partial x^2\partial y^2$ is not required—merely that the integral of this for x is $\partial^3 f/\partial x\partial y^2$ and similarly for y.

Similar results hold for any number of variables, the right-hand and left-hand series corresponding to (1) converging absolutely.

The conditions imposed upon $f(x, y)$ are rather stringent, but they suffice for most of the results which I have in view. It will be seen that a number of Hecke's most striking results appear in a more obvious and general way, and really with not a great deal of calculation if we bear in mind that the results are very general and that the proofs are not formal in character. For simplicity I generally state the results for two variables. Most of the results are true for any number. Thus, for three variables, the formula corresponding to (1) requires the continuity, etc., of $\frac{\partial^{\alpha+\beta+\gamma} f(x, y, z)}{\partial x^{\alpha} \partial y^{\beta} \partial z^{\gamma}}$ ($\alpha, \beta, \gamma = 0, 1, 2$, excluding $\alpha = \beta = \gamma = 2$), and the convergence of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Sigma\left(|f| + \left|\frac{\partial^2 f}{\partial x^2}\right| + \left|\frac{\partial^4 f}{\partial x^2 \partial y^2}\right| + \left|\frac{\partial^6 f}{\partial x^2 \partial y^2 \partial z^2}\right|\right) dx\, dy\, dz.$$

In § 2 is given the summation formula under the conditions and in the form in which it will be required for the applications. § 2 also contains some well known results for justifying certain processes. The first application is given in § 3. It deals with a result of which a particular case was communicated by F. Bernstein to Siegel. The general result has proved an important one in the subsequent researches of Siegel and Hecke, of whom the former published the first proof. § 4 includes the classic formula of Hecke for the theta functions associated with an algebraic field. § 5 is devoted to the most general zeta function in an algebraic field and to similar functions. It is practically a self-contained account requiring a minimum knowledge of algebraic numbers. Thus the *characters* occurring in the general term of the zeta function are defined in a way which is different from that of other writers and is more general. Further, the theory makes no use of the properties of the theta functions which play so fundamental a part in the researches of Hecke, Siegel, and others. § 6 contains a *proof* of a general result from which Hecke's fundamental formulae for his new modular functions and allied functions follow *formally*, immediately, without using the zeta functions. It is from these formulae that Hecke made the surprising discovery that modular functions can be constructed by means of doubly infinite series in which the exponents are *indefinite* quadratic forms in the variables of summation. § 7 contains the justification of the application in § 6 by showing that the formulae of § 6 are valid for other classes of functions. The section concludes by indicating the possibility of certain generalizations.

This résumé is sufficient to show how many important results follow from a systematic application of the summation formula. The generality

of many of them means that some attention is required to details. I hope, however, that this memoir will enable the reader with comparatively little previous acquaintance with the subject to acquire not only a knowledge of my own results, but also of many recent important developments, with less trouble than if he had to consult the numerous references mentioned herein.

The fundamental summation formulae.

2. *Suppose that* a, b, c, a', b', c' *are any numbers such that*

$$\Delta = ab' - a'b \neq 0$$

and put

$$\mu = am + bn + c, \quad \mu' = a'm + b'n + c',$$

$$\Delta\lambda = b'm - a'n, \quad \Delta\lambda' = -bm + an.$$

Then, if a, b, c, a', b', c' *are all real and* $f(x, y)$ *satisfies the conditions* (1 A),

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(\mu, \mu') = \frac{1}{|\Delta|} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i[\lambda(\xi-c)+\lambda'(\eta-c')]} f(\xi, \eta)\, d\xi\, d\eta. \tag{2.1}$$

But, if a, a', b, b', c, c' *respectively are pairs of conjugate complex numbers and* $f(x+iy, x-iy)$ *satisfies the conditions* (1 A), *then*

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(\mu, \mu') = \frac{2}{|\Delta|} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i[\lambda(\xi+i\eta-c)+\lambda'(\xi-i\eta-c')]} f(\xi+i\eta, \xi-i\eta)\, d\xi\, d\eta. \tag{2.2}$$

We shall often write for such a summation in m, n simply $\sum_{\mu} f(\mu, \mu')$, and use $\sum_{\mu}'$ to indicate the omission of $\mu = \mu' = 0$ should this set arise.

To prove (2.1), replace $f(x, y)$ in (1) by $f(ax+by+c, a'x+b'y+c')$, then

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(\mu, \mu') = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i(mx+ny)} f(ax+by+c, a'x+b'y+c')\, dx\, dy.$$

Change the variables of integration to ξ, η, where

$$ax+by+c=\xi, \quad a'x+b'y+c'=\eta,$$

so that

$$\Delta x = b'(\xi-c)-b(\eta-c'), \quad \Delta y = -a'(\xi-c)+a(\eta-c'),$$

$$\Delta(mx+ny) = (b'm-a'n)(\xi-c)+(-bm+an)(\eta-c'),$$

and the result (2.1) easily follows.

We prove (2.2) by writing

$$ax+by+c=\xi+i\eta, \quad a'x+b'y+c'=\xi-i\eta.$$

These results take an interesting form when referred to algebraic fields. Thus, let k be a quadratic field in which a, b are the base of an ideal $\mathfrak{a}$, a', b' are the conjugates of a, b, and c, c' are conjugate numbers in k. Then μ takes the set of values such that $\mu-c$ is a number belonging to the ideal $\mathfrak{a}$, *i.e.* $\mu \equiv c \pmod{\mathfrak{a}}$, while μ' is the conjugate of μ. The set of numbers λ is the fractional ideal* $1/(\mathfrak{a}\mathfrak{d})$, where $\mathfrak{d}$ is the *fundamental ideal* of the field whose norm is d, the absolute value of the discriminant of the field. Then

$$\Delta = N(\mathfrak{a})\, d^{\frac{1}{2}} \quad (d^{\frac{1}{2}} > 0).$$

Hence (2.1) becomes

$$\sum_{\mu \atop c \,(\mathrm{mod}\,\mathfrak{a})} f(\mu, \mu') = \frac{1}{N(\mathfrak{a})\, d^{\frac{1}{2}}} \sum_{\lambda \text{ in } 1/(\mathfrak{a}\mathfrak{d})} e^{-2\pi i(c\lambda + c'\lambda')} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(\lambda\xi+\lambda'\eta)} f(\xi, \eta)\, d\xi\, d\eta. \tag{2.3}$$

The corresponding result for (2.2) will require the insertion of a factor 2 on the right-hand side and the replacing of ξ, η in the integrand (but not in $d\xi\, d\eta$) by $\xi+i\eta$, $\xi-i\eta$.

The general theorem of which these are particular cases may be stated as follows.

Let k be any algebraic field of degree n and discriminant d in which $\mathfrak{d}$ is the fundamental ideal and $\mathfrak{a}$ is any ideal. Let $k^{(1)} = k$, $k^{(2)}$, ..., $k^{(n)}$ be the conjugate fields of which the first r_1 are real and the remaining $2r_2$ are imaginary, conjugate complex pairs being denoted by indices whose difference is r_2. Suppose that c_1, c_2, ..., c_n are any numbers, not necessarily conjugate numbers in k, of which the first r_1 are real, while the remainder,

* See Hecke, **13**, page 113 for fractional ideals and page 131 *et seq.* for the properties of the ideal $1/\mathfrak{d}$.

with a notation similar to that for the fields, give r_2 pairs of conjugate complex numbers. Then

$$\sum_{\mu \equiv c \pmod{\mathfrak{a}}} f(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)})$$

$$= \frac{2^{r_2}}{N(\mathfrak{a})\, d^{\frac{1}{2}}} \sum_{\lambda \text{ in } 1/(\mathfrak{a}\mathfrak{d})} e^{-2\pi i (c_1 \lambda^{(1)} + \ldots c_n \lambda^{(n)})}$$

$$\times \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} e^{2\pi i (\lambda^{(1)} \eta_1 + \ldots \lambda^{(n)} \eta_n)} f(\eta_1, \eta_2, \ldots, \eta_n)\, d\xi_1 d\xi_2 \ldots d\xi_n, \quad (2.4)$$

where

$$\eta_m = \xi_m \quad (m = 1, 2, \ldots, r_1);$$

$$\eta_{r_1+m},\ \eta_{r_1+r_2+m} = \xi_{r_1+m} \pm i\xi_{r_1+r_2+m} \quad (m = 1, 2, \ldots, r_2),$$

and the ξ's are real variables, and the summation in μ means that $\mu^{(1)} - c^{(1)}$, $\mu^{(2)} - c^{(2)}$, etc., are conjugate algebraic numbers in k of the ideals $\mathfrak{a}$, etc.; provided that, on writing $f \equiv f(\eta_1, \eta_2, \ldots, \eta_n)$,

$$\frac{\partial^{a_1+a_2+\ldots+a_n} f}{\partial \xi_1^{a_1} \partial \xi_2^{a_2} \ldots \partial \xi_n^{a_n}}$$

are continuous functions of $\xi_1, \xi_2, \ldots, \xi_n$ ($a_1, a_2, \ldots, a_n = 0, 1, 2$), and excluding $a_1 = a_2 = \ldots = a_n = 2$, vanish when any variable tends to $\pm\infty$, and that the integrals

$$\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \left| \frac{\partial^{2\beta_1+2\beta_2+\ldots+2\beta_n} f}{\partial \xi_1^{2\beta_1} \ldots \partial \xi_n^{2\beta_n}} \right| d\xi_1 \ldots d\xi_n \quad (\beta_1, \beta_2, \ldots, \beta_n = 0, 1)$$

converge.

Some similar types of theorems have been given by Müntz*. He is, I believe, the first writer to give valid, *i.e.* not formal, proofs of similar general formulae. His results have forms and conditions different from mine, especially in the case when all the conjugate fields are not real. They would not be convenient for my applications, though some of his conditions are less stringent than mine.

It will be convenient to give here some theorems required later for justifying certain processes.

The proof of the formula (2.1) shows that the series

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f(\mu, \mu')$$

converges absolutely. We note a crude test for the convergence of such

* Müntz, **1**, 2.

a series when f is an arbitrary function. Recall that

$$\mu = am+bn+c, \quad \mu' = a'm+b'n+c',$$

and suppose, for example, that a, b, c, a', b', c' are all real. Group together the terms for which

$$M \leqslant |\mu| < M+1, \quad N \leqslant |\mu'| < N+1. \tag{2.5}$$

Then, for given M, N, the number of values of μ, μ' is finite and has a maximum value independent of the values of M, N. For (2.5) defines in the m, n plane four parallelograms whose sides have directions and lengths independent of M, N. It often happens that $\sum_{N=0}^{\infty} \sum_{M=0}^{\infty} |f(M, N)|$ can be used as a comparison series. Sometimes the terms arising from $M = 0$ or $N = 0$ must be dealt with separately. Thus, take the series

$$\sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \frac{\sin^2 \pi\mu \sin^2 \pi\mu'}{\mu^2 \mu'^2}.$$

For the terms corresponding to $M = 0$, use $\sin^2 x < x^2$, and then the given series converges by comparison with

$$\sum_{N=1}^{\infty} \sum_{M=1}^{\infty} \frac{1}{M^2 N^2} + \sum_{N=1}^{\infty} \frac{1}{N^2} + \sum_{M=1}^{\infty} \frac{1}{M^2}.$$

Note next the well known result :

If, for $a \leqslant x \leqslant b$, where $-\infty \leqslant a \leqslant b \leqslant \infty$,

$$f = f_1 + f_2 + f_3 + \cdots$$

where f_1, f_2, ... are positive continuous functions of x, and if f is continuous except for isolated discontinuities in any finite part of the range of integration, then

$$\int_a^b f\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx + \ldots.$$

Also de la Vallée Poussin's theorem is useful*, namely that

If a, A, b, B are any constants, finite or infinite, then

$$\int_a^A dx \int_b^B f(x, y)\,dy = \int_b^B dy \int_a^A f(x, y)\,dx,$$

if, within the rectangle of integration, $f(x, y)$ does not change sign, and has its discontinuities lying upon a finite number of continuous curves

* *Cours d'analyse*, 2, ch. 1.

each with only a finite number of maxima and minima, and if the inner integrals are, except for isolated points, continuous functions of x and y respectively.

The extension in both cases to absolute convergence is obvious.

A formula of Hecke and Siegel.

3. Take for the first application of (2 . 3),

$$f(\mu, \mu') = e^{2\pi i(\tau\mu+\tau'\mu')}\mu^{s_1-1}\mu'^{s_2-1} \quad \text{if} \quad \mu \geqslant 0, \quad \mu' \geqslant 0,$$

$$= 0 \quad \text{otherwise,}$$

where μ^{s_1-1} means $\exp[(s_1-1)\log\mu]$ with the arithmetic value of $\log\mu$. The conditions (1A) are satisfied if* $R(s_1)>3$, $R(s_2)>3$, $I(\tau)>0$, $I(\tau')>0$. The double integral in (2 . 3) becomes

$$\int_0^\infty\int_0^\infty e^{2\pi i(\tau\xi+\tau'\eta)+2\pi i(\lambda\xi+\lambda'\eta)}\xi^{s_1-1}\eta^{s_2-1}\,d\xi\,d\eta.$$

When $a>0$, $R(n)>0$, and b is real,

$$\int_0^\infty e^{-(a+ib)x}x^{n-1}\,dx = \frac{\Gamma(n)}{(a+ib)^n},$$

where $$-\tfrac{1}{2}\pi < \arg(a+ib) < \tfrac{1}{2}\pi;$$

hence the double integral becomes

$$\frac{\Gamma(s_1)\Gamma(s_2)}{[-2\pi i(\tau+\lambda)]^{s_1}[-2\pi i(\tau'+\lambda')]^{s_2}} = \frac{e^{2\pi i(s_1+s_2)}\Gamma(s_1)\Gamma(s_2)}{(2\pi)^{s_1+s_2}(\tau+\lambda)^{s_1}(\tau'+\lambda')^{s_2}},$$

where $$0 < \arg(\tau+\lambda) < \pi, \quad 0 < \arg(\tau'+\lambda') < \pi.$$

Hence, for a real quadratic field and *any* real numbers c, c',

$$\sum_{\substack{\mu \geqslant 0 \\ \mu \equiv c \pmod{\mathfrak{a}}}} e^{2\pi i(\tau\mu+\tau'\mu')}\mu^{s_1-1}\mu'^{s_2-1}$$

$$= \frac{e^{\frac{1}{2}\pi i(s_1+s_2)}\Gamma(s_1)\Gamma(s_2)}{(2\pi)^{s_1+s_2}N(\mathfrak{a})\,d^{\frac{1}{2}}} \sum_{\lambda \text{ in } 1/(\mathfrak{a}\mathfrak{d})} \frac{e^{-2\pi i\lambda c-2\pi i\lambda' c'}}{(\tau+\lambda)^{s_1}(\tau'+\lambda')^{s_2}}. \qquad (3\,.\,2)$$

The summation in μ means that $\mu-c$, $\mu'-c'$ are conjugate numbers of the ideals $\mathfrak{a}$, $\mathfrak{a}'$ such that $\mu \geqslant 0$, $\mu' \geqslant 0$. This result, proved for $R(s_1)>3$, $R(s_2)>3$, clearly holds for $R(s_1)>1$, $R(s_2)>1$, since both sides of (3 . 2) converge uniformly for

$$R(s_1) \geqslant 1+\delta_1 > 1, \quad R(s_2) \geqslant 1+\delta_2 > 1,$$

* If $z = x+iy$ where x, y are real, $R(z)=x$, $I(z)=y$.

and represent analytic functions of s_1, s_2 for $R(s_1)>1$, $R(s_2)>1$. It can be obviously extended to any totally real field and, when $s_1=s_2$, is a well known result first published by Siegel*. The result (3.2) was also discovered by Hecke†.

Take next

$$f(\mu,\mu')=e^{2\pi i(\tau|\mu|+\tau'|\mu'|)}\mu^{s_1-1}\mu'^{s_2-1}.$$

We are led to the integral

$$\int_{-\infty}^{\infty}e^{2\pi i\tau|\xi|+2\pi i\mu\xi}\xi^{s_1-1}d\xi=\int_0^{\infty}e^{2\pi i(\tau+\mu)\xi}\xi^{s_1-1}d\xi+\int_0^{\infty}e^{2\pi i(\tau-\mu)\xi}(-\xi)^{s_1-1}d\xi$$

$$=\frac{\Gamma(s_1)}{(2\pi)^{s_1}}\left(\frac{1}{[-i(\tau+\mu)]^{s_1}}+\frac{e^{\pi i(s_1-1)}}{[-i(\tau-\mu)]^{s_2}}\right),$$

whence

$$\sum_{\mu\equiv c\,(\mathrm{mod}\,\mathfrak{a})}e^{2\pi i(\tau|\mu|+\tau'|\mu'|)}\mu^{s_1-1}\mu'^{s_2-1}$$

$$=\frac{e^{\frac{1}{2}\pi i(s_1+s_2)}\Gamma(s_1)\Gamma(s_2)}{(2\pi)^{s_1+s_2}N(\mathfrak{a})\,d^{\frac{1}{2}}}\sum_{\lambda\text{ in }1/(\mathfrak{a}\mathfrak{d})}e^{-2\pi i\lambda c-2\pi i\lambda' c}$$

$$\times\left(\frac{1}{(\tau+\lambda)^{s_1}}+\frac{e^{\pi i(s_1-1)}}{(\tau-\lambda)^{s_1}}\right)\left(\frac{1}{(\tau'+\lambda')^{s_2}}+\frac{e^{\pi i(s_2-1)}}{(\tau'-\lambda')^{s_2}}\right)$$

under the same conditions as (3.2), from which, of course, it is easily deduced.

Hecke's theta functions.

4. We discuss now the theta functions associated with an algebraic field. As already remarked, the first results are due to Hecke. A more general result was given by Siegel‡, who remarks that it gives a proof of the law of quadratic reciprocity in any algebraic field. The result and the application are given by Hecke§ in his book. The former can be stated more generally in the notation of (2.4) as follows. Let

$$\vartheta(t,z,\omega;\mathfrak{a})=\sum_{\mathfrak{a}|\mu}\exp\left\{-\pi\sum_{p=1}^{n}\left[t_p|\mu^{(p)}+z_p|^2-2i\omega_p(\mu^{(p)}+z_p)^2\right]\right\},\quad(4.1)$$

where the summation refers to all the numbers μ of the ideal $\mathfrak{a}$, *i.e.* μ in $\mathfrak{a}$. It is supposed that z_p is real for $p=1,2,\ldots,r_1$, and that z_p, z_{p+r_2}

---

* Siegel, **1** (I), 5–9.
† Hecke, **8** (I), 109–111.
‡ Siegel, **1** (II), 186–188.
§ Hecke, **13**, 234–240.

are conjugate imaginaries for $p = r_1+1, \ldots, r_1+r_2$. The constants ω_p, t_p are arbitrary except that $t_p = t_{p+r_2}$ for $p = r_1+1, \ldots, r_1+r_2$ (this involves no loss of generality),

$$R(t_p+2i\omega_p) > 0 \quad \text{for} \quad p = 1, 2, \ldots, r_1,$$

and
$$R\{t_p xy + i\omega_p(x+iy)^2 + i\omega_{p+r_2}(x-iy)^2\}$$

is a positive definite form in x, y. Then the result is

$$\vartheta(t, z, \omega; \mathfrak{a}) = \sum_{\lambda \equiv 0 \pmod{\mathfrak{a}\mathfrak{d}}} a(\lambda_1, \lambda_2, \ldots, \lambda_n)\, e^{2\pi i(\lambda_1 z_1 + \ldots + \lambda_n z_n)}, \tag{4.2}$$

where

$$a(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{d^{\frac{1}{2}} N(\mathfrak{a})\, W(t, \omega)} \exp\left\{-\pi \sum_{p=1}^{n} \tau_p |\lambda^{(p)}|^2 + 2\pi i \sum_{p=1}^{n} \kappa_p \lambda^{(p)2}\right\}$$

and
$$\tau_p = \frac{t_p}{t_p^2+4\omega_p^2}, \quad \kappa_p = -\frac{\omega_p}{t_p^2+4\omega_p^2} \quad (p = 1, 2, \ldots, r_1),$$

$$\tau_p = \frac{t_p}{t_p^2+4\omega_p\omega_{p+r_2}}, \quad \kappa_p = -\frac{\omega_{p+r_2}}{t_p^2+4\omega_p\omega_{p+r_2}}; \quad \kappa_{p+r_2} = -\frac{\omega_p}{t_p^2+4\omega_p\omega_{p+r_2}}$$
$$(p = r_1+1, \ldots, r_1+r_2).$$

Also

$$W(t, \omega) = \prod_{p=1}^{r_1} (t_p - 2i\omega_p)^{\frac{1}{2}} \prod_{p=r_1+1}^{r_1+r_2} (t_p^2+4\omega_p\omega_{p+r_2})^{\frac{1}{2}},$$

where each root is taken with a positive real part.

With some unimportant restrictions, such as $\omega_p = \omega_{p+r_2}$, which simplify a little the expressions for τ_p, κ_p, Siegel's proof assumes the usual transformation formula for the ordinary theta function, while Hecke's is tantamount to the use of the formula (2.4). In fact, it suffices to take f in this formula to be an exponential of a quadratic form in $\mu^{(1)}, \ldots, \mu^{(n)}$, say

$$f(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)}) = \exp\left\{-\pi \sum_{p=1}^{n} A_p \mu^{(p)2} - 2\pi \sum_{p=r_1+1}^{r_1+r_2} B_p \mu^{(p)} \mu^{(p+r_2)}\right\},$$

where A_p, B_p are arbitrary constants, $R(A_p) > 0$ for $p = 1, 2, \ldots, r_1$, and

$$R(A_p x^2 + 2B_p xy + A_{p+r_2} y^2)$$

is a positive definite form in x, y for $p = r_1+1, \ldots, r_1+r_2$.

The multiple integral in (2.4) splits into a product of r_1 single integrals and r_2 double integrals. The former are typified by

$$\int_{-\infty}^{\infty} e^{-\pi A x^2 + 2\pi i \lambda x}\, dx = A^{-\frac{1}{2}} e^{-\pi\lambda^2/A} \quad [R(A) > 0, \quad R(A^{-\frac{1}{2}}) > 0], \tag{4.3}$$

and the latter by, say,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\pi\{A(x+iy)^2+A'(x-iy)^2+2B(x^2+y^2)+2\pi i\lambda(x+iy) \right.$$
$$\left. +2\pi i\lambda'(x-iy)\}\right]dx\,dy. \quad (4.4)$$

Now, if $$R(ax^2+2hxy+by^2)$$

is a positive definite form in x, y,

$$(ab-h^2)^{\frac{1}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\pi(ax^2+2hxy+by^2+2fy+2gx)\}\,dx\,dy$$
$$= \exp\left\{\frac{\pi(af^2+bg^2-2fgh)}{ab-h^2}\right\},$$

where the radical is taken with a positive real part. For, on integrating first with respect to x and writing $ax+hy+g=X$, the path of integration in the new variable X can be deformed into the real axis of X. The integral then splits into the product of two of the type (4.3), whence the result. If a, h, b were real, the result would, of course, be given immediately by a linear transformation.

Since

$$a=2B+A+A', \quad b=2B-A-A', \quad h=i(A-A'),$$

we have $$ab-h^2=4B^2-4AA',$$

$$af^2+bg^2-2fgh$$
$$= (2B+A+A')(\lambda-\lambda')^2-(2B-A-A')(\lambda+\lambda')^2-2(A-A')(\lambda^2-\lambda'^2)$$
$$= 4(A'\lambda^2+A\lambda'^2-B\lambda\lambda').$$

Hence the value of the double integral (4.4) is

$$\frac{1}{2(B^2-AA')^{\frac{1}{2}}}\exp\left\{-\frac{\pi(A'\lambda^2+A\lambda'^2-B\lambda\lambda')}{AA'-B^2}\right\}.$$

We have then the following results. Put

$$Q(x)=\sum_{p=1}^{n} A_p x_p^2+2\sum_{p=r_1+1}^{r_1+r_2} B_p x_p x_{p+r_2},$$

$$\bar{Q}(x)=\sum_{p=1}^{r_1}\frac{x_p^2}{A_p}+\sum_{p=r_1+1}^{r_1+r_2}\frac{(A_{p+r_2}x_p^2-2B_p x_p x_{p+r_2}+A_p x_{p+r_2}^2)}{A_p A_{p+r_2}-B_p^2}$$

$$\vartheta(z,\mathfrak{a},Q)=\sum_{\mathfrak{a}|\mu}\exp\{-\pi Q(\mu+z)\}.$$

Then

$$\vartheta(z, \mathfrak{a}, Q) = \frac{1}{N(\mathfrak{a})\, d^{\frac{1}{2}} W} \sum_{\lambda \equiv 0 \pmod{\mathfrak{a}\mathfrak{d}}} \exp\{2\pi i \lambda^{(1)} z_1 + \ldots 2\pi i \lambda^{(n)} z_n + \bar{Q}(\lambda)\}, \quad (4.5)$$

where $$W = \prod_{p=1}^{r_1} (A_p)^{-\frac{1}{2}} \prod_{p=r_1+1}^{r_1+r_2} (B_p^2 - A_p A_{p+r_2})^{-\frac{1}{2}}.$$

The result of Siegel and Hecke now follows on putting

$$A_p = t_p - 2i\omega_p \quad (p = 1, 2, \ldots, r_1),$$
$$A_p = -2i\omega_p \quad (p = r_1+1, \ldots, r_1+r_2),$$
$$B_p = t_p \quad (p = r_1+1, \ldots, r_1+r_2).$$

If we put $A_p = 0$ when $p > r_1$, we have Hecke's original fundamental formula for the transformation of the theta function associated with an ideal. For, using A_p instead of B_p,

$$Q(x) = \sum_{p=1}^{n} A_p |x_p|^2, \quad \bar{Q}(x) = \sum_{p=1}^{n} \frac{|x_p|^2}{A_p}, \quad W = \prod_{p=1}^{n} A_p^{-\frac{1}{2}},$$

where $A_p = A_{p+r_2}$ for $p = r_1+1, \ldots, r_1+r_2$, and each radical has a positive real part. Then the result is given by (4.5).

The generalized "zeta" function.

5. An illustration of the advantages resulting from the summation formula is given by an application to some of the more important functions usually associated with an algebraic field k of degree n. Suppose that the first r_1 ($\geqslant 0$) conjugate fields are real and the remaining $2r_2$ ($\geqslant 0$) are imaginary. Let a group U of units ϵ in k be defined by

$$\epsilon = \pm \xi^{\rho} \epsilon_1^{\rho_1} \epsilon_2^{\rho_2} \ldots \epsilon_r^{\rho_r} \quad (r = r_1+r_2-1), \quad (5.1)$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ are r independent generating units of U (*i.e.* ϵ cannot be unity unless $\rho_1 = \rho_2 = \ldots = \rho_r = 0$); ξ is a root of unity in k; $\rho_1, \rho_2, \ldots, \rho_r$ take all integer values from $-\infty$ to ∞, and ρ is an integer satisfying $0 \leqslant \rho < w$, w being the least positive integer such that $\xi^w = \pm 1$. If $r_1 = 0$, $r_2 = 1$, we take $\epsilon = \pm \xi^{\rho}$. If $r_1 = 1$, $r_2 = 0$, when the field is that of the rational numbers, we take $\epsilon = \pm 1$. Thus all the units of k can be expressed uniquely in this way when $\epsilon_1, \ldots, \epsilon_r$ are the *fundamental* units.

Two numbers μ, μ_1 of k are said to be *associated* or *equivalent* with respect to U if a unit ϵ of U exists such that $\mu_1 = \epsilon\mu$, and we then write $\mu_1 \sim \mu \bmod U$. So two pairs of numbers μ, ν and μ_1, ν_1 of k are called *associated* with respect to U if, for a unit ϵ of U, $\mu_1 = \epsilon\mu$, $\nu_1 = \epsilon\nu$.

Let $\mathfrak{a}$, $\mathfrak{b}$ be any two ideals in k with norms $N(\mathfrak{a})$, $N(\mathfrak{b})$. They are said to be equivalent (in the widest sense) if integers μ, ν exist in k such that $(\mu)\mathfrak{a} = (\nu)\mathfrak{b}$. All the ideals equivalent to a given ideal $\mathfrak{a}$ are said to form a class, say the class $\mathfrak{K}(\mathfrak{a})$. Any other kind of equivalence of the ideals $\mathfrak{a}$ and $\mathfrak{b}$ may be defined by selecting an appropriate group U and imposing on μ, ν the further condition $\mu \sim \nu \bmod U$. Hence the results given here are the most general possible.

A summation in μ extended over the non-associated values of $\mu \bmod U$ will be indicated by $(\mu)_U$ or simply (μ) placed under the summation sign. A dash is used to indicate the omission of $\mu = 0$. So a summation in μ, ν extended over the pairs of values μ, ν non-associated $\bmod U$ will be indicated by (μ, ν). A dash indicates the omission of the combination $\mu = \nu = 0$.

Then the Dedekind zeta function in k is defined, when $R(s) > 1$, by

$$\zeta(s, \mathfrak{K}) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}, \tag{5.2}$$

where the summation is extended to all the ideals $\mathfrak{a}$ of k. So we can write

$$\zeta\{s, \mathfrak{K}(\mathfrak{a}^{-1})\} = \sum_{\mathfrak{c}} N(\mathfrak{c})^{-s}, \tag{5.3}$$

where the summation is extended to all the ideals of the class $\mathfrak{K}(\mathfrak{a}^{-1})$. This becomes, on putting $\mathfrak{c}\mathfrak{a} = \mu$,

$$N(\mathfrak{a}^{-s})\zeta\{s, \mathfrak{K}(\mathfrak{a}^{-1})\} = \sum_{\mathfrak{a}|(\mu)}{}' |N\mu|^{-s}, \tag{5.4}$$

where the summation is extended to the numbers $\mu \neq 0$, non-associated with respect to the units of k, and divisible by $\mathfrak{a}$, *i.e.* the μ are numbers of $\mathfrak{a}$. These zeta functions are meromorphic functions of s throughout the whole s plane and satisfy functional equations of a well known type. Similar results hold for the functions obtained on multiplying the general term above by certain complex numbers of modulus unity, *e.g.* the roots of unity known as the *characters* of μ.

As already remarked, Hecke discovered the most important and fundamental results. Some of them are given in the book* by Landau, who has also written a paper† generalizing them by using Hecke's method. Some proofs of a different kind, and allied functions from which some of the results follow, were given by Siegel.‡ It will, however, be more con-

* Landau, **1**, 64–76.
† Landau, **2**.
‡ Siegel, **5**.

venient if I explain my notation and prove my own results before dealing with those of others.

It will be seen that the results still hold when the characters are replaced by more general functions which are defined as follows. Let $\mathfrak{f}$ be a given integral ideal in k and $\mathfrak{a}$ any given ideal. Suppose that $F(\mu, \mathfrak{f}, \mathfrak{a}, U)$, or say $F(\mu)$, is a function of μ such that, if ϵ is any unit of U,

$$F(\mu) = e^{\pi i F_\epsilon} F(\nu) \quad \text{if} \quad \mu \equiv \epsilon\nu \pmod{\mathfrak{a}\mathfrak{f}}, \tag{5.5}$$

where F_ϵ is real and independent of ν, but depends on U, $\mathfrak{a}$, $\mathfrak{f}$. Clearly

$$F(\epsilon) = e^{\pi i F_\epsilon} F(1), \quad F_\epsilon + F_{\epsilon^{-1}} \equiv 0 \pmod 2,$$

and F_{-1} is a rational integer. $F(\mu)$ depends on the residue of μ mod $\mathfrak{a}\mathfrak{f}$.

Put
$$\sum_{\substack{\mu \bmod \mathfrak{a}\mathfrak{f} \\ \mu \equiv 0\,(\mathfrak{a})}} F(\mu)\, e^{2\pi i S(\lambda\mu)} = \bar{F}(\lambda), \tag{5.6}$$

where the summation in μ refers to those residues divisible by $\mathfrak{a}$ of a complete set (mod $\mathfrak{a}\mathfrak{f}$). Here λ is 0 or any number in the ideal $1/(\mathfrak{a}\mathfrak{f}\mathfrak{d})$, and $\bar{F}(\lambda)$ is supposed not to be zero for all such λ. The sum of $\lambda\mu$ and its conjugates in k is denoted by $S(\lambda\mu)$. So $N(\lambda\mu)$ denotes the product of $\lambda\mu$ and its conjugates in k. This equation merely defines $\bar{F}(\lambda)$, thus

$$\sum_{\substack{\mu \bmod \mathfrak{a}\mathfrak{f} \\ \mu \equiv 0\,(\mathfrak{a})}} F(\mu) = \bar{F}(0), \tag{5.7}$$

where $\bar{F}(0)$ may or may not be zero.

If $F(\mu)$ were a character, as usually defined, then $|F(\mu)| = 0$ or 1, $F(\mu\nu) = F(\mu)F(\nu)$, and $\bar{F}(\lambda)$ involves the character conjugate to $F(\lambda)$; but none of these properties is required or is assumed to hold.

If we write $\epsilon\lambda$ for λ and $\epsilon^{-1}\mu$ for μ, where ϵ is any unit of U, and note (5.5), the definition (5.6) shows that

$$\bar{F}(\epsilon\lambda) = e^{-\pi i F_\epsilon} \bar{F}(\lambda). \tag{5.8}$$

Also $\bar{F}(\lambda_1) = \bar{F}(\lambda_2)$ if $\lambda_1 \equiv \lambda_2 \ [\mathrm{mod}\ 1/(\mathfrak{a}\mathfrak{d})]$,

since then*

$$(\lambda_1 - \lambda_2)\mu \equiv 0 \pmod{1/\mathfrak{d}}, \quad S[(\lambda_1 - \lambda_2)\mu] \equiv 0 \pmod 1$$

{*i.e.* $S[(\lambda_1 - \lambda_2)\mu]$ is a rational integer}.

We require an expression for F in terms of $\bar{F}$. Let α be any number in the ideal $\mathfrak{a}$. Multiply (5.6) by $e^{-2\pi i S(\alpha\lambda)}$ and sum for λ over the

* See Hecke, **13**, p. 113, for fractional ideals, and p. 131 *et seq.* for the properties of the ideal $1/\mathfrak{d}$.

residues divisible by $1/(\mathfrak{a}\mathfrak{f}\mathfrak{d})$ of a complete set mod $1/(\mathfrak{a}\mathfrak{d})$. Since

$$\sum_{\lambda} e^{2\pi i S[\lambda(\mu-\alpha)]} = N(\mathfrak{f}) \quad \text{if} \quad \mu \equiv \alpha \pmod{\mathfrak{a}\mathfrak{f}},$$

$$= 0 \qquad \text{if} \quad \mu \not\equiv \alpha \pmod{\mathfrak{a}\mathfrak{f}},$$

we have $$N(\mathfrak{f})F(\alpha) = \sum_{\lambda} \bar{F}(\lambda) e^{-2\pi i S(\alpha\lambda)}.$$

Change α into $-\mu$ and we have

$$\sum_{\substack{\lambda \bmod 1/(\mathfrak{a}\mathfrak{d}) \\ \lambda \equiv 0 \ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d})}} \bar{F}(\lambda)\, e^{2\pi i S(\mu\lambda)} = N(\mathfrak{f})F(-\mu). \tag{5.9}$$

The comparison of (5.6) and (5.9) shows that if $\mathfrak{a}$ is changed into $1/(\mathfrak{a}\mathfrak{f}\mathfrak{d})$ and $F(\mu)$ into $\bar{F}(\lambda)$, then $\bar{F}(\lambda)$ is changed into $N(\mathfrak{f})F(-\mu)$, and in particular $\bar{F}(0)$ becomes $N(\mathfrak{f})F(0)$.

Let there be given the following sets of n numbers: rational integers $a_1, a_2, \ldots, a_n \geqslant 0$, where $a_1, a_2, \ldots, a_{r_1}$ are either 0 or 1 and such that, in the notation of (2.4),

$$a_t a_{t+r_2} = 0 \text{ for } t = r_1+1, \ldots, r_1+r_2;$$

$a_1', a_2', \ldots, a_n'$ where $a_t' = a_t$ for $t = 1, 2, \ldots, r_1$; $a_t' = a_{t+r_2}$, $a_{t+r_2}' = a_t$, for $t = r_1+1, \ldots, r_1+r_2$;

$s_1, s_2, \ldots, s_n$, the s's being complex numbers with $s_t = s_{t+r_2}$ for $t = r_1+1, \ldots, r_1+r_2$.

Then the general zeta function is defined by

$$\zeta(s, \mathfrak{a}, F, \alpha) = \sum_{\mathfrak{a}|(\mu)}{}' F(\mu) N(\mu/|\mu|)^{\alpha} N|\mu|^{-s}, \tag{5.10}$$

where, as indicated, the summation extends over all the numbers $\mu \neq 0$ of the ideal $\mathfrak{a}$ which are non-associated mod U.

We write for shortness

$$N|\mu|^{-s} = |\mu^{(1)}|^{-s_1} |\mu^{(2)}|^{-s_2} \ldots |\mu^{(n)}|^{-s_n}, \tag{5.11}$$

where $|\mu^{(1)}|^{-s_1}$ means $\exp(-s_1 \log|\mu^{(1)}|)$ with the arithmetic value of $\log|\mu^{(1)}|$; and similarly in other cases for the symbol $N|\xi|^{-s}$.

Put $$s_t = s + i\sigma_t \quad \text{for} \quad t = 1, 2, \ldots, n,$$

where we assume that $R(s) > 1$ and σ_t is real. No confusion will arise if s be used to denote both the group of n variables $s_1, s_2, \ldots, s_n$ as well as the variable s in s_t, so long as we note that $N|\mu|^{-s}$ is defined as in (5.11).

In order that the summation in μ should have a meaning, it must be immaterial which of the associated values of μ mod U is taken. Hence

the general term must remain unaltered when μ is replaced by $\epsilon\mu$ where ϵ is any unit of U. This will be so from (5.1) if it holds when $\epsilon = -1, \xi, \epsilon_t$, where $t = 1, 2, \ldots, r$. Hence, from the condition

$$e^{\pi i F_\epsilon} N(e^{ia \arg \epsilon}) N|\epsilon|^{-s} = 1, \tag{5.12}$$

we have

$$F_{-1} + \sum_{t=1}^{n} a_t \equiv 0 \pmod 2. \tag{5.13}$$

$$\tfrac{1}{2} w F_\xi + \sum_{t=1}^{n} a_t c_t \equiv 0 \pmod w,$$

where $c_1, c_2, \ldots, c_n$ are given rational integers depending on the field, *e.g.* when $\epsilon^{(1)} = \xi = e^{2\pi i/w}$, $c_1 = 1$, and $\epsilon^{(2)} = \xi^{c_2}$, etc. Also

$$s_1 \log|\epsilon_t^{(1)}| + \ldots + s_n \log|\epsilon_t^{(n)}| = \pi i F_\epsilon + i \sum_{v=1}^{n} a_v \arg \epsilon_t^{(v)} + 2\pi i m_t$$

for $t = 1, 2, \ldots, r$, where m_t may be any rational integer. Since $N|\epsilon_t| = 1$, this becomes, in terms of the σ,

$$\sigma_1 \log|\epsilon_t^{(1)}| + \ldots + \sigma_n \log|\epsilon_t^{(n)}| = \pi F_\epsilon + \sum_{v=1}^{n} a_v \arg \epsilon_t^{(v)} + 2\pi m_t.$$

These $r_1 + r_2 - 1$ equations together with the r_2 equations $\sigma_t = \sigma_{t+r_2}$ for $t = r_1 + 1, \ldots, r_1 + r_2$ do not determine the σ uniquely. Put

$$\sigma_1 + \ldots + \sigma_n = n\sigma,$$

where we may take $\sigma = 0$, or, if not, we need only replace σ_t by $\sigma + \sigma_t$. These equations then determine the sigmas uniquely, but there is no need to write down their values. The general term of the zeta function could be written as

$$F(\mu) N(\mu/|\mu|)^a \chi(\mu) |\mu^{(1)} \mu^{(2)} \ldots \mu^{(n)}|^{-s},$$

where the part

$$N|\mu|^{i\sigma} = e^{iS(\sigma \log|\mu|)} = \chi^{-1}(\mu)$$

has a modulus unity and is a kind of character of μ. There is, however, no need to do this.

The zeta function is an analytic function of s for $R(s) > 1$. For, since the summation is restricted to non-associated values of μ, we may suppose that $|\mu|$ and all its conjugates in k, excluding $|\mu^{(1)}|$ if there are real conjugate fields, and if there are none excluding one pair of conjugate imaginary μ's, are bounded both above and below. For the $r-1$ rational

integers $v_1, v_2, \ldots, v_{r-1}$ can be found so that this applies to the product $\mu\epsilon_1^{v_1}\epsilon_2^{v_2}\ldots\epsilon_{r-1}^{v_{r-1}}$, where $\epsilon_1, \ldots, \epsilon_{r-1}$ are the special units in U, since only one of each pair of conjugate imaginary fields need be considered. If we note also that the numbers of values of $\mu^{(1)}$, say, for which now $N < |\mu^{(1)}| \leqslant N+1$, $N = 0, 1, 2, \ldots$, has a maximum value independent of the value of N, the uniform convergence of the zeta series for $R(s) \geqslant 1+\delta > 1$ follows as in § 3 on comparison with $\sum_{N=1}^{\infty} N^{-s}$.

It will be shown that the zeta function defined by the series for $R(s) > 1$ is an analytic function of s throughout the whole s plane, except that when $a_t = 0$, $\sigma_t = 0$ for $t = 1, 2, \ldots, n$, $\bar{F}(0) \neq 0$, there may be a simple pole at $s = 1$. It satisfies the functional equation

$$\xi\{1-s,\ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\ \bar{F},\ a'\} = \xi(s,\ \mathfrak{a},\ F,\ a)\, d^{\frac{1}{2}} N(\mathfrak{a}\mathfrak{f}) \prod_{t=1}^{r_1} \frac{\Gamma(\frac{1}{2}s_t+\frac{1}{2}a_t)}{\Gamma(\frac{1}{2}-\frac{1}{2}s_t+\frac{1}{2}a_t)} \times \prod_{t=r_1+1}^{r_1+r_2} \frac{2^{1-2s_t}\Gamma(s_t+\frac{1}{2}a_t+\frac{1}{2}a_t')}{\Gamma(1-s_t+\frac{1}{2}a_t+\frac{1}{2}a_t')}. \quad (5.14)$$

There is no need in the definition of the ξ function to take $s_{t+r_2} = s_t$ for $t = r_1+1, \ldots, r_1+r_2$, an assumption which, however, has not lessened the generality, since only the sum $s_t+s_{t+r_2}$ enters into the general term. Then by $\xi(s', \ldots)$, where $s'_{t+r_2} \neq s'$ for $t = r_1+1, \ldots, r_1+r_2$, we mean $\xi(s, \ldots)$ where

$$s_t = s_{t+r_2} = \tfrac{1}{2}s'_{t+r_2}+\tfrac{1}{2}s'_t \quad \text{for} \quad t = r_1+1, \ldots, r_1+r_2;$$

$$s_t = s'_t \quad \text{for} \quad t = 1, 2, \ldots, r_1.$$

These results include all the results of this kind on the zeta function previously given*. The general term of the zeta function is a function of the ideal (μ), and can be put in a form which shows its dependence upon the ideal (μ) and the class of (μ) for the method of division into classes determined by the group of units U. This has been done by Hecke† for his zeta functions. My form of the result, however, requires a minimum of previous knowledge.

We now define the Φ function. *Let $p_1, p_2, \ldots, p_n$, where $p_t = p_{t+r_2}$, $t = r_1+1, \ldots, r_1+r_2$, and similarly $q_1, q_2, \ldots, q_n$, be two sets of real numbers all greater than zero. Let b be a set of the kind typified by a.*

---

* Artin's new and important L series are not defined by a series of the type (5.10).

† Hecke, **6**, II.

Put, with the usual meaning of the multiple-valued functions,

$$\Phi\begin{pmatrix} s, \mathfrak{a}, \mathfrak{b} \\ F, G, a, b \end{pmatrix} = \sum'_{\substack{\mathfrak{a}\,|\,\mu \\ \mathfrak{b}\,|\,\nu}} F(\mu)\,G(\nu)\,N(\mu^a\nu^b)\,N(p\,|\,\mu\,|^2+q\,|\,\nu\,|^2)^{-s}, \quad (5.15)$$

where the summation refers to all non-associated pairs of numbers μ, ν, mod U *of the ideals* $\mathfrak{a}$, $\mathfrak{b}$ *respectively, and the combination* $\mu = \nu = 0$ *is excluded. When* $\mu = 0$, *we take for* ν *the non-associated numbers* mod U *of the ideal* $\mathfrak{b}$, *and similarly when* $\nu = 0$. *The character* $G(\nu)$ *is defined with reference to an ideal* $\mathfrak{g}$ *as* $F(\mu)$ *is with respect to* $\mathfrak{f}$. *Let*

$$s_t = s + i\sigma_t + \tfrac{1}{2}a_t + \tfrac{1}{2}b_t \quad (t = 1, \ldots, n),$$

and suppose that σ_t *is real and* $R(s) > 1$.

Here again, if the summation is to have a meaning, first the general term must remain unaltered when μ, ν are replaced by $\epsilon\mu$, $\epsilon\nu$, where ϵ is any unit of U, and so

$$e^{\pi i F_\epsilon + \pi i G_\epsilon} N(\epsilon^{a+b})\,N(|\,\epsilon\,|^{-2s}) = 1. \quad (5.16)$$

This gives

$$\left.\begin{aligned} F_{-1} + G_{-1} + \sum_1^n (a_t + b_t) &\equiv 0 \pmod 2 \\ \tfrac{1}{2}w(F_\xi + G_\xi) + \sum_1^n c_t(a_t + b_t) &\equiv 0 \pmod w \end{aligned}\right\}, \quad (5.16')$$

$$\sigma_1 \log|\,\epsilon_t^{(1)}\,| + \ldots + \sigma_n \log|\,\epsilon_t^{(n)}\,| = \tfrac{1}{2}\sum_{v=1}^{n} (a_v + b_v) \arg \epsilon_t^{(v)} + \tfrac{1}{2}\pi F_\epsilon + \tfrac{1}{2}\pi G_\epsilon + \pi n_t,$$

for $t = 1, 2, \ldots, r$, where n_t may be any rational integer.

We have also

$$\sigma_t + \tfrac{1}{2}a_t + \tfrac{1}{2}b_t = \sigma_{t+r_2} + \tfrac{1}{2}a_{t+r_2} + \tfrac{1}{2}b_{t+r_2} \quad (t = r_1+1, \ldots, r_1+r_2);$$

and if, as before, we assume that

$$\sigma_1 + \sigma_2 + \ldots + \sigma_n = 0,$$

we can determine the sigmas uniquely, but their values are not required.

Secondly, when $\mu = 0$, we must express the fact that the general term of Φ is unaltered when ν is replaced by $\epsilon\nu$.

The term is zero, and no condition is required, unless all the a are zero and $F(0) \neq 0$. If all the a are zero, and $F(0) \neq 0$, the condition is

$$e^{\pi i G_\epsilon} N(\epsilon^b)\,N|\,\epsilon\,|^{-2s} = 1;$$

and this is included in (5.16) since $F(0) = e^{\pi i F_\epsilon} F(0)$, and so $e^{\pi i F_\epsilon} = 1$.

Similarly when $\nu = 0$ and μ is replaced by $\epsilon\mu$.

It will be shown that the Φ *function is defined by the series when* $R(s) > 1$, *and that it is an analytic function of* s *throughout the whole* s *plane except that when* $a_t = b_t = \sigma_t = 0$ *for* $t = 1, 2, \ldots, n$ *and* $\bar{F}(0) \neq 0$, $\bar{G}(0) \neq 0$, *there may be a simple pole for* $s = 1$; *and also at* $s = \frac{1}{2}$, *the latter not being a pole if* $F(0) = G(0) = 0$.

It satisfies the functional equation

$$\Phi\begin{pmatrix} 1-s+E, & 1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}), & 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}) \\ \bar{G}, & \bar{F}, & b', & a' \end{pmatrix}$$

$$= \Phi\begin{pmatrix} s, & \mathfrak{a}, & \mathfrak{b} \\ F, & G, & a, & b \end{pmatrix} d\, N(\mathfrak{a}\mathfrak{b}\mathfrak{f}\mathfrak{g})\, N(\pi^{-2s+1+E} i^{a+b} p^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'} q^{s-\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}a'})$$

$$\times \prod_{t=1}^{r_1} \frac{\Gamma(s_t)}{\Gamma(1-s_t+E_t)} \prod_{t=r_1+1}^{r_1+r_2} \frac{2^{2-4s_t+2E_t}\Gamma(2s_t)}{\Gamma(2-2s_t+2E_t)}, \quad (5.17)$$

where $$2E_t = a_t + a_t' + b_t + b_t'.$$

The introduction of the Φ function is due to Siegel* in the particular case when $\mathfrak{b} = \mathfrak{a}$, $a = b = 0$, $F(\mu) = G(\mu) = 1$, $s_t = s$. He also uses Hecke's formula (4.6) for the transformation of the theta function, with the result that my presentation differs from his. Siegel mentions that his result can be extended to the zeta functions of the type due to Lerch and Epstein, and gives the result when in (5.15) $p|\mu|^2 + q|\nu|^2$ is replaced by $a|\mu|^2 + 2h|\mu||\nu| + b|\nu|^2$, where $ab - h^2 > 0$ and the field and its conjugates are all real. I should mention here that my methods give results when no restriction is made as to the definiteness of the real parts of the quadratic forms.

An account of this, and applications to the Epstein zeta function, was given in a recent paper†.

We can write (5.15) as

$$\Phi\begin{pmatrix} s, & \mathfrak{a}, & \mathfrak{b} \\ F, & G, & a, & b \end{pmatrix} = \sum_{\substack{\mathfrak{a}|(\mu) \\ \mathfrak{b}|\nu}} F(\mu)\, G(\nu)\, N(\mu^a \nu^b)\, N(p|\mu|^2 + q|\nu|^2)^{-s}, \quad (5.18)$$

where the summation in ν refers to *all* the numbers of the ideal $\mathfrak{b}$ including $\nu = 0$, but only non-associated values of ν mod U are taken when $\mu = 0$. The summation in μ refers to the non-associated numbers mod U of the ideal $\mathfrak{a}$ and includes $\mu = 0$. The combination $\mu = \nu = 0$ is still

---

* Siegel, **5**, 25–31.

† Mordell, **7**.

excluded. For given μ, the partial series in ν certainly converges absolutely if $R(s) > \frac{1}{2}$. When μ is zero, we have the terms

$$F(0)\,N(0^a) \sum_{\mathfrak{b}\,|\,(\nu)}{}' G(\nu)\,N\{\nu/|\nu|\}^b N(q^{-s})\,N|\nu|^{-2s+b}$$
$$= F(0)\,N(0^a)\,N(q^{-s})\,\xi(2s-b,\ \mathfrak{b},\ G,\ b),$$

where $N(0^a) = 0$ unless all the a are zero, when it is 1. Next, when $\mu \neq 0$, the summation for ν is split into a finite number of others by writing $\nu \equiv \beta \pmod{\mathfrak{b}\mathfrak{g}}$, and making β run through the residues divisible by $\mathfrak{b}$ of a complete set (mod $\mathfrak{b}\mathfrak{g}$). The summation formula (2.4) can be applied to the summation $\nu \equiv \beta \pmod{\mathfrak{b}\mathfrak{g}}$ for given β, since, when ν is complex,

$$|\nu|^2 = \nu_t \nu_{t+\nu_2}, \quad \text{say,}$$

is merely the product of two variables. The multiple integral in (2.4) becomes the product of r_1 single integrals and r_2 double integrals. We must discuss these.

Write
$$\phi(h,\ s) = \int_{-\infty}^{\infty} \frac{e^{2ihx}\,dx}{(x^2+1)^s}, \tag{5.19}$$

where h is real. The integral converges when $h \neq 0$ if $R(s) > 0$, and when $h = 0$ if $R(s) > \frac{1}{2}$. Clearly $\phi(h,\ s)$ is an even function of h, and

$$\phi(0,\ s) = \frac{\Gamma(\frac{1}{2})\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}.$$

Suppose in the first place that $\lambda \neq 0$. Then the single integrals are typified by

$$\int_{-\infty}^{\infty} \frac{e^{2\pi i\lambda\tau} x^b\,dx}{(p|\mu|^2+qx^2)^s} = \frac{1}{|\mu|^{2s-1-b}\,p^{s-\frac{1}{2}(1+b)}\,q^{\frac{1}{2}(1+b)}} \int_{-\infty}^{\infty} \frac{e^{2\pi i\lambda|\mu|\sqrt{(p/q)}x}\,x^b\,dx}{(x^2+1)^s}. \tag{5.19'}$$

If $b = 0$, the value of the integral is $\phi\{\pi|\lambda\mu|\sqrt{(p/q)},\ s\}$, since λ is real. If $b = 1$, on integrating by parts, the integral becomes

$$\frac{2\pi i\lambda|\mu|\sqrt{(p/q)}}{2(s-1)}\,\phi\{\pi|\lambda\mu|\sqrt{(p/q)},\ s-1\}.$$

Hence, in both cases, the right-hand side becomes

$$\frac{\Gamma(s-b)}{\Gamma(s)}\,\frac{\{\pi i\lambda\sqrt{(p/q)}\}^b}{|\mu|^{2s-1-2b}}\,\frac{1}{p^{s-\frac{1}{2}(1+b)}\,q^{\frac{1}{2}(1+b)}}\,\phi\{\pi|\lambda\mu|\sqrt{(p/q)},\ s-b\}$$
$$= \frac{\Gamma(s-b)}{\Gamma(s)}\,\frac{(\pi i\lambda)^b\,\phi\{\pi|\lambda\mu|\sqrt{(p/q)},\ s-b\}}{|\mu|^{2s-1-b-b'}\,p^{s-\frac{1}{2}(1+b+b')}\,q^{\frac{1}{2}(1+b+b')}}. \tag{5.19''}$$

The double integrals are typified by

$$J=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{2\pi i\lambda(x+iy)+2\pi i\lambda'(x-iy)}(x+iy)^b(x-iy)^{b'}}{(p|\mu|^2+qx^2+qy^2)^{2s}}\,dx\,dy,$$

where $bb'=0$. We show that, if $\lambda\lambda'\neq 0$,

$$J=\frac{(2\pi i)^{b+b'}\lambda'^b\lambda^{b'}}{|\mu|^{4s-2-2b-2b'}p^{2s-1-b-b'}q^{1+b+b'}}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(2s-\frac{1}{2}-b-b')}{\Gamma(2s)}$$

$$\times\phi\{2\pi|\lambda\mu|\sqrt{(p/q)},\ 2s-\tfrac{1}{2}-b-b'\}. \quad (5.20)$$

It clearly suffices to put $b'=0$, $p=|\mu|=q=1$; so that what really must be proved is that, if $b\geqslant 0$ is an integer and $\lambda\lambda'\neq 0$, then

$$J(b)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{2\pi i\lambda(x+iy)+2\pi i\lambda'(x-iy)}(x+iy)^b}{(1+x^2+y^2)^{2s}}\,dx\,dy$$

$$=\frac{(2\pi i)^b\lambda'^b\Gamma(\frac{1}{2})\,\Gamma(2s-\frac{1}{2}-b)}{\Gamma(2s)}\int_{-\infty}^{\infty}\frac{e^{4\pi i(\lambda\lambda')^{\frac{1}{2}}z}\,dz}{(z^2+1)^{2s-\frac{1}{2}-b}}. \quad (5.20')$$

This is true when $b=0$. For write

$$\lambda=\xi+i\eta,\quad \lambda'=\xi-i\eta,\quad |\lambda\lambda'|^{\frac{1}{2}}=(\xi^2+\eta^2)^{\frac{1}{2}},$$

and change the variable x, y of integration to X, Y, where

$$\xi x-\eta y=X(\xi^2+\eta^2)^{\frac{1}{2}},\quad \eta x+\xi y=Y(\xi^2+\eta^2)^{\frac{1}{2}},$$

so that
$$(\xi+i\eta)(x+iy)=(\xi^2+\eta^2)^{\frac{1}{2}}(X+iY).$$

Then
$$J(0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{4\pi i(\lambda\lambda')^{\frac{1}{2}}X}}{(1+X^2+Y^2)^{2s}}\,dX\,dY.$$

Write $Y(X^2+1)^{\frac{1}{2}}$ for Y and this becomes

$$J(0)=\int_{-\infty}^{\infty}\frac{dY}{(Y^2+1)^{2s}}\int_{-\infty}^{\infty}\frac{e^{4\pi i(\lambda\lambda')^{\frac{1}{2}}X}\,dX}{(X^2+1)^{2s-\frac{1}{2}}},$$

which is (5.20′) with $b=0$.

We prove (5.20′) by induction by differentiating both sides with respect to λ. The differentiation under the integral sign is clearly valid if $R(s)$ is sufficiently great, and then by the theory of analytical continuation for all values of s for which the integrals have a meaning.

Now
$$\frac{d}{d\lambda}[J(b)]=2\pi iJ(b+1).$$

Also

$$\frac{d}{d\lambda}\,\phi\{2\pi(\lambda\lambda')^{\frac{1}{2}},\ 2s-\tfrac{1}{2}-b\} = 2\pi i\left(\frac{\lambda'}{\lambda}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{e^{4\pi i(\lambda\lambda')^{\frac{1}{2}}z}\,z\,dz}{(z^2+1)^{2s-\frac{1}{2}-b}}$$

$$= -\frac{\pi i(\lambda'/\lambda)^{\frac{1}{2}}}{2s-\frac{3}{2}-b}\int_{-\infty}^{\infty} e^{4\pi i(\lambda\lambda')^{\frac{1}{2}}z}\,d\,\frac{1}{(z^2+1)^{2s-\frac{3}{2}-b}}$$

$$= \frac{(2\pi i)^2\lambda'}{2s-\frac{3}{2}-b}\int_{-\infty}^{\infty}\frac{e^{4\pi i(\lambda\lambda')^{\frac{1}{2}}z}}{(z^2+1)^{2s-\frac{3}{2}-b}}\,dz.$$

This proves the result since

$$\Gamma(2s-\tfrac{1}{2}-b) = (2s-\tfrac{3}{2}-b)\,\Gamma(2s-\tfrac{3}{2}-b).$$

We next consider the case $\lambda = 0$. The single integral in (5.19′) reduces to

$$\int_{-\infty}^{\infty}\frac{x^b\,dx}{(p|\mu|^2+qx^2)^s} = \frac{1}{|\mu|^{2s-1}p^{s-\frac{1}{2}}q^{\frac{1}{2}}}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(s-\frac{1}{2})}{\Gamma(s)},$$

if $b = 0$, and 0 if $b = 1$. The value in both cases, since $b = b'$, can be written as

$$\frac{0^b}{|\mu|^{2s-1-b-b'}p^{s-\frac{1}{2}(1+b+b')}q^{\frac{1}{2}(1+b+b')}}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(s-\frac{1}{2})}{\Gamma(s)},$$

where $0^b = 1$ if $b = 0$.

The double integral, when $\lambda = 0$, is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x+iy)^b(x-iy)^{b'}\,dx\,dy}{(p|\mu|^2+qx^2+qy^2)^{2s}}$$

$$= \frac{1}{|\mu|^{4s-2-b-b'}p^{2s-1-\frac{1}{2}(b+b')}q^{1+\frac{1}{2}(b+b')}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(x+iy)^b(x-iy)^{b'}\,dx\,dy}{(1+x^2+y^2)^s}.$$

Change to polar coordinates; since $bb' = 0$, the last integral is

$$\int_0^{\infty}\int_0^{2\pi}\frac{r^{b+b'+1}e^{i\theta(b-b')}}{(r^2+1)^{2s}}\,dr\,d\theta = 0 \qquad (b\neq 0 \text{ or } b'\neq 0),$$

$$= \frac{\pi}{2s-1} \qquad (b=0,\ b'=0).$$

Hence the double integral becomes

$$\frac{0^{b+b'}\pi^{1-b-b'}}{|\mu|^{4s-2-b-b'}p^{2s-(1+b+b')}q^{1+b+b'}}\,\frac{\Gamma(2s+b+b'-1)}{\Gamma(2s)}.$$

The corresponding term of the summation when $\lambda = 0$ can then be written as

$$\frac{N(0^b)\,N(p^{-s+\frac{1}{2}(1+b+b')}q^{-\frac{1}{2}(1+b+b')})}{N|\mu|^{2s-1-b}} \prod_{t=1}^{r_1} \frac{\Gamma(\frac{1}{2})\,\Gamma(s_t-\frac{1}{2})}{\Gamma(s_t)}$$

$$\times \prod_{t=r_1+1}^{r_1+r_2} \frac{(2\pi)^{1-b_t-b_t}\,\Gamma(2s_t+b_t+b_t'-1)}{\Gamma(2s_t)}. \quad (5.21)$$

We now require a few simple properties of the integral

$$\phi(s) = \phi(g, s) = \int_{-\infty}^{\infty} \frac{e^{2igx}\,dx}{(x^2+1)^s},$$

where $g \neq 0$ is real and may be assumed positive since $\phi(g, s)$ is an even function of g. I prove that*

(5 A) $\phi(s)$, defined by the integral for $R(s) > 0$, is an integral function of s;

(5 B) $\phi(s) = O(g^P e^{-g})$ for large g, where the constant P and that implied in O are independent of s for bounded s.

(5 C) $$\Gamma(s)\,\phi(s) = g^{2s-1}\,\Gamma(1-s)\,\phi(1-s).$$

The integral for $\phi(s)$ obviously converges uniformly for

$$R(s) \geqslant 1+\delta > 0.$$

Integrate by parts and so

$$\phi(s) = \frac{-1}{2ig} \int_{-\infty}^{\infty} e^{2igx} \frac{d}{dx}(x^2+1)^{-s}\,dx.$$

This integral obviously converges uniformly for bounded s and

$$R(s) \geqslant \delta > 0,$$

and defines $\phi(s)$ for these values. Then

$$\phi(s) = \frac{1}{(2ig)^2} \int_{-\infty}^{\infty} e^{2igx} \frac{d^2}{dx^2}(x^2+1)^{-s}\,dx$$

for bounded s and $R(s) \geqslant \delta > -1$. By continuing this process, we see that $\phi(s)$ is defined for all values of s. It is also easy to see† that $\phi(s)$ is an integral function of s.

---

* Proofs can also be based on properties of modified Bessel functions.

† *Cf.* Landau, *Zahlentheorie* 1, 198.

Next, if $R(s) > 0$,

$$\frac{\Gamma(s)}{(x^2+1)^s} = \int_0^\infty e^{-t(x^2+1)}\, t^{s-1}\, dt,$$

so that $$\Gamma(s)\,\phi(s) = \int_0^\infty dx \int_{-\infty}^\infty e^{-t(x^2+1)+2igx}\, t^{s-1}\, dt.$$

Integrate first with respect to x, an inversion which is justified by de la Vallée Poussin's theorem, certainly when $R(s) > \frac{1}{2}$, since the double integral then converges absolutely. We have

$$\Gamma(s)\,\phi(s) = \pi^{\frac{1}{2}} \int_0^\infty e^{-t-g^2/t}\, t^{s-\frac{3}{2}}\, dt,$$

and this integral converges uniformly for all bounded s, since, for $0 < t < \infty$,

$$t^{s-\frac{3}{2}} = O(e^{\frac{1}{2}t+g^2/2t}),$$

the implied constant depending only on g.

Hence the new integral defines $\phi(s)$ for all s. Next change t into g^2/t, whence

$$\Gamma(s)\,\phi(s) = \pi^{\frac{1}{2}} g^{2s-1} \int_0^\infty e^{-t-g^2/t}\, t^{-s-\frac{3}{2}}\, dt,$$

and this is (5 c).

To prove (5 B), suppose that s is bounded and write

$$\Gamma(s)\,\phi(s) = \pi^{\frac{1}{2}}(I_1+I_2),$$

where $$I_1 = \int_g^\infty e^{-t-g^2/t}\, t^{s-\frac{3}{2}}\, dt, \quad I_2 = \int_0^g e^{-t-g^2/t}\, t^{s-\frac{3}{2}}\, dt.$$

Suppose first that $R(s) \geqslant \frac{1}{2}+\delta > \frac{1}{2}$. Then, since $t^2+g^2 > 2tg$,

$$|I_2| < e^{-2g} \int_0^g t^{R(s)-\frac{3}{2}}\, dt.$$

Also, since $$\tfrac{1}{2}t + \frac{g^2}{t} > g\sqrt{2},$$

$$|I_1| < e^{-g\sqrt{2}} \int_0^\infty e^{-\frac{1}{2}t}\, t^{R(s)-\frac{3}{2}}\, dt.$$

Hence $$|I_1|+|I_2| = O[e^{-g}(1+g^{R(s)-\frac{1}{2}})],$$

the implied constant being independent of s.

When $R(s) \leqslant \frac{1}{2}-\delta$, we see from (5 c) that

$$\phi(s) = g^{2s-1}\, O[e^{-g}(1+g^{-R(s)+\frac{1}{2}})].$$

Finally, when $\frac{1}{2}-\delta \leqslant R(s) \leqslant \frac{1}{2}+\delta$,

$$|I_1| \leqslant \int_g^\infty e^{-t-g^2/t}\frac{dt}{t^{1-\delta}} < e^{-g}\int_g^\infty e^{-\frac{1}{2}(t+g^2/t)}\frac{dt}{t^{1-\delta}}$$

$$< \frac{e^{-g}}{g^{1-\delta}}\int_g^\infty e^{-\frac{1}{2}t}\,dt.$$

Also $$|I_2| < \int_0^g e^{-t-g^2/t}\frac{dt}{t^{\frac{3}{2}-R(s)}} = \frac{1}{g^{1-2R(s)}}\int_g^\infty e^{-t-g^2/t}\frac{dt}{t^{\frac{1}{2}+R(s)}}$$

on writing t for g^2/t. These together prove (5 B)*.

We now deduce the uniform convergence of a series of the type of (5.22) below for bounded s. The method is illustrated sufficiently by taking a cubic field with real conjugates. On noting the inequality (5 B) and that $N|\lambda|$, $N|\mu|$ are bounded below, we need deal only with a series such as

$$\sum_{\substack{\mathfrak{a}\,|\,(\mu)\\ \mathfrak{b}\,|\,\lambda}} |N(\lambda)^A N(\mu)^B| e^{-2C(|\lambda^{(1)}\mu^{(1)}|+|\lambda^{(2)}\mu^{(2)}|+|\lambda^{(3)}\mu^{(3)}|)},$$

where $C>0$, $A\geqslant 0$, $B\geqslant 0$ are constants, and $\lambda=0$ and $\mu=0$ are omitted from the summation.

Since, for x, y, z all positive,

$$x+y+z \geqslant 3(xyz)^{\frac{1}{3}},$$

it suffices to consider

$$\sum_{\substack{\mathfrak{a}\,|\,(\mu)\\ \mathfrak{b}\,|\,\lambda}} |N(\lambda)^A N(\mu)^B| e^{-3C|N(\lambda)N(\mu)|^{\frac{1}{3}}-C(|\lambda^{(1)}\mu^{(1)}|+|\lambda^{(2)}\mu^{(2)}|+|\lambda^{(3)}\mu^{(3)}|)}.$$

Since $|N(\lambda)|$, $|N(\mu)|$ have finite lower bounds which are not zero, and depend only on the ideals $\mathfrak{b}$, $\mathfrak{a}$ respectively,

$$|N(\lambda)^A N(\mu)^B| = O|N(\lambda\mu)^{A+B}|,$$

$$N(\lambda)^A N(\mu)^B e^{-3C|N(\lambda\mu)|^{\frac{1}{3}}} = O\left(\frac{1}{|N(\lambda\mu)^2|}\right) = O\left(\frac{1}{|N(\mu)^2|}\right),$$

where the constants implied in O are independent of λ, μ.

Now group together the terms given by

$$L_1 < |\lambda^{(1)}| \leqslant L_1+1, \quad L_2 < |\lambda^{(2)}| \leqslant L_2+1, \quad L_3 < |\lambda^{(3)}| \leqslant L_3+1,$$

so that the number of terms in the group has a maximum value independent of L_1, L_2, L_3. Take first the case when L_1, L_2, L_3 run through all the integers 1 to ∞.

Then $$\sum_{L_1=1}^{\infty} e^{-C|\mu^{(1)}|L_1} = \frac{1}{e^{C|\mu^{(1)}|}-1} < \frac{1}{C|\mu^{(1)}|},$$

* Since $\phi(s)$ is an even function of g, (5 B) holds for all real g.

and so this sum in L_1, L_2, L_3 gives a series $\sum'_{\mathfrak{a}\,|\,(\mu)} O[|N(\mu)|^{-3}]$ which converges absolutely.

We consider next the case when some of the L's are zero.

Suppose, for example, that $L_1 = L_2 = 0$. We note the inequalities

$$e^{-C|\lambda^{(1)}\mu^{(1)}|-C|\lambda^{(2)}\mu^{(2)}|} < 1, \quad \sum_{L_3=1}^{\infty} e^{-C|\mu^{(3)}|L_3} < \frac{1}{C|\mu^{(3)}|},$$

so that we must discuss

$$\sum'_{\mathfrak{a}\,|\,(\mu)} \frac{1}{|\mu^{(3)}||N(\mu)^2|}.$$

Since the summation refers to non-associated numbers μ, two of the conjugates of μ are bounded above and below, so that, writing, if need be, $1/\mu^{(3)} = \mu^{(1)}\mu^{(2)}/N(\mu)$, the convergence follows from that of $\sum_{\mathfrak{a}\,|\,(\mu)} |N(\mu)|^{-2}$.

Similarly, if only $L_1 = 0$; while, if $L_1 = L_2 = L_3 = 0$, no question of convergence arises since the corresponding terms are finite in number. This proves then that the series in (5 . 22) below is an integral function of s.

After having applied the summation formula to $\sum_{\nu \equiv \beta}$ in (5 . 18), we multiply the result by $G(\beta)$ and sum in β for the residues divisible by $\mathfrak{b}$ of a complete set mod $\mathfrak{bg}$. Multiply the result by $F(\mu)\mu^{\alpha}$ and sum for μ as indicated in (5 . 18). We have, on noting the terms corresponding to $\mu = 0$, $\lambda = 0$, and (5 . 19″), (5 . 20), (5 . 21),

$$\Phi\begin{pmatrix} s, & \mathfrak{a}, & \mathfrak{b} \\ F, & G, & a, & b \end{pmatrix} = F(0)\,N(0^a)\,N(q^{-s})\,\zeta(2s-b,\ \mathfrak{b},\ G,\ b)$$

$$+ \frac{\bar{G}(0)\,N(0^b)}{d^{\frac{1}{2}}N(\mathfrak{bg})\,N(p^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'}q^{\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}b'})} \prod_{t=1}^{r_1} \frac{\Gamma(\frac{1}{2})\,\Gamma(s_t-\frac{1}{2})}{\Gamma(s_t)}$$

$$\times \prod_{t=r_1+1}^{r_1+r_2} (2\pi)^{1-b_t-b'_t} \frac{\Gamma(2s_t+b_t+b'_t-1)}{\Gamma(2s_t)}\,\zeta(2s-1-a-b-b',\ \mathfrak{a},\ F,\ a)$$

$$+ \sum_{\substack{\mathfrak{a}\,|\,(\mu) \\ 1/(\mathfrak{bgd})\,|\,\lambda}} \frac{F(\mu)\,\bar{G}(-\lambda)\,N\{(\pi i)^b\lambda^{b'}\mu^a\}}{d^{\frac{1}{2}}N(\mathfrak{bg})\,N(p^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'}q^{\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}b'})\,N|\mu|^{2s-1-b-b'}}$$

$$\times \prod_{t=1}^{r_1} \frac{\Gamma(s_t-b_t)}{\Gamma(s_t)} \prod_{t=r_1+1}^{r_1+r_2} \frac{\Gamma(\frac{1}{2})\,\Gamma(2s_t-\frac{1}{2}-b_t-b'_t)}{\Gamma(2s_t)}$$

$$\times 2^{1+b_t+b'_t}\,\chi(s-b,\ 2s-\tfrac{1}{2}-b-b',\ \lambda,\ \mu), \qquad (5\ .\ 22)$$

where

$$\chi(s-b,\ 2s-\tfrac{1}{2}-b-b',\ \lambda,\ \mu)$$

$$= \prod_{t=1}^{r_1} \phi(\pi\,|\,\lambda_t\mu_t\,|\,p_t^{\frac{1}{2}}q_t^{-\frac{1}{2}},\ s_t-b_t) \prod_{t=r_1+1}^{r_1+r_2} \phi(2\pi\,|\,\lambda_t\mu_t\,|\,p_t^{\frac{1}{2}}q_t^{-\frac{1}{2}},\ 2s_t-\tfrac{1}{2}-b_t-b'_t).$$

The summation in μ refers to all the non-associated numbers mod U of the ideal $\mathfrak{a}$, while that in λ refers to all the numbers of the ideal $1/(\mathfrak{bgd})$, except that $\lambda = 0$ is excluded from the summation, as is also $\mu = 0$.

Since it has been shown that the series in (5.22), when the two products of gamma functions of s_t are omitted from the numerator, is an integral function of s, converging absolutely for all s, the properties of the zeta and Φ functions are now easily studied.

We require a slight transformation of (5.22). On applying (5 c), we have

$$\begin{aligned}
&\chi(s-b,\ 2s-\tfrac{1}{2}-b-b',\ \lambda,\ \mu)\\
&\quad=\prod_{t=1}^{r_1}\frac{\Gamma(1+b_t-s_t)}{\Gamma(s_t-b_t)}\,\phi(\pi\,|\,\lambda_t\mu_t\,|\,p_t^{\frac{1}{2}}q_t^{-\frac{1}{2}},\ 1+b_t-s_t)(\pi\,|\,\lambda_t\mu_t\,|\,p_t^{\frac{1}{2}}q_t^{-\frac{1}{2}})^{2s_t-2b_t-1}\\
&\qquad\times\prod_{t=r_1+1}^{r_1+r_2}\frac{\Gamma(\frac{3}{2}+b_t+b_t'-2s)}{\Gamma(2s_t-\frac{1}{2}-b_t-b_t')}\,\phi(2\pi\,|\,\lambda_t\mu_t\,|\,p_t^{\frac{1}{2}}q_t^{-\frac{1}{2}},\ \tfrac{3}{2}+b_t+b_t'-2s)\\
&\qquad\times(2\pi\,|\,\lambda_t\mu_t\,|\,p_t^{\frac{1}{2}}q_t^{-\frac{1}{2}})^{4s_t-2b_t-2b_t'-2}\\
&\quad=N(\pi\,|\,\lambda\mu\,|\,p^{\frac{1}{2}}q^{-\frac{1}{2}})^{2s-b-b'-1}\prod_{t=1}^{r_1}\frac{\Gamma(1+b_t-s_t)}{\Gamma(s_t-b_t)}\\
&\quad\times\prod_{t=r_1+1}^{r_1+r_2}2^{4s_t-2b_t-2b_t'-2}\frac{\Gamma(\frac{3}{2}+b_t+b_t'-2s_t)}{\Gamma(2s_t-\frac{1}{2}-b_t-b_t')}\,\chi(1+b-s,\ \tfrac{3}{2}+b+b'-2s,\ \lambda,\ \mu).
\end{aligned}$$

Substitute this in (5.22) and note that, in the symbol of summation $\mathfrak{a}\,|\,(\mu)$, $1/(\mathfrak{bgd})\,|\,\lambda$ can be replaced by $\mathfrak{a}\,|\,\mu$, $1/(\mathfrak{bgd})\,|\,(\lambda)$. For the change of μ into $\epsilon\mu$ and λ into $\epsilon^{-1}\lambda$, where ϵ is a unit of U, leaves the general term unaltered since ϕ involves the product $\lambda\mu$ and

$$e^{\pi i F_\epsilon+\pi i G_\epsilon}N(\epsilon^a\epsilon^{-b'})=N\,|\,\epsilon\,|^{2s-1-b-b'}$$

follows easily from (5.16). Hence (5.22) can be written

$$\begin{aligned}
&\phi\begin{pmatrix}s,\ \mathfrak{a},\ \mathfrak{b}\\ F,\ G,\ a,\ b\end{pmatrix}\\
&=F(0)\,N(0^a)\,N(q^{-s})\,\xi(2s-b,\ \mathfrak{b},\ G,\ b)+\frac{N(0^b)\,\bar G(0)}{d^{\frac{1}{2}}\,N(\mathfrak{bg})\,N(p^{s-\frac{1}{2}(1+b+b')}q^{\frac{1}{2}(1+b+b')})}\\
&\qquad\times\prod_{t=1}^{r_1}\frac{\Gamma(\frac{1}{2})\,\Gamma(s_t-\frac{1}{2})}{\Gamma(s_t)}\prod_{t=r_1+1}^{r_1+r_2}(2\pi)^{1-b_t-b_t'}\frac{\Gamma(2s_t+b_t+b'-1)}{\Gamma(2s_t)}\\
&\qquad\times\xi(2s-1-a-b-b',\ \mathfrak{a},\ F,\ a)\\
&+\sum_{\substack{\mathfrak{a}\,|\,\mu\\ 1/(\mathfrak{bgd})\,|\,(\lambda)}}\frac{F(\mu)\,\bar G(-\lambda)\,N(\pi^{2s-1-\frac{1}{2}b-\frac{1}{2}b'}i^b\lambda^{b'}\mu^a)}{d^{\frac{1}{2}}\,N(\mathfrak{bg})\,N(q^s)\,N\,|\,\lambda\,|^{-2s+b+b'+1}}\prod_{t=1}^{r_1}\frac{\Gamma(1+b_t-s_t)}{\Gamma(s_t)}\\
&\qquad\times\prod_{t=r_1+1}^{r_1+r_2}2^{4s_t-1-b_t-b_t'}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2}+b_t+b_t'-2s_t)}{\Gamma(2s_t)}\\
&\qquad\times\chi(1+b-s,\ \tfrac{3}{2}+b+b'-2s,\ \lambda,\ \mu).\qquad(5.23)
\end{aligned}$$

Here again the series, omitting the product of the gamma functions in the numerator, is an integral function of s.

But the series such as those in (5.22), (5.23) still represent integral functions of s even when multiplied by the gamma functions occurring therein. Suppose that in (5.22) a number of the factors $\Gamma(s_t-b_t)$, $\Gamma(2s_t-\frac{1}{2}-b_t-b_t')$ become infinite for the same value of s. On applying (5 c) to the corresponding ϕ functions [as was done for example with all the ϕ functions to obtain (5.23)], these factors are replaced by $\Gamma(1+b_t-s_t)$, $\Gamma(\frac{3}{2}+b_t+b_t'-2s_t)$, which do not become infinite for the particular value of s.

We can now discuss the behaviour of the zeta and Φ functions. In (5.22), interchange $\mathfrak{a}$, $\mathfrak{b}$; F, G; a, b; p, q; and then subtract so that the left-hand side vanishes.

Suppose first of all that all the a and b are zero. Put $\mathfrak{a}=\mathfrak{b}$, $F=G$, and we have

$$F(0)\{N(q^{-s})-N(p^{-s})\}\,\zeta(2s,\mathfrak{a},F,0)+\frac{\bar{F}(0)}{d^{\frac{1}{2}}N(\mathfrak{a}\mathfrak{f})}\prod_{t=1}^{r_1}\frac{\Gamma(\frac{1}{2})\,\Gamma(s_t-\frac{1}{2})}{\Gamma(s_t)}$$
$$\times\prod_{t=r_1+1}^{r_1+r_2}\frac{2\pi}{2s_t-1}\,[N(p^{-s+\frac{1}{2}}q^{-\frac{1}{2}})-N(q^{-s+\frac{1}{2}}p^{-\frac{1}{2}})]\,\zeta(2s-1,\mathfrak{a},F,0)=G_1(s), \tag{5.24}$$

say, where $G_1(s)$ is an integral function of s.

If $F(0)=0$, $\bar{F}(0)\neq 0$, write $\frac{1}{2}+\frac{1}{2}s_t$ for s_t; then $\zeta(s,\mathfrak{a},F,0)$ is regular for all s except possibly for the poles of $\Gamma(\frac{1}{2}+\frac{1}{2}s_t)$ $(t=1,2,\ldots,r_1)$; and for the zeros of $N(q^{-\frac{1}{2}s+\frac{1}{2}})-N(p^{-\frac{1}{2}s+\frac{1}{2}})$ independent of p and q. Take all the q equal to 1, and then all the p equal to 1 with one exception if there are real fields among k and its conjugates, or with two exceptions if k and its conjugates are all imaginary. Hence there may be a simple pole at $s=1$ if $\sigma_t=0$ for $t=1,2,\ldots,n$. It will be seen in a moment that no singularities arise from the gamma function $\Gamma(\frac{1}{2}+\frac{1}{2}s_t)$.

Next suppose that $\bar{F}(0)=0$, $F(0)\neq 0$, and write $\frac{1}{2}s_t$ for s_t; hence $\zeta(s,\mathfrak{a},F,0)$ is regular for all s except possibly for $s=0$ when $\sigma_t=0$ $(t=1,2,\ldots,n)$ arising from the zeros of $N(q^{-\frac{1}{2}s})-N(p^{-\frac{1}{2}s})$ independent of p and q. But then $s=0$ cannot be a pole; for the functional equation (5.14), which will be proved later, and now takes the form

$$\zeta\{1-s,1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\bar{F},0\}$$
$$=\zeta(s,\mathfrak{a},F,0)\,d^{\frac{1}{2}}N(\mathfrak{a}\mathfrak{f})\,N(\pi^{\frac{1}{2}-s})\prod_{t=1}^{r_1}\frac{\Gamma(\frac{1}{2}s_t)}{\Gamma(\frac{1}{2}-\frac{1}{2}s_t)}\prod_{t=r_1+1}^{r_1+r_2}2^{1-2s_t}\,\frac{\Gamma(s_t)}{\Gamma(1-s_t)},$$

shows that $\zeta\{s,1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\bar{F},0\}$ would have a pole of order $1+r_1+r_2$ at $s=1$; this contradicts the result just found that it is at most a pole

of order 1. So, if poles arose in the first case from the gamma function $\Gamma(\frac{1}{2}+\frac{1}{2}s_t)$, these would imply the existence of poles $s \neq 0$ in the second case.

If $F(0) \neq 0$, $\bar{F}(0) \neq 0$, then (5.24) is really a recurrence formula for the zeta function, defined so far when $R(s) > 1$. Change s_t into $\frac{1}{2}s_t+\frac{1}{2}$ and we see that $\zeta(s, \mathfrak{a}, F, 0)$ is a meromorphic function of s, first for $R(s) > 0$, and then also for $R(s) > -1, -2, \ldots$. Its possible poles for $R(s) > 0$ are given by the zeros independent of p and q of

$$N(p^{-\frac{1}{2}s+\frac{1}{2}})-N(q^{-\frac{1}{2}s+\frac{1}{2}}).$$

Hence $s = 1$ may be a simple pole if $\sigma_t = 0$ $(t = 1, 2, \ldots, n)$. The functional equation (5.14) then shows that $\zeta(s, \mathfrak{a}, F, 0)$ is analytic for $R(s) \leqslant 0$ and has a zero at $s = 0$ of order r_1+r_2-1 when $s = 1$ is a pole. When $s = 1$ is not a pole, the zero at $s = 0$ is of multiplicity τ, where τ is the number of σ equal to zero among $\sigma_1, \sigma_2, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \ldots, \sigma_{r_1+r_2}$.

If $F(0) = 0$, $\bar{F}(0) = 0$, and also when all the a are not zero, $\zeta(s, \mathfrak{a}, F, a)$ is an integral function of s. For, after the interchange of $\mathfrak{a}, \mathfrak{b}$, etc., in (5.22) and subtraction, instead of taking $\mathfrak{a} = \mathfrak{b}$, $F = G$, and deducing (5.24), we take all the $b = 0$; hence, since $F(0)N(0^a) = 0$,

$$-G(0)N(p^{-s})\zeta(2s-a, \mathfrak{a}, F, a)+\frac{\bar{G}(0)N(p^{-s+\frac{1}{2}}q^{-\frac{1}{2}})}{d^{\frac{1}{2}}N(\mathfrak{b}\mathfrak{g})}\prod_{t=1}^{r_1}\frac{\Gamma(\frac{1}{2})\Gamma(s_t-\frac{1}{2})}{\Gamma(s_t)}$$

$$\times\prod_{t=r_1+1}^{r_1+r_2}\frac{2\pi}{2s_t-1}\zeta(2s-1-a, \mathfrak{a}, F, a) = G_2(s), \quad (5.25)$$

where $G_2(s)$ is an integral function of s. Since the character G is arbitrary we may suppose for the moment* that $\bar{G}(0) = 0$, $G(0) \neq 0$, whence $\zeta(2s-a, \mathfrak{a}, F, a)$ is an integral function of s, and the result follows.

Hence $\zeta(s, \mathfrak{a}, F, a)$ is an analytic function of s throughout the whole s plane, except possibly for a simple pole at $s = 1$ when $\bar{F}(0) \neq 0$ and $a_t = \sigma_t = 0$ for $t = 1, 2, \ldots, n$. If $F(\mu) \geqslant 0$, then $s = 1$ is actually a pole, for otherwise the series for the zeta function, having its terms positive, would converge throughout the s plane by Landau's theorem, whereas it clearly diverges for $R(s) < 0$.

Next take the Φ function. We see from (5.22) that Φ is an integral function of s if all the b are not zero and so by symmetry if all the a are not zero. Suppose, then, that all the a and b are zero, so that (5.22)

* From (5.8) we may take $\bar{G}(\nu) = 0$ if $\nu \equiv 0 \pmod{1/\mathfrak{b}\mathfrak{b}}$, $\bar{G}(\nu) = 1$ if $\nu \not\equiv 0 \pmod{1/\mathfrak{b}\mathfrak{b}}$, and then from (5.9) $N(\mathfrak{g})G(0) = N(\mathfrak{g})-1$.

becomes

$$\Phi\begin{pmatrix} s, \mathfrak{a}, \mathfrak{b} \\ F, G, 0, 0 \end{pmatrix} = F(0)\,N(q^{-s})\,\xi(2s, \mathfrak{b}, G, 0) + \frac{\bar{G}(0)}{d^{\frac{1}{2}}N(\mathfrak{b}\mathfrak{g})\,N(p^{s-\frac{1}{2}}q^{\frac{1}{2}})}$$
$$\times \prod_{t=1}^{r_1} \frac{\Gamma(\frac{1}{2})\,\Gamma(s_t-\frac{1}{2})}{\Gamma(s_t)} \prod_{t=r_1+1}^{r_1+r_2} \frac{2\pi}{2s_t-1}\,\xi(2s-1, \mathfrak{a}, F, 0) + G_3(s), \tag{5.26}$$

where $G_3(s)$ is an integral function of s. Hence Φ is now an integral function of s if $\bar{G}(0) = 0$, and by symmetry if $\bar{F}(0) = 0$. Suppose then that $\bar{F}(0) \neq 0$, $\bar{G}(0) \neq 0$. Clearly Φ may have a simple pole at $s = 1$ if $\sigma_t = 0$ for $t = 1, 2, \ldots, n$. Excluding $s = 1$, it is seen from (5.25), on taking therein $a = 0$, that the middle term of (5.26) is a regular function of s, except possibly for a simple pole at $s = \frac{1}{2}$, if $\sigma_t = 0$ when $t = 1, 2, \ldots, n$. Clearly $s = \frac{1}{2}$ is not a pole if $F(0) = G(0) = 0$.

Hence $\Phi\begin{pmatrix} s, \mathfrak{a}, \mathfrak{b} \\ F, G, a, b \end{pmatrix}$ is an analytic function of s throughout the whole s plane except that when $\bar{F}(0) \neq 0$, $\bar{G}(0) \neq 0$, $a_t = b_t = \sigma_t = 0$, for $t = 1, 2, \ldots, n$, there may be a simple pole at $s = 1$ and also at $s = \frac{1}{2}$, the latter not being a pole if $F(0) = G(0) = 0$.

We come finally to the functional equations. In (5.22) replace

$$\mathfrak{a},\ \mathfrak{b},\ a,\ b,\ F(\mu),\ \bar{G}(\lambda),\ F(0),\ \bar{G}(0),\ s_t$$

by

$$1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}),\ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\ b',\ a',\ \bar{G}(\lambda),\ N(\mathfrak{f})\,F(-\mu),\ \bar{G}(0),\ N(\mathfrak{f})\,F(0),\ 1-s_t+E_t$$

where $$2E_t = a_t + a_t' + b_t + b_t',$$

so that for $t = 1, 2, \ldots, r_1$, $E_t = a_t + b_t$. Write then $-\mu$ for μ, and we have

$$\Phi\begin{pmatrix} 1-s+E, & 1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}), & 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}) \\ \bar{G}, & \bar{F}, & b', a' \end{pmatrix}$$
$$= \bar{G}(0)\,N(0^{b'})\,N(q^{-1+s-E})\,\xi\{2-2s+a+b+b',\ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\ \bar{F},\ a'\}$$
$$+ \frac{F(0)\,N(0^{a'})\,N(\mathfrak{a}\mathfrak{f})\,d^{\frac{1}{2}}}{N(p^{\frac{1}{2}-s+\frac{1}{2}b+\frac{1}{2}b'}q^{\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}a'})} \prod_{t=1}^{r_1} \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-s_t+E_t)}{\Gamma(1-s_t+E_t)}$$
$$\times \prod_{t=r_1+1}^{r_1+r_2} (2\pi)^{1-a_t-a_t'} \frac{\Gamma(1-2s_t+2E_t+a_t+a_t')}{\Gamma(2-2s_t+2E_t)}\,\xi\{1-2s+b,\ 1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}),\ \bar{G},\ b'\}$$
$$+ \sum_{\substack{1/(\mathfrak{b}\mathfrak{g}\mathfrak{d})\mid(\lambda) \\ \mathfrak{a}\ \ \mu}} \frac{\bar{G}(\lambda)\,F(-\mu)\,N(\mathfrak{a}\mathfrak{f})\,d^{\frac{1}{2}}}{N(p^{\frac{1}{2}-s+\frac{1}{2}b+\frac{1}{2}b'}q^{\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}a'})} \frac{N(\pi^{\frac{1}{2}a+\frac{1}{2}a'}i^{-a}\mu^a\lambda^{b'})}{N|\lambda|^{-2s+b+b'+1}}$$
$$\times \prod_{t=1}^{r_1} \frac{\Gamma(1-s_t+E_t-a_t')}{\Gamma(1-s_t+E_t)} \prod_{t=r_1+1}^{r_1+r_2} \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2}-2s_t+b_t+b_t')}{\Gamma(2-2s_t+2E_t)}\,2^{1+a_t+a_t'}$$
$$\times \chi(1+b-s,\ \tfrac{3}{2}+b+b'-2s,\ \lambda,\ \mu).$$

Write this and (5 . 23) as

$$\Phi = C_1 N(q^{-s})\,\xi(2s-b,\ \mathfrak{b},\ G,\ b)+C_2 N(p^{-s+\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}b'}q^{-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'})$$
$$\times\xi(2s-1-a-b-b',\ \mathfrak{a},\ F,\ a)+C_3 N(q^{-s})\,\Sigma,$$

$$\Phi' = C_4 N(q^{-1+s-E})\,\zeta\{2-2s+a+b+b',\ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\ \bar{F},\ a'\}$$
$$+C_5 N(p^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'}q^{-\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}a'})\,\xi\{1-2s+b,\ 1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}),\ \bar{G},\ b'\}$$
$$+C_6 N(p^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'}q^{-\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}a'})\,\Sigma,$$

where Σ denotes in both cases the *same* series, since

$$F(-\mu) = (-1)^{F-1}F(\mu),\quad G(-\mu) = (-1)^{G-1}G(\mu),$$

and the C's are independent of p, q.

Hence, eliminating this series Σ,

$$C_6[N(q^s)\,\Phi - C_1\xi(2s-b,\ \mathfrak{b},\ G,\ b)$$
$$-C_2 N(q/p)^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'}\xi(2s-1-a-b-b',\ \mathfrak{a},\ F,\ a)]$$
$$= C_3[N(q^{\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}a'}p^{-s+\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}b'})\,\Phi' - C_4 N(q/p)^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'}$$
$$\times\zeta\{2-2s+a+b+b',\ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\ \bar{F},\ a'\} - C_5\xi\{1-2s+b,\ 1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}),\ \bar{G},\ b'\}].$$

Hence

$$[C_6 N(q^s)\,\Phi - C_3 N(q^{\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}a'}p^{-s+\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}b'})\,\Phi']$$
$$= [C_1 C_6\xi(2s-b,\ \mathfrak{b},\ G,\ b) - C_3 C_5\xi\{1-2s+b,\ 1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}),\ \bar{G},\ b'\}]$$
$$+N(q/p)^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'}[C_2 C_6\xi(2s-1-a-b-b',\ \mathfrak{a},\ F,\ a)$$
$$-C_3 C_4\xi\{2-2s+a+b+b',\ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\ \bar{F},\ a'\}].$$

Multiply by $N(p^s)$ and take $\mathfrak{b}=\mathfrak{a}$, $G=F$, $b=a$. Then, since Φ, Φ' become symmetrical functions of p and q, the right-hand side must also be symmetrical in p and q. Hence each of the three square brackets is zero since $\frac{1}{2}+\frac{1}{2}b+\frac{1}{2}b'\neq 0$. Write s for $2s-b$ in the second, and s for $2s-1-a-b-b'$ in the third, and then

$$\Phi\begin{pmatrix}1-s+\frac{1}{2}E,\ 1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}),\ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d})\\ \bar{G},\ \bar{F},\ b'\ a'\end{pmatrix} = D_1 N(p^{s-\frac{1}{2}-\frac{1}{2}b-\frac{1}{2}b'}q^{s-\frac{1}{2}-\frac{1}{2}a-\frac{1}{2}a'})\,\Phi\begin{pmatrix}s,\ \mathfrak{a},\ \mathfrak{b}\\ F,\ G,\ a,\ b\end{pmatrix},$$

$$\xi\{1-s,\ 1/(\mathfrak{b}\mathfrak{g}\mathfrak{d}),\ \bar{G},\ b'\} = D_2\xi(s,\ \mathfrak{b},\ G,\ b),$$

$$\xi\{1-s,\ 1/(\mathfrak{a}\mathfrak{f}\mathfrak{d}),\ \bar{F},\ a'\} = D_3\xi(s,\ \mathfrak{a},\ F,\ a),$$

where $D_1 = C_6/C_3$, $D_2 = C_1C_6/C_3C_5$, with s replaced by $\frac{1}{2}(s+b)$, or rather by $\frac{1}{2}s+\frac{1}{4}(b+b')$, since really s_t is replaced by $\frac{1}{2}s_t+\frac{1}{4}(b_t+b_t')$; $D_3 = C_2C_6/C_3C_4$, with s replaced by $\frac{1}{2}(s+1+a+b+b')$. Unless all the a are zero, D_2 takes the form $0/0$, and so does D_3 unless all the b are zero.

On noting from (5 . 16) that

$$(-1)^{F_{-1}+G_{-1}}N(i^{-a-b}) = N(i^{a+b}),$$

we have

$$D_1 = d\,N(\pi^{-2s+1+E}i^{a+b})N(\mathfrak{abfg})\prod_{t=1}^{r_1}\frac{\Gamma(s_t)}{\Gamma(1-s_t+E_t)} \times \prod_{t=r_1+1}^{r_1+r_2} 2^{2E_t+2-4s_t}\frac{\Gamma(2s_t)}{\Gamma(2-2s_t+2E_t)},$$

and this is (5 . 17).

Take next $a=0$, $F(0)=1$, so that $F_{-1}\equiv 0 \pmod 2$, since $F(0) = e^{\pi i F_{-1}}F(0)$. We have then to replace s by $\frac{1}{2}s+\frac{1}{4}(b+b')$ in

$$D_2' = \frac{C_1}{C_5}D_1 = (-1)G_{-1}N(i^{-b})N(\mathfrak{bg})\,d^{\frac{1}{2}}\frac{N(\pi^{1-2s+\frac{1}{2}(b+b')})}{\pi^{\frac{1}{2}r_1+r_2}}\prod_{t=1}^{r_1}\frac{\Gamma(s_t)}{\Gamma(\frac{1}{2}-s_t+b_t)} \times \prod_{t=r_1+1}^{r_1+r_2} 2^{b_t+b_t'+1-4s_t}\frac{\Gamma(2s_t)}{\Gamma(1-2s_t+b_t+b_t')}.$$

Noting (5 . 13), we have

$$D_2 = d^{\frac{1}{2}}N(\pi^{\frac{1}{2}-s}i^{b})N(\mathfrak{bg})\prod_{t=1}^{r_1}\frac{\Gamma(\frac{1}{2}s_t+\frac{1}{2}b_t)}{\Gamma(\frac{1}{2}+\frac{1}{2}b_t-\frac{1}{2}s_t)}\prod_{t=r_1+1}^{r_1+r_2}2^{1-2s_t}\frac{\Gamma(s_t+\frac{1}{2}b_t+\frac{1}{2}b_t')}{\Gamma(1-s_t+\frac{1}{2}b_t+\frac{1}{2}b_t')},$$

and this is (5 . 14).

A method of Hecke.

Some of the summations in this section refer to the values of μ non-associated with respect to U. We close the section by mentioning the very useful idea employed by Hecke to transform a sum extended over all the numbers associated with μ. Take, for example, the case of a real quadratic field, where $\epsilon > 1$ is a unit and the conjugate unit ϵ' is given by $\epsilon\epsilon' = 1$, and consider

$$\sum_{n=-\infty}^{\infty} f(|\mu|\epsilon^n,\ |\mu'|\epsilon'^n). \tag{5 . 27'}$$

If conditions are satisfied permitting the application of Poisson's formula

to the summation in n, this series becomes

$$\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(|\mu|\epsilon^x, |\mu'|\epsilon'^x) e^{2n\pi i x} dx.$$

Write $$x \log \epsilon = v + \tfrac{1}{2} \log \left| \frac{\mu'}{\mu} \right|,$$

and this becomes

$$\frac{1}{\log \epsilon} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(|\mu\mu'|^{\frac{1}{2}} e^v, |\mu\mu'|^{\frac{1}{2}} e^{-v}) \exp\left\{ \frac{n\pi i}{\log \epsilon} \log \left| \frac{\mu'}{\mu} \right| + \frac{2n\pi i v}{\log \epsilon} \right\} dv.$$

In many applications $f(x, y)$ is homogeneous of degree $-s$ in the variables x, y, so that this becomes

$$\frac{1}{\log \epsilon} \sum_{n=-\infty}^{\infty} \frac{\exp\left\{ \frac{n\pi i}{\log \epsilon} \log \left| \frac{\mu'}{\mu} \right| \right\}}{|\mu\mu'|^{\frac{1}{2}s}} \int_{-\infty}^{\infty} f(e^v, e^{-v}) \exp\left\{ \frac{2n\pi i v}{\log \epsilon} \right\} dv.$$

Write $I(n)$ for the integral.

Suppose now that $\mathfrak{a}$ is a given ideal and that μ is any number in $\mathfrak{a}$ excluding $\mu = 0$. The application of (5 . 27′) gives

$$\sum_{\mathfrak{a}|\mu}{}' f(\mu, \mu') = \frac{1}{\log \epsilon} \sum_{n=-\infty}^{\infty} I(n) \zeta\left(\frac{s}{2}, n\right), \qquad (5 . 27)$$

where $$\zeta(s, n) = \sum_{\mathfrak{a}.(\mu)}{}' \frac{\exp\left\{ i\pi n \log \left| \frac{\mu'}{\mu} \right| \Big/ \log \epsilon \right\}}{|N(\mu)|^s},$$

the summation in μ, a number of $\mathfrak{a}$, referring here to non-associated values of $|\mu|$ with respect to the group ϵ^m. We assume that the interchange of the summation in (μ) and n is allowed in (5 . 27). Hecke* uses this type of result to discuss the behaviour of the left-hand side as a function of a certain parameter, *e.g.*, s. The $\zeta(s, n)$ is merely a special case of the more general zeta function when we take

$$s_1 = s - \frac{i\pi n}{\log \epsilon}, \quad s_2 = s + \frac{i\pi n}{\log \epsilon}.$$

Similar results hold for any number of variables. A generalization of these results of Hecke has been given by Behnke†.

---

* Hecke, **7**.

† Behnke, **1**.

The formulae for Hecke's modular and allied functions.

6. We come now to the second class of results, which lie considerably deeper than the first, and make clearer the advantage of emphasizing the explicit enunciation of the summation formula. One result takes the following form for a real quadratic field $k = k(\sqrt{D})$ of discriminant D in which $\mathfrak{a}$ is any ideal and ϵ is any given totally positive unit, *i.e.* ϵ and its conjugate are both positive. There is no loss of generality in supposing that $\epsilon > 1$.

Let $\phi(x, y)$, or ϕ, say, be defined for all real values of x, y, and suppose that

$$\phi(\epsilon x, \epsilon' y) = \phi(x, y); \tag{6 A}$$

$$\frac{\partial^{\alpha+\beta}\phi}{\partial x^{\alpha} dy^{\beta}}, \quad (\alpha, \beta = 0, 1, 2), \tag{6 B}$$

are continuous functions of x, y, and, excluding $\alpha = \beta = 2$, vanish when either $x = 1$ or $|y| = \infty$, $1 \leqslant x < \epsilon$; also suppose that

$$\int_{-\infty}^{\infty}\int_{1}^{\epsilon}\left(|\phi| + \left|\frac{\partial^2\phi}{\partial x^2}\right| + \left|\frac{\partial^2\phi}{\partial y^2}\right| + \left|\frac{\partial^4\phi}{\partial x^2\partial y^2}\right|\right)dx\,dy, \tag{6 C}$$

converges.

Then, using dashes to indicate conjugate numbers, ideals, etc., we have the theorem.

Let c, c' be any two conjugate numbers of k such that

$$c(1-\epsilon) \equiv 0 \pmod{\mathfrak{a}}; \tag{6.1}$$

then

$$\sum_{\substack{\mu \equiv c(\mathfrak{a}) \\ (\mu)_*}}{}' \phi(\mu, \mu') = \frac{1}{N(\mathfrak{a})\sqrt{D}} \sum_{\substack{1 \\ (|\lambda|)_*}}{}' e^{-2\pi i(\lambda c + \lambda' c')} \int_0^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(\lambda\xi + \lambda'\eta)}\phi(\xi, \eta)\,d\eta\,d\xi$$
$$+ \frac{1}{N(\mathfrak{a})\sqrt{D}} \int_1^{\epsilon}\int_{-\infty}^{\infty} \phi(\xi, \eta)\,d\eta\,d\xi. \tag{6.2}$$

The notation indicates that only non-associated values of μ with respect to the group of units $1, \epsilon, \epsilon^2, \epsilon^3, \ldots$ are taken, so that we may suppose that $1 \leqslant \mu < \epsilon$ since $\mu = 0$ is to be omitted. The corresponding restriction on λ means that $1 \leqslant |\lambda| < \epsilon$, so that BOTH *$\lambda$ and $-\lambda$ occur in the summation if λ does.*

Similarly

$$\sum_{\substack{\mu \equiv c(\mathfrak{a}) \\ (|\mu|)_*}}{}' \phi(\mu, \mu') = \frac{1}{N(\mathfrak{a})\sqrt{D}} \sum_{\substack{1 \\ (|\lambda|)_*}}{}' e^{-2\pi i(\lambda c + \lambda' c')} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2\pi i(\lambda\xi + \lambda'\eta)}\phi(\xi, \eta)\,d\eta\,d\xi$$
$$+ \frac{1}{N(\mathfrak{a})\sqrt{D}} \int_{1 \leqslant |\xi| < \epsilon}\int_{-\infty}^{\infty} \phi(\xi, \eta)\,d\eta\,d\xi, \tag{6.3}$$

where the limits for ξ are given by $-\epsilon < \xi \leqq -1$ and $1 \leqq \xi < \epsilon$, provided that $\phi(x, y)$ satisfies the conditions (6 A), (6 B), (6 C), *when in* (6 B) *we read $|x| = 1$, or $|y| = \infty$, $1 \leqq |x| < \epsilon$, instead of $x = 1$, etc., and in* (6 C) *we replace* $\int_1^\epsilon$ *by* $\int_{1 \leqq |x| < \epsilon}$.

The existence of Hecke's new modular functions and similar results follow at once *formally* from the second of these results by taking $\phi(x, y) = e^{-\tau|xy|} \operatorname{sgn} x$, where τ is a complex variable with $R(\tau) > 0$. Since $\phi(x, y)$ does not satisfy the conditions (6 B), some justification must be given. This will be done by showing that the formulae (6 . 2), (6 . 3) are still valid for other sets of conditions. Thus Hecke's results are particular cases of a general expansion, and are found independently of both his theta and zeta functions.

The expansion (6 . 2) is included in the following.

Let a, b, c, a', b', c', $\epsilon > 1$, $\epsilon' = 1/\epsilon$ be any real numbers such that

$$\Delta = ab' - a'b \neq 0,$$

and neither a/b nor a'/b' is a rational number. Put

$$\mu = am + bn + c, \quad \mu' = a'm + b'n + c',$$

$$\Delta\lambda = b'm - a'n, \quad \Delta\lambda' = -bm + an.$$

Suppose that, as m, n run through all integer values from $-\infty$ to ∞,

(6 D) *the sets of values $(\epsilon\lambda, \epsilon'\lambda')$ and (λ, λ') are the same ;*

(6 E) $$\lambda c + \lambda' c' \equiv \epsilon\lambda c + \epsilon'\lambda' c' \pmod 1.$$

This means that, if t is any integer,

$$\lambda c + \lambda' c' \equiv \epsilon^t \lambda c + \epsilon'^t \lambda' c' \pmod 1.$$

Let also $\phi(x, y)$ satisfy the conditions (6 A), (6 B), (6 C). *Then*

$$\sum_{(\mu)_\epsilon} \phi(\mu, \mu') = \frac{1}{|\Delta|} \sum_{(|\lambda|)_\epsilon} e^{-2\pi i(\lambda c + \lambda' c')} \int_0^\infty \int_- e^{2\pi i(\lambda\xi + \lambda'\eta)} \phi(\xi, \eta)\, d\eta\, d\xi$$

$$+ \frac{1}{|\Delta|} \int_1^\epsilon \int_{-\infty}^{\infty} \phi(\xi, \eta)\, d\eta\, d\xi. \quad (6 . 4)$$

The left-hand summation is extended to all integer values of m, n which give non-associated values of $\mu \pmod \epsilon$, so that we may suppose that $1 \leqq \mu < \epsilon$. Similarly for λ, where, however, only non-associated values of $|\lambda|$ are taken.

This expansion is deduced from (2.1) by taking $f(x, y) = \phi(x, y)$ when $1 \leqq x \leqq \epsilon$, and $f(x, y) = 0$ for other values of x. This gives

$$\sum_{(\mu)_\epsilon} \phi(\mu, \mu') = \frac{1}{|\Delta|} \sum_{\lambda} e^{-2\pi i(\lambda c + \lambda' c')} \int_{-\infty}^{\infty} \int_{1}^{\epsilon} e^{2\pi i(\lambda\xi + \lambda'\eta)} \phi(\xi, \eta) d\xi d\eta, \quad (6.5)$$

where, on the left, the use of the symbol $(\mu)_\epsilon$, which implies that $1 \leqq \mu < \epsilon$, is allowable, since $\phi(\epsilon, \mu') = 0$. Isolate the term given by $\lambda = \lambda' = 0$, which is the last term in (6.4). The λ series (6.5) converges absolutely, so the terms can be rearranged in groups given by

$$\epsilon^t \leqq |\lambda| < \epsilon^{t+1} \quad (t = 0, \pm 1, \pm 2, \ldots). \quad (6.6)$$

For these, use (6 E) to write the general term of the right-hand side of (6.5) as

$$e^{-2\pi i(\lambda\epsilon^{-t}c + \lambda'\epsilon'^{-t}c)} \int_{-\infty}^{\infty} \int_{1}^{\epsilon} e^{2\pi i(\lambda\xi + \lambda'\eta)} \phi(\xi, \eta)\, d\xi\, d\eta.$$

Put $\epsilon^{-t}\xi$, $\epsilon'^{-t}\eta$ for ξ, η, and this becomes, from (6 A),

$$e^{-2\pi i(\lambda\epsilon^{-t}c + \lambda'\epsilon'^{-t}c')} \int_{-\infty}^{\infty} \int_{\epsilon^t}^{\epsilon^{t+1}} e^{2\pi i(\lambda\epsilon^{-t}\xi + \lambda'\epsilon^{-t}\eta)} \phi(\xi, \eta)\, d\xi\, d\eta.$$

By (6 D), the set of values $(\lambda\epsilon^{-t}, \lambda'\epsilon'^{-t})$ is merely a rearrangement of the set (λ, λ'), but now $1 \leqq |\lambda\epsilon^{-t}| < \epsilon$. Writing λ, λ' for $\lambda\epsilon^{-t}$, $\lambda'\epsilon'^{-t}$, we see that the group of terms (6.6) contributes to (6.5) a sum

$$\sum_{\lambda} e^{-2\pi i(\lambda c + \lambda' c')} \int_{-\infty}^{\infty} \int_{\epsilon^t}^{\epsilon^{t+1}} e^{2\pi i(\lambda\xi + \lambda'\eta)} \phi(\xi, \eta)\, d\xi\, d\eta,$$

where $1 \leqq |\lambda| < \epsilon$. On summing now for $t = 0, \pm 1, \pm 2, \ldots$, and noting that the order of the integrations is indifferent, the sum of all the groups gives

$$\sum_{(\pm\lambda)_\epsilon} e^{-2\pi i(\lambda c + \lambda' c')} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i(\lambda\xi + \lambda'\eta)} \phi(\xi, \eta)\, d\eta\, d\xi,$$

the integration with respect to η being now effected first. This proves (6.4). Then (6.2) follows by taking a, b as the base of the ideal, a', b' as their conjugates, and ϵ, ϵ' as conjugate units in k. The condition (6 E) becomes

$$\lambda c(1-\epsilon) + \lambda' c'(1-\epsilon') \equiv 0 \pmod 1,$$

and if c, c' are conjugate numbers in k, this requires

$$\lambda c(1-\epsilon) \equiv 0 \pmod{1/\mathfrak{d}}.$$

Since λ refers to numbers in the ideal $1/(\mathfrak{a}\mathfrak{d})$, this means that

$$c(1-\epsilon) \equiv 0 \pmod{\mathfrak{a}}.$$

An obvious slight modification of the work gives the formula analogous to (6 . 4), from which (6 . 3) is derived.

We now apply the modified form of (6 . 4) to the function

$$\chi(x, y) = (\operatorname{sgn} x)^{\alpha} (\operatorname{sgn} y)^{\beta} e^{-2\tau|xy|},$$

where τ is a complex variable and $R(\tau) > 0$, and α, β are 0 or 1. This function does not satisfy the conditions (6 B), and so the result will subsequently require justification. First

$$\int_{-\infty}^{\infty} \int_{1 \leqslant |\xi| \leqslant \epsilon} e^{-2\tau|\xi\eta|} (\operatorname{sgn} \xi)^{\alpha} (\operatorname{sgn} \eta)^{\beta} \, d\xi \, d\eta$$

$$= \int_0^{\infty} \int_1^{\epsilon} e^{-2\tau\xi\eta} [1+(-1)^{\alpha}][1+(-1)^{\beta}] \, d\xi \, d\eta$$

$$= \frac{2 \log \epsilon}{\tau} \quad (\alpha = \beta = 1),$$

$$= 0 \qquad \text{otherwise.}$$

Next, the double integral occurring in the general term of the expansion is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\operatorname{sgn} \xi)^{\alpha} (\operatorname{sgn} \eta)^{\beta} \, e^{-2\tau|\xi\eta| + 2\pi i\lambda\xi + 2\pi i\lambda'\eta} \, d\xi \, d\eta.$$

On splitting the range of integration $-\infty$, ∞ into 0, $+\infty$ and $-\infty$, 0, this integral splits into four. These are

$$\int_0^{\infty} \int_0^{\infty} e^{-2\tau\xi\eta + 2\pi i\lambda\xi + 2\pi i\lambda'\eta} \, d\xi \, d\eta = \int_0^{\infty} \frac{e^{2\pi i\lambda'\eta}}{2\tau\eta - 2\pi i\lambda} \, d\eta,$$

$$(-1)^{\alpha+\beta} \int_0^{\infty} \int_0^{\infty} e^{-2\tau\xi\eta - 2\pi i\lambda\xi - 2\pi i\lambda'\eta} \, d\xi \, d\eta = (-1)^{\alpha+\beta} \int_0^{\infty} \frac{e^{-2\pi i\lambda'\eta}}{2\tau\eta + 2\pi i\lambda} \, d\eta,$$

$$(-1)^{\beta} \int_0^{\infty} \int_0^{\infty} e^{-2\tau\xi\eta + 2\pi i\lambda\xi - 2\pi i\lambda'\eta} \, d\xi \, d\eta = (-1)^{\beta} \int_0^{\infty} \frac{e^{-2\pi i\lambda'\eta}}{2\tau\eta - 2\pi i\lambda} \, d\eta,$$

$$(-1)^{\alpha} \int_0^{\infty} \int_0^{\infty} e^{-2\tau\xi\eta - 2\pi i\lambda\xi + 2\pi i\lambda'\eta} \, d\xi \, d\eta = (-1)^{\alpha} \int_0^{\infty} \frac{e^{2\pi i\lambda'\eta}}{2\tau\eta + 2\pi i\lambda} \, d\eta.$$

These integrals are transformed by deforming the path of integration into the imaginary η axis from 0 to $i\infty$, or from 0 to $-i\infty$. We have then, if v is a positive constant and $R(g) \neq 0$,

$$\int_0^{\infty} \frac{e^{2\pi i v\eta} \, d\eta}{\eta + g} = \int_0^{i\infty} \frac{e^{2\pi i v\eta} \, d\eta}{\eta + g} + [2\pi i e^{-2\pi i v g}]$$

$$= \int_0^{\infty} \frac{e^{-2\pi v\eta} \, d\eta}{\eta - ig} + [2\pi i e^{-2\pi i v g}],$$

the bracketed term being added only when $R(g)<0$, $I(g)<0$. So

$$\int_0^\infty \frac{e^{-2\pi i v\eta}\,d\eta}{\eta+g} = \int_0^{-i\infty} \frac{e^{-2\pi i v\eta}\,d\eta}{\eta+g} - [2\pi i e^{2\pi i v g}]$$

$$= \int_0^\infty \frac{e^{-2\pi v\eta}\,d\eta}{\eta+ig} - [2\pi i e^{2\pi i v g}],$$

the bracketed term being added only when $R(g)<0$, $vI(g)<0$.

The combination of these results means that, if $v\neq 0$ is *any* real constant and $R(g)\neq 0$, then

$$\int_0^\infty \frac{e^{2\pi i v\eta}}{\eta+g}\,d\eta = \int_0^\infty \frac{e^{-2\pi|v|\eta}\,d\eta}{\eta-ig\,\operatorname{sgn} v} + [2\pi i\,\operatorname{sgn} v e^{-2\pi i v g}]_{R(g)<0,\ vI(g)<0}, \quad (6.7)$$

the bracketed term being added only when $R(g)<0$, $vI(g)<0$.

Replace τ by $\pi i/\tau$, so that subsequently $I(\tau)>0$. We have then the expansion

$$\sum_{(\mu)_\epsilon} (\operatorname{sgn}\mu)^\alpha (\operatorname{sgn}\mu')^\beta e^{-2\pi i|\mu\mu'|\tau} = -\frac{i\tau}{2\pi}\{1+(-1)^\alpha\}\{1+(-1)^\beta\}\log\epsilon$$

$$+\frac{\tau}{2\pi i|\Delta|}\sum_{(\lambda)_\epsilon} e^{-2\pi i(\lambda\epsilon+\lambda'\epsilon')}(I_1+I_2+I_3+I_4).$$

Here
$$I_1 = \int_0^\infty \frac{e^{2\pi i\lambda'\eta}\,d\eta}{\eta-\tau\lambda} = \int_0^\infty \frac{e^{2\pi i\lambda'|\lambda|\eta}\,d\eta}{\eta-\tau\operatorname{sgn}\lambda};$$

hence, from (6.7),

$$I_1 = \int_0^\infty \frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta+i\tau\operatorname{sgn}\lambda\lambda'} + [2\pi i\operatorname{sgn}\lambda' e^{2\pi i\tau|\lambda\lambda'|}]_{\lambda\lambda'>0,\ \lambda' R(\tau)>0};$$

$$I_2 = (-1)^{\alpha+\beta}\int_0^\infty \frac{e^{-2\pi i\lambda'|\lambda|\eta}\,d\eta}{\eta+\tau\operatorname{sgn}\lambda}$$

$$= (-1)^{\alpha+\beta}\int_0^\infty \frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta+i\tau\operatorname{sgn}\lambda\lambda'} + (-1)^{\alpha+\beta}[-2\pi i\operatorname{sgn}\lambda' e^{2\pi i\tau\cdot\lambda\lambda'}]_{\lambda\lambda'>0,\ \lambda' R(\tau)<0};$$

$$I_3 = (-1)^\beta\int_0^\infty \frac{e^{-2\pi i\lambda'|\lambda|\eta}\,d\eta}{\eta-\tau\operatorname{sgn}\lambda}$$

$$= (-)^\beta\int_0^{\prime} \frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta-i\tau\operatorname{sgn}\lambda\lambda'} + (-1)^\beta[-2\pi i\operatorname{sgn}\lambda' e^{2\pi i\tau\cdot\lambda\lambda'}]_{\lambda\lambda'<0,\ \lambda' R(\tau)<0};$$

$$I_4 = (-1)^\alpha\int_0^\infty \frac{e^{2\pi i\lambda'|\lambda|\eta}\,d\eta}{\eta+\tau\operatorname{sgn}\eta}$$

$$= (-1)^\alpha\int_0^\infty \frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta-i\tau\operatorname{sgn}\lambda\lambda'} + (-1)^\alpha[2\pi i\operatorname{sgn}\lambda' e^{2\pi i\tau|\lambda\lambda'|}]_{\lambda\lambda'<0,\ \lambda' R(\tau)>0}.$$

Take first the case when $\alpha+\beta=1$. The sum of the four new integrals for I_1, ... vanishes. The sum of the four terms in the brackets reduces to

$$2\pi i\,(\operatorname{sgn}\lambda)^{\alpha}(\operatorname{sgn}\lambda')^{\beta}e^{2\pi i\tau|\lambda\lambda'|},$$

e.g., if $R(\tau)>0$, I_1, I_2, I_3, I_4 contribute terms

$$\left[2\pi i e^{2\pi i\tau|\lambda\lambda'|}\right]_{\lambda>0,\ \lambda'>0}, \qquad (-1)^{\alpha+\beta}\left[2\pi i e^{2\pi i\tau|\lambda\lambda'|}\right]_{\lambda<0,\ \lambda'<0},$$

$$(-1)^{\beta}\left[2\pi i e^{2\pi i\tau|\lambda\lambda'|}\right]_{\lambda>0,\ \lambda'<0}, \quad (-1)^{\alpha}\left[2\pi i e^{2\pi i\tau|\lambda\lambda'|}\right]_{\lambda<0,\ \lambda'>0},$$

respectively, and these are included in the term above.

So if $R(\tau)<0$, I_1 contributes a term $\left[-2\pi i e^{2\pi i\tau|\lambda\lambda'|}\right]_{\lambda<0,\ \lambda'<0}$, etc.

This means that, if $\alpha+\beta=1$, $I(\tau)>0$, then

$$\sum_{(|\mu|)_\epsilon}{}'(\operatorname{sgn}\mu)^{\alpha}(\operatorname{sgn}\mu')^{\beta}e^{-2\pi i|\mu\mu'|/\tau}$$

$$=\frac{\tau}{|\Delta|}\sum_{(|\lambda|)_\epsilon}(\operatorname{sgn}\lambda)^{\alpha}(\operatorname{sgn}\lambda')^{\beta}e^{-2\pi i(\lambda c+\lambda' c')+2\pi i\tau|\lambda\lambda'|}. \quad (6.8)$$

Considerations of continuity show that $R(\tau)=0$ may also be included.

Next suppose that $\alpha+\beta=0$ or 2. The sum of the four new integrals gives

$$2\int_0^{\infty}\frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta+i\tau\operatorname{sgn}\lambda\lambda'}+2(-1)^{\alpha}\int_0^{\infty}\frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta-i\tau\operatorname{sgn}\lambda\lambda'}$$

$$=2(\operatorname{sgn}\lambda\lambda')^{\alpha}\left[\int_0^{\infty}\frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta+i\tau}+(-1)^{\alpha}\int_0^{\infty}\frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta-i\tau}\right].$$

The sum of the four terms in the brackets reduces to

$$2\pi i\operatorname{sgn}R(\tau)(\operatorname{sgn}\lambda)^{\alpha}(\operatorname{sgn}\lambda')^{\beta}e^{2\pi i\tau|\lambda\lambda'|},$$

e.g. if $R(\tau)<0$, I_1 gives a term $\left[-2\pi i e^{2\pi i\tau|\lambda\lambda'|}\right]_{\lambda'>0,\ \lambda>0}$, etc. Hence, if $\alpha+\beta=0$ or 2, that is, if we can take $\beta=\alpha$,

$$\sum_{(|\mu|)_\epsilon}(\operatorname{sgn}\mu\mu')^{\alpha}e^{-2\pi i|\mu\mu'|/\tau}$$

$$=-\frac{i\tau\log\epsilon}{2\pi|\Delta|}\{1+(-1)^{\alpha}\}^2+\frac{\tau}{|\Delta|}\operatorname{sgn}R(\tau)\sum_{(|\lambda|)_\epsilon}(\operatorname{sgn}\lambda\lambda')^{\alpha}e^{-2\pi i(\lambda c+\lambda' c')+2\pi i\tau|\lambda\lambda'|}$$

$$+\frac{\tau}{\pi i|\Delta|}\sum_{(|\lambda|)_\epsilon}(\operatorname{sgn}\lambda\lambda')^{\alpha}e^{-2\pi i(\lambda c+\lambda' c')}\left[\int_0^{\infty}\frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta+i\tau}+(-1)^{\alpha}\int_0^{\infty}\frac{e^{-2\pi|\lambda\lambda'|\eta}\,d\eta}{\eta-i\tau}\right]. \quad (6.9)$$

Take now $\alpha=0$, and write in (6.9)

$$-i\tau\left(\frac{1}{\eta+i\tau}+\frac{1}{\eta-i\tau}\right)=\frac{\eta}{\eta+i\tau}-\frac{\eta}{\eta-i\tau}.$$

The order of integration and summation can then be interchanged, since the series

$$F(\eta) = \sum_{(|\lambda|)_\epsilon} \eta e^{-2\pi|\lambda\lambda'|\eta} \tag{6.9'}$$

converges uniformly for $\eta \geqq \delta > 0$, and is bounded for $\eta \geqq \delta \geqq 0$, where δ is arbitrarily small. For, since

$$\Delta\lambda' = -bm+an, \quad \Delta\lambda = b'm-a'n,$$

and $1 \leqq |\lambda| < \epsilon$, we compare the series (6.9') first with

$$\sum_{(|\lambda|)_\epsilon} \eta e^{-2\pi|\lambda'|\eta}.$$

For a given integer $|n|$, the number of solutions in m of $1 \leqq |\lambda| < \epsilon$ has an upper bound independent of $|n|$, while for $n \neq 0$, $|\lambda'/n|$ has a lower bound independent of $|n|$. Compare the last series with

$$\sum_{n=-\infty}^{\infty} \eta\epsilon^{-2\pi|n|\eta},$$

and the required result follows.

It is well to note that for $\eta \geqq \eta_0$, say, and an appropriate constant $c(\eta_0)$,

$$|F(\eta)| \leqq e^{-\eta c(\eta_0)}.$$

Hence

$$\sum_{(|\mu|)_\epsilon} e^{-2\pi i|\mu\mu'|/\tau} = -\frac{2\pi i\tau \log\epsilon}{\pi|\Delta|} + \frac{\tau}{|\Delta|}\operatorname{sgn} R(\tau) \sum_{(|\lambda|)_\epsilon} e^{-2\pi i(\lambda c+\lambda' c')+2\pi i\tau|\lambda\lambda'}$$
$$+\frac{1}{\pi|\Delta|}\int_0^\infty \sum_{(|\lambda|)_\epsilon} \frac{e^{-2\pi i(\lambda c+\lambda' c')-2\pi|\lambda\lambda'|\eta}\eta\, d\eta}{\eta+i\tau}$$
$$-\frac{1}{\pi|\Delta|}\int_0^\infty \sum_{(|\lambda|)_\epsilon} \frac{e^{-2\pi i(\lambda c+\lambda' c')-2\pi|\lambda\lambda'|\eta}\eta\, d\eta}{\eta-i\tau}. \tag{6.10}$$

Take, finally, $\alpha = 1$ in (6.9), whence

$$\sum_{(|\mu|)_\epsilon} \operatorname{sgn}\mu\mu' e^{-2\pi i|\mu\mu'|/\tau}$$
$$= \frac{\tau}{|\Delta|}\operatorname{sgn} R(\tau) \sum_{(|\lambda|)_\epsilon} \operatorname{sgn}\lambda\lambda' e^{-2\pi i(\lambda c+\lambda' c')+2\pi i\tau|\lambda\lambda'|}$$
$$+\frac{\tau}{\pi i|\Delta|}\sum_{(|\lambda|)_\epsilon} \operatorname{sgn}\lambda\lambda' e^{-2\pi i(\lambda c+\lambda' c')}\left[\int_0^\infty \frac{e^{-2\pi|\lambda\lambda'|\eta}d\eta}{\eta+i\tau} - \int_0^\infty \frac{e^{-2\pi|\lambda\lambda'|\eta}d\eta}{\eta-i\tau}\right]. \tag{6.11}$$

The validity of the interchange of the order of integration and summation is not now so simple a question, since the integrability of

$$\sum_{(|\lambda|)_\epsilon} \operatorname{sgn}\lambda\lambda' e^{-2\pi i(\lambda c+\lambda' c')-2\pi|\lambda\lambda'|\eta}$$

in an interval including $\eta = 0$ is not obvious. But writing

$$-i\tau\left(\frac{1}{\eta+i\tau}-\frac{1}{\eta-i\tau}\right)=\frac{\eta}{\eta+i\tau}+\frac{\eta}{\eta-i\tau}-2,$$

the -2 corresponds to a constant term independent of τ in the expansion (6.11). The interchange for the other terms is permitted, and it is *suggested* that

$$\sum_{(\mu)_\epsilon} \operatorname{sgn}\mu\mu' e^{-2\pi i|\mu\mu'|/\tau} = C+\frac{\tau}{|\Delta|}\operatorname{sgn}R(\tau)\sum_{(\lambda)_\epsilon}\operatorname{sgn}\lambda\lambda' e^{-2\pi i(\lambda c+\lambda' c')+2\pi i\tau,\ \lambda\lambda'|}$$
$$+\frac{1}{\pi|\Delta|}\int_0^\infty \sum_{(\lambda)_\epsilon}\operatorname{sgn}\lambda\lambda'\frac{e^{-2\pi i(\lambda c+\lambda' c')-2\pi|\lambda\lambda'|\eta}\eta\,d\eta}{\eta+i\tau}$$
$$+\frac{1}{\pi|\Delta|}\int_0^\infty \sum_{(\lambda)_\epsilon}\operatorname{sgn}\lambda\lambda'\frac{e^{-2\pi i(\lambda c+\lambda' c')-2\pi\ \lambda\lambda'\ \eta}\eta\,d\eta}{\eta-i\tau}, \quad (6.12)$$

where C is independent of τ. Make $I(\tau)\to\infty$; then the sum of both integrals tends to zero, since

$$\frac{1}{|\eta^2+\tau^2|}<\frac{1}{|2R(\tau)I(\tau)|}$$

and $\int_0^\infty \sum_{(\lambda)_\epsilon}\eta^2 e^{-2\pi|\lambda\lambda'|\eta}d\eta$ is bounded. Hence, writing $-1/\tau$ for τ,

$$C=\lim_{\tau\to 0}\sum_{(\mu)_\epsilon}\operatorname{sgn}\mu\mu' e^{2\pi i\tau|\mu\mu'|}\quad[I(\tau)>0]. \quad (6.13)$$

We express these results in terms of a real quadratic field $k=k(\sqrt{D})$ of discriminant D. Suppose that $\mathfrak{a}$ is any ideal in k with norm $A=N(\mathfrak{a})$, ρ is any number of $\mathfrak{a}$ except zero, so that $\rho\equiv 0\pmod{\mathfrak{a}}$, Q is a positive rational number, and ϵ is a unit of k such that

$$\epsilon\equiv 1\pmod{Q\sqrt{D}},\quad 0<\epsilon'=1/\epsilon<1.$$

In the formula (6.8) put $c=\rho$, take a, b to be the base of the ideal $\mathfrak{a}Q\sqrt{D}$, and replace τ by τAQD. We have, from (6.1),

$$\rho\equiv 0\pmod{\mathfrak{a}}.$$

If $I(\tau)>0$, we put

$$\chi_{\alpha,\beta}(\tau;\ \rho,\mathfrak{a})=\sum_{\mu=\rho(\mathfrak{a}Q\sqrt{D})\atop(\mu)_\epsilon}(\operatorname{sgn}\mu)^\alpha(\operatorname{sgn}\mu')^\beta e^{2\pi i\tau|\mu\mu'|/AQD}. \quad (6.14)$$

Suppose first that $\alpha+\beta=1$. Then, since

$$|\Delta|=N(\mathfrak{a}Q\sqrt{D})=AQ^2D,$$

we have, from (6.8),

$$\chi_{\alpha,\beta}\Big(-\frac{1}{\tau};\ \rho,\ \mathfrak{a}\Big) = \frac{\tau}{Q\sqrt{D}} \sum_{\substack{\lambda \bmod \mathfrak{a}QD \\ (\lambda)_\epsilon}} (\operatorname{sgn}\lambda)^\alpha (\operatorname{sgn}\lambda')^\beta e^{-2\pi i(\lambda\rho+\lambda'\rho')} e^{2\pi i\tau AQD|\lambda\lambda'|}$$

$$= -\frac{\tau}{Q\sqrt{D}} \sum_{\substack{\mathfrak{a}'|\lambda \\ (\lambda')_\epsilon}} (\operatorname{sgn}\lambda)^\alpha (\operatorname{sgn}\lambda')^\beta e^{2\pi i S(\lambda\rho)/AQD} e^{2\pi i\tau|\lambda\lambda'|/AQD},$$

on replacing λ by $-\lambda/AQD$, and writing $S(\lambda\rho) = \lambda\rho+\lambda'\rho'$. The λ sum is split into partial sums by writing $\lambda \equiv \sigma' \pmod{\mathfrak{a}'Q\sqrt{D}}$, and making σ' run through those residues $\equiv 0 \pmod{\mathfrak{a}'}$ of a complete set mod $\mathfrak{a}'Q\sqrt{D}$. For a given σ',

$$S(\lambda\rho) \equiv S(\sigma'\rho) \pmod{\mathfrak{a}QD},$$

whence

$$\chi_{\alpha,\beta}\Big(-\frac{1}{\tau};\ \rho,\ \mathfrak{a}\Big) = -\frac{\tau}{Q\sqrt{D}} \sum_{\substack{\sigma' \bmod \mathfrak{a}'Q\sqrt{D} \\ \sigma' \equiv 0(\mathfrak{a}')}} e^{2\pi i S(\rho\sigma')/AQD} \chi_{\alpha,\beta}(\tau;\ \sigma',\ \mathfrak{a}'), \quad (6.15)$$

where clearly the dashes may be removed under the Σ.

Hecke's* new modular function is then really

$$\vartheta(\tau;\ \rho,\ \mathfrak{a}) = \sum_{\substack{\mu \equiv \rho(\mathfrak{a}Q\sqrt{D}) \\ (\mu)_\alpha,\ \mu\mu'>0}} (\operatorname{sgn}\mu)^\alpha (\operatorname{sgn}\mu')^\beta e^{2\pi i\tau\ \mu\mu'/AQD}, \quad (6.16)$$

where $\alpha = 0$, $\beta = 1$, or $\alpha = 1$, $\beta = 0$. Clearly

$$2\vartheta(\tau;\ \rho,\ \mathfrak{a}) = \chi_{\alpha,\beta}(\tau;\ \rho,\ \mathfrak{a}) + \chi_{\alpha,\beta}(\tau;\ \rho',\ \mathfrak{a}'),$$

since

$$\chi_{\alpha,\beta}(\tau;\ \rho',\ \mathfrak{a}') = \sum_{\substack{\mu \equiv \rho'(\mathfrak{a}'Q\sqrt{D}) \\ (\mu)_\epsilon}} (\operatorname{sgn}\mu)^\alpha (\operatorname{sgn}\mu')^\beta e^{2\pi i\tau\ \mu\mu'/AQD}$$

$$= \sum_{\substack{\mu \equiv \rho(\mathfrak{a}Q\sqrt{D}) \\ (\mu)_\epsilon}} (\operatorname{sgn}\mu)^\beta (\operatorname{sgn}\mu')^\alpha e^{2\pi i\tau\ \mu\mu'/AQD},$$

on putting μ' for μ. On addition, the terms cancel when $\operatorname{sgn}\mu+\operatorname{sgn}\mu' = 0$, *i.e.* when $\mu\mu' < 0$. Hence, finally, from (6.15),

$$\vartheta\Big(-\frac{1}{\tau};\ \rho,\ \mathfrak{a}\Big) = -\frac{\tau}{Q\sqrt{D}} \sum_{\substack{\sigma \bmod \mathfrak{a}Q\sqrt{D} \\ \sigma \equiv 0(\mathfrak{a})}} e^{2\pi i S(\rho\sigma')/AQD} \vartheta(\tau;\ \sigma,\ \mathfrak{a}). \quad (6.17)$$

Obviously $$\vartheta(\tau+1;\ \rho,\ \mathfrak{a}) = e^{2\pi i\rho\rho'/AQD}\vartheta(\tau;\ \sigma,\ \mathfrak{a}). \quad (6.18)$$

* Hecke, **11**.

By a characteristically elegant method, Hecke* deduces from (6.17) and (6.18) that his function leads at once to invariants of congruence subgroups of the modular group when $\mathfrak{a}$ is an integral ideal and Q a positive rational integer, the cases for which he proves his formula.

So from (6.10) and (6.12) we find the following formula, where the summation in σ' refers to the residues divisible by $\mathfrak{a}'$ of a complete set mod $\mathfrak{a}'Q\sqrt{D}$.

$$\chi_{00}\left(-\frac{1}{\tau};\ \rho,\ \mathfrak{a}\right) = -\frac{2i\tau\log\epsilon}{\pi Q\sqrt{D}} + \frac{\tau}{Q\sqrt{D}}\operatorname{sgn} R(\tau)\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\chi_{00}(\tau;\ \sigma',\ \mathfrak{a}')$$
$$+\frac{1}{\pi Q\sqrt{D}}\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\int_0^\infty \frac{\eta\chi_{00}(i\eta;\ \sigma',\ \mathfrak{a}')\,d\eta}{\eta+i\tau}$$
$$-\frac{1}{\pi Q\sqrt{D}}\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\int_0^\infty \frac{\eta\chi_{00}(i\eta;\ \sigma',\ \mathfrak{a}')\,d\eta}{\eta-i\tau}, \quad (6.19)$$

$$\chi_{11}\left(-\frac{1}{\tau};\ \rho,\ \mathfrak{a}\right) - C_{11} = \frac{\tau}{Q\sqrt{D}}\operatorname{sgn} R(\tau)\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\chi_{11}(\tau;\ \sigma',\ \mathfrak{a}')$$
$$+\frac{1}{\pi Q\sqrt{D}}\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\int_0^\infty \frac{\eta\chi_{11}(i\eta;\ \sigma',\ \mathfrak{a}')\,d\eta}{\eta+i\tau}$$
$$+\frac{1}{\pi Q\sqrt{D}}\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\int_0^\infty \frac{\eta\chi_{11}(i\eta;\ \sigma',\ \mathfrak{a}')\,d\eta}{\eta-i\tau}, \quad (6.20)$$

where [with $I(\tau) > 0$]

$$C_{11} = \lim_{\tau\to 0}\chi_{11}(\tau;\ \rho,\ \mathfrak{a}) = \lim_{\tau\to 0}\sum_{\substack{\mu\equiv\rho(\mathfrak{a}Q\sqrt{D})\\(|\mu|)_\epsilon}}\operatorname{sgn}\mu\mu'\, e^{2\pi i\tau|\mu\mu'|/AQD}. \quad (6.21)$$

Hecke† gives the transformation formula for the function defined by

$$D(\tau;\ \rho,\ \mathfrak{a}) = \sum_{\substack{\mu\equiv\rho(\mathfrak{a}Q\sqrt{D})\\(|\mu|)_\epsilon,\ \mu\mu'>0}} e^{2\pi i\tau|\mu\mu'|/AQD}. \quad (6.22)$$

Write also $$D_1(\tau;\ \rho,\ \mathfrak{a}) = \sum_{\substack{\mu=-\rho(\mathfrak{a}Q\sqrt{D})\\(|\mu|)_\epsilon,\ \mu\mu'<0}} e^{2\pi i\tau|\mu\mu'|/AQD}.$$

From the definition

$$D(\tau;\ \rho',\ \mathfrak{a}') = D(\tau;\ \rho,\ \mathfrak{a}),\quad D_1(\tau;\ \rho',\ \mathfrak{a}') = D_1(\tau;\ \rho,\ \mathfrak{a}),$$

since the $(|\mu|)_\epsilon$ in (6.22) can be replaced by $(|\mu'|)_\epsilon$. Also

$$D_1(\tau;\ \rho,\ \mathfrak{a}) = D(\tau;\ \rho\sqrt{D},\ \mathfrak{a}\sqrt{D}),$$

---

* Hecke, **11.**

† Hecke, **12.**

since, if μ is replaced by $\mu\sqrt{D}$, μ' is replaced by $-\mu'\sqrt{D}$. Clearly

$$2D(\tau;\ \rho,\ \mathfrak{a}) = \chi_{00}(\tau;\ \rho,\ \mathfrak{a}) + \chi_{11}(\tau;\ \rho,\ \mathfrak{a}),$$

whence (6.19), (6.20) give

$$D\Big(-\frac{1}{\tau};\ \rho,\ \mathfrak{a}\Big) = \tfrac{1}{2}C_{11} - \frac{c\tau\log\epsilon}{\pi Q\sqrt{D}} + \frac{\tau}{Q\sqrt{D}}\operatorname{sgn} R(\tau)\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\, D(\tau;\ \sigma',\ \mathfrak{a}')$$
$$+ \frac{1}{\pi Q\sqrt{D}}\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\int_0^\infty \frac{\eta D(i\eta;\ \rho',\ \mathfrak{a}')\,d\eta}{\eta+i\tau}$$
$$- \frac{1}{\pi Q\sqrt{D}}\sum_{\sigma'} e^{2\pi i S(\rho'\sigma)/AQD}\int_0^\infty \frac{\eta D_1(i\eta;\ \sigma',\ \mathfrak{a}')\,d\eta}{\eta-i\tau}. \quad (6.23)$$

This is Hecke's* result given, however, for the case $R(\tau)<0$, and with the omission of the constant C_{11}. Possibly† $C_{11}=0$, but it is not easy to fill in the details of Hecke's work and to verify his result.

Validity of the formulae.

7. The applications will now be justified by proving (6.3) for other classes of functions. The conditions given below are designed to meet these applications and are not so simple as those in § 6. There, however, the restriction that $\partial^{\alpha+\beta}\phi/\partial x^\alpha \partial y^\beta = 0$ for $x=1$ causes the difficulty. It would be of considerable interest to find other conditions which suffice when this restriction is removed.

Suppose, for all real values of $t>0$, x, y, that $\psi(x, y, t)$ is defined as a continuous function of t. Let $\epsilon>1$, $\epsilon'=1/\epsilon$, c, c' be constants such that the sets of numbers (μ, μ') and (λ, λ'), defined in § 2, take the same pairs of values as the sets $(\epsilon\mu, \epsilon'\mu')$ and $(\epsilon\lambda, \epsilon'\lambda')$ respectively, and that, for all λ, λ',

$$c\lambda + c'\lambda' \equiv c\epsilon\lambda + c'\epsilon'\lambda' \pmod 1.$$

Suppose, also, that

(7 A) $$\psi(\epsilon x, \epsilon' y, t) = \epsilon\psi(x, y, \epsilon t);$$

(7 B) the series $\sum_\mu \psi(\mu, \mu', t)$ converges absolutely for $1\leqslant t\leqslant\epsilon$, and also uniformly in the sense that, for any given number $\delta>0$, a number $M(\delta)$ exists independently of t, so that

$$\sum_\mu |\psi(\mu, \mu', t)| < \delta,$$

---

* Hecke, **12**, 166.

† Clearly $C_{11}=0$ if ϵ is the square of a unit of norm -1.

where the summation refers to the values of μ for which either $|\mu| > M(\delta)$ or $|\mu'| > M(\delta)$. The series then clearly converges for $t > 0$ in any finite interval on dividing it into a finite number of the form $\epsilon^\rho \leqslant t < \epsilon^{\rho+1}$ and noting (7 A);

(7 C) the summation formula (2.1) holds for the ψ function, *i.e.*

$$\sum_\mu \psi(\mu, \mu', t) = \frac{1}{|\Delta|} \sum_\lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i[\lambda(\xi-c)+\lambda'(\eta-c')]} \psi(\xi, \eta, t) d\xi d\eta,$$

for $1 \leqslant t \leqslant \epsilon$, and, further, that the series of moduli of the terms on the right-hand series converges uniformly for $1 \leqslant t \leqslant \epsilon$, and hence in any finite interval, $t \geqslant t^0 > 0$ in the sense just explained;

(7 D) each of the following triple integrals converges to a value independent of the order in which the integrations are carried out, except that in the first (*i.e.* 7 DA) we need not consider the integrals obtained by integrating last with respect to η;

(7 DA) $$\int_1^\epsilon dt \int_{-\infty}^{\infty} d\eta \int_{\xi_2}^{\xi_1} \psi(\xi, \eta, t) d\xi,$$

for all real ξ_1, ξ_2 with $-\infty \leqslant \xi_1, \xi_2 \leqslant \infty$;

(7 DB) $$\int_0^\infty dt \int_{-\infty}^{\infty} d\eta \int_{1 \leqslant |\xi| \leqslant \epsilon} \psi(\xi, \eta, t) d\xi;$$

(7 DC) $$\int_0^\infty dt \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} e^{2\pi i(\lambda\xi+\lambda'\eta)} \psi(\xi, \eta, t) d\xi,$$

for all real λ, λ', excluding $\lambda = 0$ or $\lambda' = 0$.

Then clearly a function $\phi(\mu, \mu')$ is defined by the relation

$$\phi(\mu, \mu') = \int_0^\infty \psi(\mu, \mu', t) dt;$$

for which it will be proved that the expansion (6.3) is valid.

For, from (7 B), since the order of integration and summation can be interchanged,

$$\sum_\mu \int_1^\epsilon \psi(\mu, \mu', t) dt = \int_1^\epsilon \sum_\mu \psi(\mu, \mu', t) dt.$$

Rearrange the absolutely convergent series under the integral sign; then the above

$$= \int_1^\epsilon \sum_{f=-\infty}^{\infty} \left[\sum_{(|\mu|)_\epsilon} \psi(\epsilon^f \mu, \epsilon'^f \mu', t) \right] dt = \int_1^\epsilon \sum_{f=-\infty}^{\infty} \chi_t dt,$$

say. The sum in f gives a continuous function of t; χ_t is a continuous function of t from (7 B) (see page 544); and so the summation in f may be interchanged with the integration. The integral becomes

$$\sum_{f=-\infty}^{\infty} \int_1^{\epsilon} \sum_{(|\mu|)_\epsilon} \psi(\epsilon^f \mu, \epsilon'^f \mu', t) dt,$$

and, from (7 A), this is equal to

$$\sum_{f=-\infty}^{\infty} \int_1^{\epsilon} \sum_{(|\mu|)_\epsilon} \psi(\mu, \mu', \epsilon^f t) \epsilon^f dt = \sum_{f=-\infty}^{\infty} \int_{\epsilon^f}^{\epsilon^{f+1}} \sum_{(|\mu|)_\epsilon} \psi(\mu, \mu', t) dt$$

$$= \int_0^{\infty} \sum_{(|\mu|)_\epsilon} \psi(\mu, \mu', t) dt.$$

If t is positive and bounded, the series $\sum_{(|\mu|)_\epsilon} \psi(\mu, \mu', t)$ converges absolutely, and is a continuous function of t. The order of integration and summation may be interchanged. The above is

$$\sum_{(|\mu|)_\epsilon} \int_0^{\infty} \psi(\mu, \mu' t) dt = \sum_{(|\mu|)_\epsilon} \phi(\mu, \mu').$$

We then have

$$\sum_{(|\mu|)_\epsilon} \phi(\mu, \mu') = \int_1^{\epsilon} \sum_{\mu} \psi(\mu, \mu', t) dt.$$

From (7 C) this is equal to

$$\frac{1}{|\Delta|} \int_1^{\epsilon} dt \left[\sum_{\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i [\lambda(\xi - c) + \lambda'(\eta - c')]} \psi(\xi, \eta, t) d\xi d\eta \right]. \qquad (7.1)$$

The term given by $\lambda = \lambda' = 0$ is

$$I = \int_1^{\epsilon} dt \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(\xi, \eta, t) d\xi d\eta.$$

Now

$$\int_1^{\epsilon} dt \int_{-\infty}^{\infty} d\eta \int_{\epsilon^f}^{\epsilon^{f+1}} \psi(\xi, \eta, t) d\xi = \int_{\epsilon^f}^{\epsilon^{f+1}} dt \int_{-\infty}^{\infty} d\eta \int_1^{\epsilon} \psi(\xi, \eta, t) d\xi,$$

on writing $\epsilon^f \xi$, $\epsilon'^f \eta$, t for ξ, η, $\epsilon^f t$ respectively, and noting (7 A). By (7 DA), the left-hand side can be integrated last with respect to ξ, so take $f = 0, \pm 1, \pm 2, \ldots$, and sum. By (7 DA), the new left-hand side can now be integrated first with respect to ξ, its limits being 0 and ∞; while, by (7 DB), the new right-hand side can be integrated first with respect to t,

its limits being 0 and ∞. Hence

$$\int_1^\epsilon dt \int_{-\infty}^{\infty} d\eta \int_0^\infty \psi(\xi, \eta, t) d\xi = \int_{-\infty}^{\infty} d\eta \int_1^\epsilon d\xi \int_0^\infty \psi(\xi, \eta, t) dt.$$

Similarly, when the limits for ξ on the left are $-\infty$, 0, those on the right become $-\epsilon$, -1. By addition

$$I = \int_{-\infty}^{\infty} \int_{1 \leqq \xi \leqq \epsilon} \phi(\xi, \eta) d\xi d\eta,$$

and this is the last term in (6 . 3).

Excluding $\lambda = 0$ from the summation in (7 . 1), and noting (7 C), we can interchange the order of integration in t and summation in λ, and get

$$\frac{1}{|\Delta|} \sum_\lambda' \int_1^\epsilon dt \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i[\lambda(\xi-c)+\lambda'(\eta-c')]} \psi(\xi, \eta, t) d\xi d\eta.$$

The sum in λ still converges absolutely, and can be written as

$$\frac{1}{|\Delta|} \sum_{(\lambda)_\epsilon}' \sum_{f=-\infty}^{\infty} \int_1^\epsilon dt \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i[\lambda\epsilon^{-f}(\xi-c)+\lambda'\epsilon'^{-f}(\eta-c')]} \psi(\xi, \eta, t) d\xi d\eta.$$

Write $\xi\epsilon^f$, $\eta\epsilon'^f$, $\epsilon^{-f}t$ for ξ, η, t respectively, and note (7 A) and (6 E). The result is

$$\frac{1}{|\Delta|} \sum_{(\lambda)_\epsilon}' \sum_{f=-\infty}^{\infty} \int_{\epsilon^f}^{\epsilon^{f+1}} dt \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i[\lambda(\xi-c)+\lambda'(\eta-c')]} \psi(\xi, \eta, t) d\xi d\eta.$$

Sum with respect to f, and then use (7 DC) to integrate the result first for t. The result is

$$\frac{1}{|\Delta|} \sum_{(\lambda)_\epsilon}' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^\infty e^{2\pi i[\lambda(\xi-c)+\lambda'(\eta-c')]} \psi(\xi, \eta, t) d\xi d\eta dt$$

$$= \frac{1}{|\Delta|} \sum_{(\lambda)_\epsilon}' \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i[\lambda(\xi-c)+\lambda'(\eta-c')]} \phi(\xi, \eta) d\xi d\eta.$$

This proves (6 . 3).

We now prove that this theorem is sufficient for the applications in § 6. It is well known that, if μ and μ' are real,

$$\operatorname{sgn} \mu e^{-2\tau|\mu\mu'|} = \frac{2\sqrt{\tau}}{\sqrt{\pi}} \int_0^\infty \mu e^{-\tau(\mu^2 t^2 + \mu'^2/t^2)} dt \quad [R(\tau) > 0, \ R(\sqrt{\tau}) > 0]$$

This can be written as

$$e^{-2\tau|\mu\mu'|} = \frac{2\sqrt{\tau}}{\sqrt{\pi}} \int_0^\infty |\mu| e^{-\tau(\mu^2 t^2 + \mu'^2/t^2)} dt,$$

unless $\mu = 0$, when the right-hand side is zero. This means that the term given by $\mu = 0 = \mu'$ is excluded from the left-hand side of (6.10), as it ought to be. Take

$$\phi(\mu, \mu') = (\operatorname{sgn}\mu)^\alpha (\operatorname{sgn}\mu')^\beta e^{-\tau(\mu^2 t^2 + \mu'^2/t^2)},$$

except that $\phi(0, 0) = 0$ when $\alpha = \beta = 0$.

Then

$$\psi(\mu, \mu', t) = \frac{2\sqrt{\tau}}{\sqrt{\pi}} |\mu| (\operatorname{sgn}\mu)^\alpha (\operatorname{sgn}\mu')^\beta e^{-\tau(\mu^2 t^2 + \mu'^2/t^2)}.$$

Suppose first that $\beta = 0$.

The conditions (7 A), (7 B) are obviously satisfied. The last part of condition (7 C) is satisfied if the first part is taken for granted for the present. For the general term there is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\xi| (\operatorname{sgn}\xi)^\alpha \exp\{2\pi i[\lambda(\xi - c) + \lambda'(\eta - c')] - \tau(\xi^2 t^2 + \eta^2/t^2)\}\, d\xi\, d\eta.$$

It is sufficient to deal with the range $0, \infty$ of ξ, *i.e.* the integral

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} \xi \exp\{2\pi i(\lambda\xi + \lambda'\eta) - \tau(\xi^2 t^2 + \eta^2/t^2)\}\, d\xi\, d\eta,$$

or, on integrating first with respect to η, with

$$te^{-\pi^2\lambda'^2 t^2/\tau}\int_0^\infty \xi e^{2\pi i\lambda\xi - \tau\xi^2 t^2}\, d\xi.$$

Integrate twice by parts using $d(e^{2\pi i\lambda\xi})$ since $\lambda \neq 0$; and then the integral is $O(1/\lambda^2)$ independently of t for $1 \leqslant t \leqslant \epsilon$. Since $|e^{-z}| < 1/R(z)$ for $R(z) > 0$, the general term is $O(\lambda^{-2}\lambda'^{-2})$ uniformly in t for $1 \leqslant t \leqslant \epsilon$. When, however, $0 < |\lambda| \leqslant 1$, the integral is bounded, and so the uniform convergence condition in (7 C) follows as in § 2.

Next the three conditions (7 D) are satisfied when

$$\psi = \psi(\xi, \eta, t) = \xi e^{-\tau(\xi^2 t^2 + \eta^2/t^2)}.$$

Take first (7 DA). Then the form of ψ as a product of a function of ξ and a function of η shows that

$$\int_1^\epsilon dt \int_{-\infty}^{\infty} d\eta \int_{\xi_2}^{\xi_1} \psi\, d\xi = \int_1^\epsilon dt \int_{\xi_2}^{\xi_1} d\xi \int_{-\infty}^{\infty} \psi\, d\eta.$$

The value of the integral $\int_{-\infty}^{\infty} \psi\, d\eta$ is obvious; whence, from de la Vallée Poussin's test, the order of the integrations in ξ and t on the right may be

interchanged. The right-hand side is

$$\int_{\xi_2}^{\xi_1} d\xi \int_1^{\epsilon} dt \int_{-\infty}^{\infty} \psi \, d\eta,$$

and here again the order of the integrations in t and η may be interchanged.

Take next (7 DB). Then, as above, the order of the integrations in ξ and η may be interchanged, and so

$$\int_0^{\infty} dt \int_{-\infty}^{\infty} d\eta \int_{1 \leqq \xi \leqq \epsilon} \psi \, d\xi = \int_0^{\infty} dt \int_{1 \leqq \xi \leqq \epsilon} d\xi \int_{-\infty}^{\infty} \psi \, d\eta,$$

and, as above, this is equal to

$$\int_{1 \leqq \xi \leqq \epsilon} d\xi \int_0^{\infty} dt \int_{-\infty}^{\infty} \psi \, d\eta,$$

and again is equal to

$$\int_{1 \leqq \xi \leqq \epsilon} d\xi \int_{-\infty}^{\infty} d\eta \int_0^{\infty} \psi \, dt.$$

The value of the integral in t is such that the order of the integrations in ξ and η may be interchanged.

Finally take (7 DC). Write ψ' for $e^{2\pi i(\lambda\xi+\lambda'\eta)}\psi(\xi, \eta, t)$. Then ψ' is also the product of a function of ξ and a function of η, and so

$$\int_0^{\infty} dt \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} \psi' \, d\xi = \int_0^{\infty} dt \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \psi' \, d\eta.$$

The value of the integral in $\int_{-\infty}^{\infty} \psi' \, d\eta$ is obvious, and from it the order of the integrations in t and ξ on the right may be interchanged. The triple integral is equal to

$$\int_{-\infty}^{\infty} d\xi \int_0^{\infty} dt \int_{-\infty}^{\infty} \psi' \, d\eta.$$

The order of the integrations in t and η may be interchanged, so that this is equal to

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_0^{\infty} \psi' \, dt.$$

The value of $\int_0^{\infty} \psi' \, dt$ is such that the integration in η may be taken last.

We must now justify the assumption that the summation formula in (7 C) holds for the function ψ. If $a = 1$, the sufficient conditions given for (2 . 1) are obviously satisfied, since $|\mu| \operatorname{sgn} \mu = \mu$. If $a = 0$, the factor $|\mu|$ introduces difficulties, since the conditions of differentiability are not

satisfied at the origin. The summation formula is easily deduced from

$$\sum_{\mu} |\mu| \mu^4 e^{-P\mu^2-Q\mu'^2} = \frac{1}{|\Delta|} \sum_{\lambda} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[2\pi i\{\lambda(\xi-c)+\lambda'(\eta-c')\} - P\xi^2-Q\eta^2]|\xi|\xi^4 d\xi d\eta, \quad (7.2)$$

since here the conditions for (2.1) are satisfied if P, Q are complex numbers with positive real parts, *i.e.* $R(P)>0$, $R(Q)>0$. It will be shown that the right-hand series converges uniformly for $P \geqq P_0 > 0$, say, and can be integrated twice term by term with respect to P from P_0 to ∞, and that the integration for P is given by integrating under the integral signs. Obviously the left-hand side can be integrated term by term, and so the summation formula in (7 c) will follow.

On the right-hand side above the integral in η can be evaluated, so that we need only consider the two series

$$\sum_{\lambda} e^{-\pi^2\lambda'^2/Q} \left| \int_0^{\infty} e^{-P\xi^2 \pm 2\pi i\lambda\xi} \xi^5 d\xi \right|,$$

resulting on splitting the integration for ξ into $-\infty$, 0, and 0, ∞. The path of integration is deformed into a line through $\xi = 0$, making an angle $\pi/6$ or $-\pi/6$ with the real ξ axis. Writing $\xi = \xi e^{\pm \pi i/6}$, we need only consider the series

$$\sum_{\lambda} e^{-\pi^2\lambda'^2/Q} \int_0^{\infty} e^{-\frac{1}{2}P\xi^2 - \pi\sqrt{3}|\lambda|\xi} \xi^5 d\xi.$$

Use $g_1, g_2, \ldots$ to denote absolute constants. Then, for $\xi \geqq 0$, $|\xi^{\frac{3}{2}} e^{-\xi}| < g_1$, and so

$$\int_0^{\infty} e^{-\frac{1}{2}P\xi^2 - \pi\sqrt{3}|\lambda|\xi} \xi^5 d\xi < g_2 \int_0^{\infty} e^{-\frac{1}{2}P\xi^2} \frac{\xi^{\frac{7}{2}}}{|\lambda|^{\frac{3}{2}}} d\xi < \frac{g_3}{|\lambda|^{\frac{3}{2}} P^2}.$$

When $|\lambda| < 1$, since $e^{-\pi\sqrt{3}|\lambda|\xi} \leqq 1$, we have

$$\int_0^{\infty} e^{-\frac{1}{2}P\xi^2 - \pi\sqrt{3}|\lambda|\xi} \xi^5 d\xi < \frac{g_4}{P^3}.$$

Hence we are led to deal with

$$\sum_{\lambda} e^{-\pi^2\lambda'^2/Q} \left(\frac{1}{|\lambda|^{\frac{3}{2}} P^{\frac{7}{2}}} \text{ or } \frac{1}{P^3} \right),$$

which obviously can be integrated twice with respect to P from P_0 to ∞.

Finally, calling ψ'' the integrand in the double integral in (7.2), then

$$\int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \int_{P_0}^{\infty} \psi'' dP = \int_{-\infty}^{\infty} d\xi \int_{P_0}^{\infty} dP \int_{-\infty}^{\infty} \psi'' d\eta,$$

since the integrand ψ'' is the product of a function of P only and a function of ξ, η. On the right-hand side, the value of the last integral in η is obvious, and so the order of the integrations in ξ and P may be interchanged. This proves that the first integration with respect to P is given by integrating under the integral signs in (7.2). Similarly for the second integration.

Take finally $\beta = 1$. The summation formula for ψ in (7c) no longer satisfies the other condition in (7c). Its general term involves

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\xi|(\operatorname{sgn}\xi)^{\alpha}(\operatorname{sgn}\eta) \times \exp\{2\pi i[\lambda(\xi-c)+\lambda'(\eta-c')]-\tau(\xi^2t^2+\eta^2/t^2)\}\,d\xi\,d\eta.$$

The integral in η reduces to

$$\int_0^{\infty} \sin 2\pi\lambda'\eta e^{-\tau\eta^2/t^2}\,d\eta = O(1/\lambda')$$

only, and so the series does not converge absolutely. The case $\beta = 1$, $\alpha = 0$ is, however, really the same as $\beta = 0$, $\alpha = 1$, and it need not be proved again, since

$$\sum_{\substack{\mu \equiv \rho(\mathfrak{a}Q\sqrt{D}) \\ (|\mu|)_\epsilon}} \operatorname{sgn}\mu'\, e^{2\pi i\tau\,\mu\mu'/|AQD} = \sum_{\substack{\mu' = \rho'(\mathfrak{a}'Q\sqrt{D}) \\ (|\mu|)_\epsilon}} \operatorname{sgn}\mu'\, e^{2\pi i\tau\,\mu\mu'/|AQD}.$$

The right-hand side, wherein $(|\mu|)_\epsilon$ can be replaced by $(|\mu'|)_\epsilon$, and then μ' by μ, becomes

$$\sum_{\substack{\mu \equiv \rho'(\mathfrak{a}'Q\sqrt{D}) \\ (|\mu|)_\epsilon}} \operatorname{sgn}\mu\, e^{2\pi i\tau|\mu\mu'/|AQD}.$$

i.e. the case $\beta = 0$, $\alpha = 1$, with $\mathfrak{a}$ replaced by $\mathfrak{a}'$.

The last case, namely $\alpha = \beta = 1$, is not disposed of so simply. We can, however, apply the general result of § 6 to the series

$$\Sigma\mu\mu' e^{-2\tau|\mu\mu'},$$

and then integrate with respect to τ. Thus, for the justification as given in the theorem of § 7, we take now the result

$$\mu\mu' e^{-2\tau|\mu\mu'|} = \frac{2\sqrt{\tau}}{\sqrt{\pi}}\int_0^{\infty} |\mu|\mu\mu' e^{-\tau(\mu^2t^2+\mu'^2/t^2)}\,dt,$$

and put

$$\phi(\mu, \mu') = \mu\mu' e^{-2\tau|\mu\mu'}, \quad \psi(\mu, \mu') = \frac{2\sqrt{\tau}}{\sqrt{\pi}}|\mu|\mu\mu' e^{-\tau(\mu^2t^2+\mu'^2/t^2)}.$$

The summation formula in (7c) still holds, and the condition there now presents no difficulty, since the integral in η is $O(1/\lambda'^2)$, and conditions

(7 A), (7 B), (7 C) are satisfied It is also easy to see that the various interchanges in (7 DA), (7 DB), (7 DC) are allowed on considering

$$\int_1^\epsilon dt \int_{-\infty}^{\infty} d\eta \int_{\xi_2}^{\xi_1} \xi^2 \eta e^{-\tau(\xi^2 t^2 + \eta^2/t^2)} d\xi,$$

etc.

Write now $\pi i/\tau$ for τ, and note that

$$\int_{-\infty}^{\infty} \int_{1 \leqslant |\xi| \leqslant \epsilon} \phi(\xi, \eta)\, d\xi\, d\eta = 0,$$

while

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \xi\eta \exp\left(-\frac{2\pi i |\xi\eta|}{\tau} + 2\pi i\lambda\xi + 2\pi i\lambda'\eta\right) d\xi d\eta$$

can be evaluated either by the same method as in § 6, or more simply by differentiating with respect to τ the result already found for

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \operatorname{sgn} \xi\eta \exp\left\{-\frac{2\pi}{\tau} |\xi\eta| + 2\pi i\lambda\xi + 2\pi i\lambda'\eta\right\} d\xi\, d\eta,$$

which is embodied in (6 . 12).

We have then, when $R(\tau) \neq 0$, $I(\tau) > 0$,

$$\sum_{(\mu)_*} \frac{d}{d\tau} \operatorname{sgn} \mu\mu' e^{-2\pi i |\mu\mu'|/\tau} = \sum_{(\lambda)_\epsilon} \frac{d}{d\tau} (T_1 + T_2)$$

$$= \sum_{(\lambda)_*} \frac{dT_1}{d\tau} + \sum_{(\lambda)_\epsilon} \frac{dT_2}{d\tau},$$

where

$$T_1 = \frac{\tau}{|\Delta|} \operatorname{sgn} R(\tau) \operatorname{sgn} \lambda\lambda' \exp\{-2\pi i(\lambda c + \lambda' c') + 2\pi i\tau |\lambda\lambda'|\},$$

$$\frac{dT_2}{d\tau} = \frac{1}{\pi |\Delta|} \int_0^\infty \frac{d}{d\tau} \left(\frac{\eta}{\eta + i\tau} + \frac{\eta}{\eta - i\tau}\right) \operatorname{sgn} \lambda\lambda'$$

$$\times \exp\{-2\pi i(\lambda c + \lambda' c') - 2\pi |\lambda\lambda'| \eta\}\, d\eta.$$

We have to justify, for the series in this identity, term by term integration with respect to τ along a path parallel to the imaginary τ-axis from τ_0, say, to $\tau_0 + i\infty$. Write $\tau = X + iY$, where $X \neq 0$ is a constant and Y goes from Y_0 to $i\infty$. No difficulty arises in interchanging the order of integration and summation on the left-hand side, since [*cf.* (6 . 9′)]

$$\sum_{(\mu)_*} \frac{|\mu\mu'|}{Y^2} e^{-2\pi |\mu\mu'|/Y}$$

is integrable with respect to Y for $Y_0 \leqq Y \leqq Y_1 < \infty$. Making $Y_1 \to \infty$ introduces the constant C of (6.12). The justification is obvious for the first series on the right-hand side because of the factor $e^{2\pi\eta\tau\ \lambda\lambda'}$, while that for the second series requires some attention.

First of all, we can interchange the order of summation in λ and integration in η since, in $dT_2/d\tau$,

$$\frac{d}{d\tau}\left(\frac{\eta}{\eta+i\tau}+\frac{\eta}{\eta-i\tau}\right) = -\frac{4\tau\eta^2}{(\eta^2+\tau^2)^2},$$

and

$$|\eta^2+\tau^2| > 2|XY|,$$

while, from (6.9′),

$$\sum_{(\lambda)_\epsilon} \eta^2 e^{-2\pi\ \lambda\lambda'|\eta}$$

is integrable with respect to η from 0 to ∞.

Next we must show that the integration in τ can be effected under the integral sign in η, *i.e.* that

$$\int_{Y_0}^{\infty} dY \int_0^{\infty} \frac{\tau\eta^2}{(\eta^2+\tau^2)^2} f(\eta)\, d\eta = \int_0^{\infty} d\eta \int_{Y_0}^{\infty} \frac{\tau\eta^2}{(\eta^2+\tau^2)^2} f(\eta)\, dY,$$

where

$$f(\eta) = \sum_{(|\lambda|)_\epsilon} \operatorname{sgn} \lambda\lambda' \exp\{-2\pi i(\lambda c+\lambda' c')-2\pi|\lambda\lambda'|\eta\}.$$

Clearly

$$|f(\eta)| \leqq \sum_{(\lambda)_\epsilon} e^{-2\pi|\lambda\lambda'|\eta},$$

and, as already proved in (6.9′), $|\eta f(\eta)|$ is bounded for $0 \leqq \eta \leqq \infty$, while for $\eta \geqq \eta_0$, say, $|f(\eta)| \leqq e^{-\eta c(\eta_0)}$, say, for a suitable constant $C(\eta_0)$.

Now for the values of τ given by $Y_0 \leqq Y \leqq Y_1 < \infty$, say, the integral

$$\int_0^{\infty} \frac{\tau\eta^2}{(\eta^2+\tau^2)^2} f(\eta)\, d\eta$$

converges uniformly, since $|\eta^2+\tau^2| > 2XY$ and

$$\int_0^{\infty} \eta^2 |f(\eta)|\, d\eta$$

converges. Hence, on integrating with respect to Y from Y_0 to Y_1,

$$\int_{Y_0}^{Y_1} dY \int_0^{\infty} \frac{\tau\eta^2}{(\eta^2+\tau^2)^2} f(\eta)\, d\eta = \int_0^{\infty} d\eta \int_{Y_0}^{Y_1} \frac{\tau\eta^2}{(\eta^2+\tau^2)^2} f(\eta)\, dY.$$

We can put $Y_1 = \infty$ here provided that the right-hand double integral then converges, and also that

$$\lim_{Y_1 \to \infty} \int_0^\infty d\eta \int_{Y_1}^\infty \frac{\tau \eta^2}{(\eta^2+\tau^2)^2} f(\eta)\, dY = 0.$$

This is obvious, since

$$2\int_{Y_1}^\infty \frac{\tau\, dY}{(\eta^2+\tau^2)^2} = \frac{1}{\eta^2+(X+iY_1)^2}$$

has modulus less than or equal to $1/2|XY_1|$, while

$$\int_0^\infty \eta^2 |f(\eta)|\, d\eta$$

converges.

This proves the result (6.12). The other results (6.8), (6.10) could also have been proved at the same time by dealing with

$$\sum_{(\mu')_\epsilon} \mu\mu' (\operatorname{sgn}\mu)^\alpha (\operatorname{sgn}\mu')^\beta e^{-2\tau|\mu\mu'|}.$$

When $\alpha = \beta = 0$, finding the arbitrary constant introduced by the integration for τ might not be easy.

I conclude by observing that the general theorems of §§ 6, 7, such as the expansion (6.2), etc., which were deduced from (2.3), can be extended by means of (2.4) to an algebraic field of degree n with r_1 real roots and r_2 pairs of conjugate complex roots. The unit ϵ will be replaced by r_1+r_2-1 totally positive units $\epsilon_1, \epsilon_2, \ldots, \epsilon_{r_1+r_2-1}$ (a unit ϵ is called totally positive if $r_1 = 0$, or if $\epsilon^{(1)}, \ldots, \epsilon^{(r_1)}$, *i.e.* the conjugates corresponding to the r_1 conjugate real fields, are all positive). The sum [*cf.* (2.4) and (6.2)] in μ will be subject to the restrictions

$$1 \leqslant \mu^{(1)} < \epsilon_1^{(1)}, \ \ldots, \ 1 \leqslant \mu^{(r_1)} < \epsilon_r^{(r_1)},$$

$$1 \leqslant |\mu|^{(r_1+1)} < |\epsilon_{r_1+1}^{(r_1+1)}|, \ \ldots, \ 1 \leqslant |\mu|^{(r_1+r_2-1)} < |\epsilon_{r_1+r_2-1}^{(r_1+r_2-1)}|,$$

and the left-hand side of the new expansion is

$$\sum_{\substack{\mu \equiv c(\mathfrak{a}) \\ (\mu)_\epsilon}} \phi[\mu^{(1)}, \ \ldots, \ \mu^{(n)}],$$

where $(\mu)_\epsilon$ refers to the restriction above. The right-hand side is the sum of

$$\frac{2^{r_2}}{N(\mathfrak{a})\, d^{\frac{1}{2}}} \sum_{1/(\mathfrak{a}\mathfrak{b})\,|\lambda|_\epsilon} e^{-2\pi i S(\lambda c)} \int_0^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty e^{2\pi i S(\lambda\eta)}\, \phi(\eta_1, \eta_2, \ldots, \eta_n)\, d\xi_1 d\xi_2 \ldots d\xi_n$$

and

$$\frac{2^{r_2}}{N(\mathfrak{a})\, d^{\frac{1}{2}}} \int \cdots \iint \phi(\eta_1, \eta_2, \ldots, \eta_n)\, d\xi_1 d\xi_2 \ldots d\xi_n,$$

where in the latter, when $\eta_\nu = \xi_\nu$ is a real variable, the limits of integration are $1 \leqslant \eta_\nu \leqslant \epsilon_\nu^{(\nu)}$, while, when $\eta_\nu = \xi_\nu + i\xi_{\nu+r_2}$ is complex, the limits are given by $1 \leqslant |\eta_\nu| \leqslant |\epsilon_\nu^{(\nu)}|$, except that for the variables in the last $|\eta_\nu|$, or for η_ν if $r_2 = 0$, the limits are $-\infty, \infty$. This expansion follows on taking, in (2.4),

$$f(x_1, x_2, \ldots, x_n) = f_1(x_1, x_2, \ldots, x_n) \text{ or } 0$$

(where f_1 is defined for all values of $x_1, x_2, \ldots$), according as $1 \leqslant x_1 \leqslant \epsilon_1$ or not. If conditions corresponding to (6 A), (6 D), (6 E) are satisfied, the right side of (2.4) is transformed so that the integral in x_1 has limits $0, \infty$ instead of $1, \epsilon_1^{(1)}$. We then assume that

$$f_1(x_1, x_2, \ldots, x_n) = f_2(x_1, x_2, \ldots, x_n) \text{ or } 0,$$

according as $1 \leqslant x_2 \leqslant \epsilon_2^{(2)}$ or not, etc. The conditions (6 B), (6 C) must also be appropriately modified.

There still remains the question of the special formulae arising from

$$f(\xi, \eta, \zeta) = e^{\pi i \tau' \xi\eta\zeta'} \operatorname{sgn}(\xi^\alpha \eta^\beta \zeta^\gamma), \quad \text{etc.},$$

and their justification. These seem to be of interest, but I have not investigated them.

References.

Behnke, **1**. "Über analytische Funktionen und algebraische Zahlen", *Hamburgische Abh.*, 2 (1923), 81–111.

Hecke, **1**. "Über die Zetafunktion beliebiger algebraischer Zahlkörper", *Göttinger Nachrichten* (1917), 77–89.

——— **2**. "Über die *L*-Funktionen und den Dirichletschen Primzahlsatz für einen beliebigen Zahlkorper", *ibid.* (1917), 299–318.

——— **3**. "Reziprozitätsgesetz und Gauss'sche Summen in quadratischen Zahlkörpern", *ibid.* (1919), 265–278.

——— **4**. "Bestimmung der Klassenzahl einer neuen Reihe von algebraischen Zahlkörpern", *ibid.* (1921), 1–23.

——— **5**. "Über einen neuen Zusammenhang zwischen elliptischen Modulfunktionen und indefiniten quadratischen Formen", *ibid.* (1925), 35–44.

——— **6**. "Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen", *Math. Zeitschrift*, 1 (1918), 357–376. (Zweite Mitteilung), *ibid.*, 6 (1920), 11–51.

——— **7**. "Über analytische Funktionen und die Verteilung von Zahlen mod eins", *Hamburgische Abh.*, 1 (1922), 54–76.

——— **8**. "Analytische Funktionen und algebraische Zahlen", I, *ibid.*, 1 (1922), 102–126. II, *ibid.*, 3 (1924), 213–236.

——— **9**. "Darstellung von Klassenzahlen als Perioden von Integralen 3 Gattung aus dem Gebiet der elliptischen Modulfunktionen", *ibid.*, 4 (1926), 211–223.

——— **10**. "Theorie der Eisensteinschen Reihen hoherer Stufe und ihre anwendung auf Funktionentheorie und Arithmetik", *ibid.*, 5 (1927), 199–224.

——— **11.** "Zur theorie der elliptischen Modulfunktionen", *Math. Annalen*, 97 (1926), 210–242.

——— **12.** "Über das Verhalten von $\sum_{m,n} \exp(2\pi i|m^2-2n^2|/8)$ und ähnlichen Funktionen bei Modulsubstitutionen", *Journal für Math.*, 157 (1927), 159–170.

——— **13.** *Theorie der algebraischen Zahlen* (Leipzig, 1923), 224–240.

Landau, **1.** *Einführung in die elementare und analytische Theorie der algebraischen Zahlen und der Ideale* (Leipzig, 1918).

——— **2.** "Über Ideale und Primideale in Idealklassen", *Math. Zeitschrift*, 2 (1918), 52–154.

Müntz, **1.** "Der Summensatz von Cauchy in beliebigen algebraischen Zahlkörpern und die Diskriminante derselben", *Math. Annalen*, 90 (1923), 279–291.

——— **2.** "Allgemeine Begründung der Theorie der höheren ζ-Funktionen", *Hamburgische Abh.*, 3 (1924), 1–11.

Mordell, **1.** "On trigonometric series involving algebraic numbers", *Proc. London Math. Soc.* (2), 21 (1923), 493–496.

——— **2.** "Poisson's summation formula and the Riemann zeta function", *Journal London Math. Soc.*, 4 (1929), 285–291.

——— **3.** "Poisson's summation formula in several variables and some applications to the theory of numbers", *Proc. Camb. Phil. Soc.*, 25 (1929), 412–420.

——— **4.** "Kronecker's fundamental limit formula in the theory of numbers and elliptic functions and similar theorems", *Proc. Royal Soc.* (A), 125 (1929), 262–276.

——— **5.** "Linear substitutions arising from series of Eisenstein's type", *Hamburgische Abh.*, 7 (1929), 164–172.

——— **6.** "The lattice points in a parallelogram", *Math. Annalen*, 103 (1930), 38–47.

——— **7.** "The zeta-functions arising from quadratic forms, and their functional equation", *Quart. J. of Math.*, 1 (1930), 77–101.

Siegel, **1.** "Additive Theorie der Zahlkörper", I, *Math. Annalen*, 87 (1922), 1–35; II, *ibid.*, 88 (1923), 184–210.

——— **2.** "Neuer Beweis für die Funktionalgleichung der Dedekindschen Zetafunktion", *ibid.*, 85 (1922), 123–128.

——— **3.** "Neuer Beweis des Satzes von Minkowski über lineare Formen", *ibid.*, 87 (1922), 36–38.

——— **4.** "Über die Diskriminanten total reeler Körper", *Göttinger Nachrichten* (1922), 17–24.

——— **5.** "Neuer Beweis für die Funktionalgleichung der Dedekindschen Zetafunktion", *ibid.* (1922), 25–31.

INDEX.

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[*The numbers refer to pages.*]

Besicovitch, A. S. On sufficient conditions for a function to be analytic, and on behaviour of analytic functions in the neighbourhood of non-isolated singular points, 1.

Besicovitch, A. S., and Walker, G. On the density of irregular linearly measurable sets of points, 142.

Bosanquet, L. S. A problem concerning mean values of analytic functions, 129.

Burkill, J. C., and Haslam-Jones, U. S. The derivates and approximate derivates of measurable functions, 346.

Cartwright, M. L. On the maximum modulus principle for functions with zeros and poles, 51.

Gilham, C. W. The complete system of the binary (2, 1, 1) form, 259.

Hardy, G. H. Some theorems concerning trigonometrical series of a special type, 441.

Hayden, H. A. Deformations of a curve, in a Riemannian n-space, which displace certain vectors parallelly at each point, 321.

——— On a generalized helix in a Riemannian n-space, 337.

Kennedy, M. D. Upper and lower Lebesgue integrals, 21.

Littlewood, D. E. The groups of the regular solids in n dimensions, 10.

——— The solution of linear congruences in quaternions, 115.

——— Identical relations satisfied in an algebra, 312.

Littlewood, D. E., and Richardson, A. R. Fermat's equation in real quaternions, 235.

Littlewood, J. E. Mathematical notes (9): On functions subharmonic in a circle (III), 222.

Mordell, L. J. On Hecke's modular functions, zeta functions, and some other analytic functions in the theory of numbers, 501.

Paley, R. E. A. C. On the k-analogues of some theorems in the theory of the Riemann ζ-function, 273.

Roth, L. Line congruences in three dimensions, 72.

——— Multiple tangents to surfaces in higher space, 154.

Ruse, H. S. Taylor's theorem in the tensor calculus, 87.

Semple, J. G. On representations of the S_k's of S_n and of the Grassmann manifolds $G(k, n)$, 200.

——— On surfaces of intersection of cubic primals, and cubic Cremona transformations of four-dimensional space, 369.

Steen, S. W. P. Divisor functions: solutions of the differential equation, 356.

Synge, J. L. A characteristic function in Riemannian space and its application to the solution of geodesic triangles, 241.

Titchmarsh, E. C. The zeros of Dirichlet's L-functions, 488.

Todd, J. A. On questions of reality for certain geometrical loci, 449.

Ursell, H. D. Parseval's theorem for almost-periodic functions, 402.

Verblunsky, S. On the limit of a function at a point, 163.

Whitehead, J. H. C. On a class of projectively flat affine connections, 93.

INDEX ACCORDING TO SUBJECTS.

[*The numbers refer to pages.*]

Aggregates: Besicovitch, A. S., and Walker, G., 142; Burkill, J. C., and Haslam-Jones, U. S., 346; Kennedy, M. D., 21; Verblunsky, S., 163.

Algebra: Gilham, C. W., 259; Littlewood, D. E., 115, 312; Littlewood, D. E., and Richardson, A. R., 235.

Almost-periodic Functions: Ursell, H. D., 402.

Analytic Functions: Besicovitch, A. S., 1; Bosanquet, L. S., 129; Cartwright, M. L., 51; Littlewood, J. E., 222.

Bessel Functions: Steen, S. W. P., 356.

Differential Calculus: Ruse, H. S., 87.

Differential Equations: Steen, S. W. P., 356.

Dirichlet Series: Mordell, L. J., 501; Paley, R. E. A. C., 273; Titchmarsh, E. C., 488.

Divisor Functions: Steen, S. W. P., 356.

Fourier Series: Hardy, G. H., 441.

Functions of a Complex Variable: Besicovitch, A. S., 1; Bosanquet, L. S., 129; Cartwright, M. L., 51; Littlewood, J. E., 222.

Functions of a Real Variable: Burkill, J. C., and Haslam-Jones, U. S., 346; Kennedy, M. D., 21; Verblunsky, S., 163.

Geometry: Hayden, H. A., 321, 337; Littlewood, D. E., 10; Roth, L., 72, 154; Semple, J. G., 200, 369; Todd, J. A., 449; Whitehead, J. H. C., 93.

Integral Calculus: Kennedy, M. D., 21.

Invariants: Gilham, C. W., 259.

Modular Functions: Mordell, L. J., 501.

Numbers, Theory of: Mordell, L. J., 501.

Quaternions: Littlewood, D. E., 115; Littlewood, D. E., and Richardson, A. R., 235.

Series, Theory of: Hardy, G. H., 441.

Sets of Points: Besicovitch, A. S., and Walker, G., 142; Burkill, J. C., and Haslam-Jones, U. S., 346; Kennedy, M. D., 21; Verblunsky, S., 163.

Tensor Calculus: Ruse, H. S., 87.

Zeta Functions: Mordell, L. J., 501, Paley, R. E. A. C., 273; Titchmarsh, E. C., 488.

CORRIGENDA.

The following corrections have been received.

LINE CONGRUENCES IN THREE DIMENSIONS.

In my paper (pp. 72-86 of Vol. 32) the following alterations should be made:—

P. 80. Formula (9) should read $c_{2:} = -a + j + c$.

P. 81. Delete formula (16). There are no pinch-points or close-points on the focal surface, nor are there pinch-planes or close-planes.

March 4, 1931. L. ROTH.

BIBL.
LONDIN.
UNIV.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 32.—PART 1.

CONTENTS.

	PAGE
ON SUFFICIENT CONDITIONS FOR A FUNCTION TO BE ANALYTIC, AND ON BEHAVIOUR OF ANALYTIC FUNCTIONS IN THE NEIGHBOURHOOD OF NON-ISOLATED SINGULAR POINTS: A. S. BESICOVITCH...	1
THE GROUPS OF THE REGULAR SOLIDS IN n DIMENSIONS: D. E. LITTLEWOOD	10
UPPER AND LOWER LEBESGUE INTEGRALS: M. D. KENNEDY	21
ON THE MAXIMUM MODULUS PRINCIPLE FOR FUNCTIONS WITH ZEROS AND POLES: MARY L. CARTWRIGHT	51
LINE CONGRUENCES IN THREE DIMENSIONS: L. ROTH...	72

Issued December 29, 1930.

LONDON:

PRINTED AND PUBLISHED FOR THE SOCIETY BY

C. F. HODGSON & SON, LTD.,

2 NEWTON STREET, KINGSWAY, W.C.2.

NOTICE.

The Royal Astronomical Society have decided to sell their sets of the *American Journal of Mathematics* and *Acta Mathematica*, and wish to offer them in the first instance to members of the London Mathematical Society before trying to dispose of them through commercial channels. The set of the *American Journal of Mathematics* consists of Vols. 1–52, of which Vols. 1–37 are bound and Vols. 38–52 are unbound. The set of *Acta Mathematica* consists of Vols. 1–43 with "Tables" of Vols. 1–35, of which Vols. 1–37 and the "Tables" are bound and Vols. 38–43 are unbound.

Any member of the Society who wishes to purchase either or both of these sets is asked to communicate with the Librarian of the London Mathematical Society, Prof. H. Hilton, Bedford College, Regent's Park, London, N.W.1.

BINDING CASES.

In order to avoid delays the Council have decided that in future the Publisher's cloth binding cases for the *Proceedings* and the *Journal* shall be issued with blanks on the back for the number of the volume and the date, so that purchasers may obtain them at any time and need not wait until an interval has elapsed after the completion of each volume. The bookbinder who binds the volume should be able to fill in the blanks in a style uniform with preceding volumes without any difficulty.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of two guineas is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is two guineas. The subscription due from a newly elected member for his first year of membership is one guinea if his election takes place between March and June inclusive: any Member may compound for his *annual* subscriptions by the payment of twenty-five guineas in one sum, or in two equal annual instalments. In certain cases the composition fee is reduced. Members of the Society who are also members of the American Mathematical Society are entitled to a reduction of subscription as regards one of the Societies. Details of the arrangements may be obtained from the Secretaries or the Treasurer of either Society.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

Annual subscriptions are due at the beginning of each Session (the first Thursday in November of each year), and may be paid to the account of the London Mathematical Society, London County, Westminster and Parr's Bank, 1 St. James's Square, London, S.W.1, or to the Treasurer, Dr. A. E. Western, 35 Essex Street, Strand, London, W.C.2.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting in November.

The Officers are a President, Vice-Presidents, a Treasurer, a Librarian, and Secretaries.

The Ordinary Meetings of the Society are now held at Burlington House, W.1, and begin at 5 o'clock in the evening. The Meetings for the Session 1930–31 are on November 13 and December 11, 1930, and January 15, February 5, March 12, April 23, May 14, and June 18, 1931. Cards giving the Programmes of the meetings can be had by Members a few days beforehand on application to C. F. Hodgson & Son, Ltd., 2 Newton Street, Kingsway, London, W.C.2.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to one or more Members, who report to the Council on its fitness for publication in the *Proceedings* or the *Journal*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers of more than about eight pages in length intended for communication to the Society are requested to furnish to the Secretaries short Abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained, for the use of the Council and Referees, not for publication. Such an Abstract should be extremely brief; as a rule its length should not exceed the length of half a page of the *Proceedings*.

Communications for the Secretaries may be forwarded to them at the following addresses:—

The University, Birmingham—G. N. Watson, Sc.D., F.R.S.
St. John's College, Cambridge—F. P. White, M.A.
London Mathematical Society, Burlington House, W.1.

Subscriptions to Journals

¶ We have a very highly organised department for the supply to Librarians, Institutions, and others, of Journals and Periodicals, both English and Foreign. We can guarantee promptitude and intelligent handling of all orders entrusted to us.

¶ We have a very efficient department for finding the unusual, the rare, and the out-of-print book.

¶ Librarians, Executors, and others, desirous of either disposing of or buying Scientific Books or Periodicals, or Books in General Literature, are invited to communicate with us. Valuers for Probate. Catalogues frequently issued and sent post free.

Remember that we are only as far from you as the nearest post-box.

We will purchase the following:

BOOKS:

Archimedes, all Works by.
Dickson's History of Numbers.
Dodgson's Euclid.
Euclid, Geometrie, Billingsley.
Galilei, all Works by.
Gilbert. On the Magnet. All editions.
Hamilton. Quaternions.
Harkness & Morley. Functions.
Mathew's Theory of Numbers.
Maxwell. Scientific Papers.
Meyer's Kinetic Theory of Gases.
Montucla. Histoire des Mathématiques.
Napier (John), all Books by.
Newton. Opera Omnia.
Newton. Principia. All editions.
Recorde, all Books by.
Routh's Dynamics of a Particle.
Russell's Principles of Mathematics. Vol. I.
Salmon. Higher Algebra.
" Higher Plane Curves.
Thomson's Vortex Rings.
Todhunter. Theory of Probability.
" Laplace's Functions.
" Elasticity.
" Attraction.
Wallis, Opera Mathematica.
Whittaker's Aether and Electricity.
Young's Sets of Points.

PERIODICALS: Complete Sets or Long Runs.

Acta Mathematica.
Annalen der Physik.
Archiv der Mathematik und Physik.
Astrophysical Journal.
Cambridge Philosophical Society.
Comptes Rendus de l'Académie.
Jahrbuch der Mathematik.
Journal de Mathématiques. Liouville.
Journal de Physique.
Journal für Mathematik. Crelle.
Journal (Quarterly) of Mathematics.
LONDON MATHEMATICAL SOCIETY PROCEEDINGS.
Mathematical Gazette.
Maths. from "The Educational Times."
Mathematische Annalen.
Mathematische Zeitschrift.
Messenger of Mathematics.
Philosophical Magazine.
Physical Society Proceedings.
R.S. Philosophical Transactions and Proceedings.
Zeitschrift für Mathematik und Physik. And all standard Mathematical Publications.

Catalogue 340. Scientific Books and Publications of Learned Societies. Catalogue 352. The Librarian's Catalogue of Periodicals, Journals, and Transactions of the Learned Societies, &c. ***Free on request.***

W. HEFFER & SONS, LTD.,

CAMBRIDGE, ENGLAND.

Phone: 862. Telegrams & Cables: "Heffer, Cambridge."